4. Electromagnetic Waves III

4.1. Diffraction and Scattering

In the previous sections, we have discussed the generation of EM waves and their propagation through a homogeneous medium. We have also discussed how the interface between two semi-infinite media interacts with the waves. In this section, we will continue our discussions to the interaction of EM waves with a nanoscale structure. We consider two general cases, the diffraction and the scattering. Diffraction is one type of scattering in which the EM wave is scattered off from a small aperture. From our previous discussions, the details of the inter-atomic and atomic-EM wave interactions can be lumped into a polarization vector (after ignoring higher order terms) as in (2.7). This process can be thought of as an ensemble of dipole moments being induced by the incoming EM wave and a new EM wave is generated from those dipole antennas. This picture, rigorous speaking, is not quite correct as the induced EM wave can in turn interact with the medium itself and generate new dipole moments. The process will iterate over and over for an infinite amount of times. But in most cases we can still use the above picture (that is to truncate the iteration process) to get a good approximation.

4.1.1. Scattering by a Small Sphere (Rayleigh scattering)

We consider the scattering of an EM wave by a small sphere. If the sphere is small enough (still > 10 nm but much smaller than the wavelength), we can approximate the induced polarization vector as an infinitesimal dipole moment. The scattered wave is the radiation generated by this dipole antenna as given by (3.26). To determine the dipole moment, we match the boundary conditions for the fields at the surface of the sphere (i.e. the interface between two macroscopic media and in this case the air and the sphere). Assume the incident wave is a plane wave given by:

$$\vec{E}_i = \hat{z}E_i e^{ikx} \approx \hat{z}E_i \quad \text{as} \quad kx \ll 1 \tag{4.1}$$

The fields (including the field inside the sphere and the scattered field) near the origin of the sphere (i.e. the location of the induced dipole moment) should have the near field nature, i.e. quasi-static. Because there are no boundaries inside the sphere and the field inside the sphere satisfies the Poisson equation, the solution is a plane wave polarized in the same direction as the incoming wave:

$$\vec{E}_t = \hat{z}E_t = \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)E_t \tag{4.2}$$

The scattered field is given by (3.29):

$$\vec{E}_s(\vec{r}) = \frac{iIl}{4\pi\omega\varepsilon}\left(\frac{1}{r}\right)^3\left[\hat{r}2\cos\theta + \hat{\theta}\sin\theta\right] \equiv \left[\hat{r}2\cos\theta + \hat{\theta}\sin\theta\right]\left(\frac{a}{r}\right)^3 E_s \tag{4.3}$$

To match the boundary condition at the surface of the sphere, we demand:

$$\begin{aligned} E_{out(\tan)} &= E_{in(\tan)} \\ D_{out(normal)} &= D_{in(normal)} \end{aligned} \tag{4.4}$$

It leads to:

$$\begin{aligned} -E_i + E_s &= -E_t \\ \varepsilon E_i + 2\varepsilon E_s &= \varepsilon_s E_t \end{aligned} \tag{4.5}$$

And we get:

$$E_s = \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} E_i \tag{4.6}$$

Combining (4.6) and (4.3), we have:

$$Il = -i4\pi ka^3 \sqrt{\frac{\varepsilon}{\mu_0}} \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right) E_i \tag{4.7}$$

Now the scattered wave is nothing but the radiation generated from an infinitesimal dipole radiator with a strength given by (4.7). In the far field region, the electric field is:

$$\vec{E}(\vec{r}) = -\hat{\theta} k^2 a^3 \frac{e^{ikr}}{r} \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right) E_i \sin\theta \equiv \hat{\theta} E_\theta \tag{4.8}$$

The magnetic field is:

$$\vec{H}(\vec{r}) = \hat{\varphi} \sqrt{\frac{\varepsilon}{\mu_0}} E_\theta \equiv \hat{\varphi} H_\varphi \tag{4.9}$$

The total (averaged) scattered power is:

$$\langle S \rangle = \frac{1}{2} \int \vec{E} \times \vec{H}^* \cdot d\vec{s} = \frac{4\pi}{3} \sqrt{\frac{\varepsilon}{\mu_0}} \left(k^2 a^3 E_i \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right)^2 \tag{4.10}$$

The scattering cross section is defined as the ratio of the total scattered power and the incident power:

$$\sigma = \frac{8\pi}{3} \left(k^2 a^3 \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right)^2 \propto \omega^4 a^6 \tag{4.11}$$

Higher frequency waves are scattered more. This explains why the sky is blue and the sunset is red.

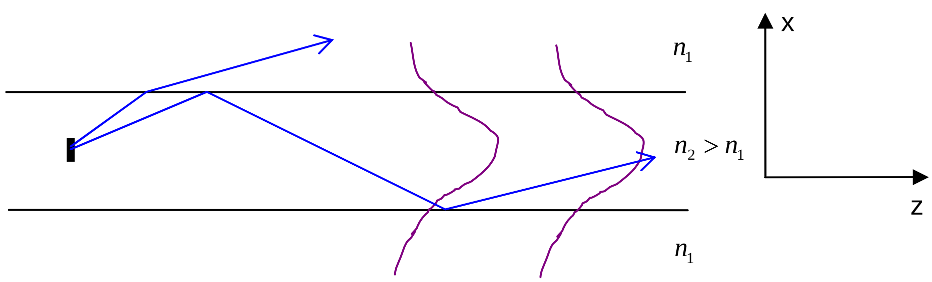
In a special case when $\varepsilon_s = -2\varepsilon$, the scattering cross section goes to infinity. Rigorously speaking, if the scattering cross section goes to infinity, we need to take into account all the iteration terms in the scattering process (i.e. scattered wave induces the dipole moment and induces the scattered wave and …) But nonetheless, it tells us a there is a resonance when $\varepsilon_s = -2\varepsilon$. This again corresponds to the surface polariton mode and will be discussed later.

4.1.2. Mie scattering

For scattering by a not-so-small sphere, we can no longer assume the scattered wave is resulted from an infinitesimal dipole radiator. This problem, however, can be solved rigorously by matching the boundary conditions for the Maxwell's equations. This is called the Mie scattering and those of you who are interested in this topic can refer to any of J. A. Kong, *Electromagnetic Wave Theory*, 2nd ed., Section 6.1.

4.2. Waveguiding

So far we have discussed how EM wave propagates and interacts with macroscopic media. We learned about reflection and refraction. We also learned when certain condition is met, we will have total internal reflection. In this section, we discussed how we can use the total internal reflection to guide and confine the EM wave along an "energy pipe", or a "waveguide." The simplest example is to stack two semi-infinite straight interfaces and with a higher refractive-index material sandwiched between two low refractive-index materials. When a radiation source (e.g. the dipole radiator) is placed inside the waveguide, the part of EM radiation that propagates at an angle greater than the total internal reflection angle will be guided inside the waveguide without energy loss through both interfaces. After a certain distance of propagation, all the radiation components that do not satisfy the total internal reflection condition will lose their energy. The part that is being guided will maintain a constant electric and magnetic field profiles along the waveguide direction. It is called the waveguide "transverse mode."



To solve the electric (and magnetic) field profile, we write down a general solution for a TE polarized field as follows.

$$E_y(x,z) = \begin{cases} A_1 e^{-\alpha_x x} e^{ik_z z} & x > d/2 \\ A_2 \cos k_{2x} x \quad \text{or} \quad A_2 \sin k_{2x} x & \\ \pm A_1 e^{\alpha_x x} e^{ik_z z} & x < -d/2 \end{cases} \tag{4.12}$$

where

$$\begin{aligned} k_{2x}^2 + k_z^2 &= \omega^2 \mu_0 \varepsilon_2 \\ (i\alpha_x)^2 + k_z^2 &= \omega^2 \mu_0 \varepsilon_1 \end{aligned} \tag{4.13}$$

The magnetic fields are:

$$H_z(x,z) = \frac{1}{i\omega\mu_0} \frac{\partial E_y}{\partial x} \tag{4.14}$$

The boundary conditions give us for even modes:

$$\frac{\alpha_x}{k_{2x}} = \tan \frac{k_{2x} d}{2} \tag{4.15}$$

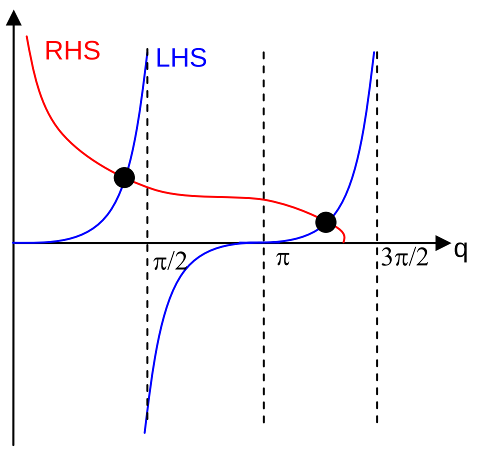
And for odd modes:

$$\frac{\alpha_x}{k_{2x}} = -\cot\frac{k_{2x}d}{2} \tag{4.16}$$

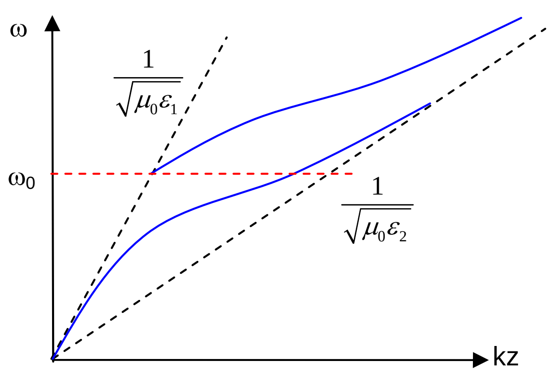
Using $\alpha_x^2 = \omega^2\mu_0(\varepsilon_2 - \varepsilon_1) - k_{2x}^2$, we can solve (4.15) and (4.16) by graphical methods. For example, (4.15) can be rewritten as:

$$\tan q = \frac{\sqrt{(\Delta kd/2)^2 - q^2}}{q} \tag{4.17}$$

where $q = k_{2x}d/2$ and $\Delta k^2 = \omega^2\mu_0(\varepsilon_2 - \varepsilon_1)$.



If we increase the index contrast between n_1 and n_2, the RHS curve will move upward and more solutions will be possible. After we obtain the solution for k_{2x} using the graphical method, we can solve for the propagation constant k_z via (4.13). The plot of ω vs k_z is called the dispersion relation.



Dispersion relation tells us what are the possible propagation constants in a particular structure. For example for frequency < ω_0, there is only one possible waveguide mode to be supported by the slab waveguide structure. Waveguides that are designed such as it support only a single transverse mode are called single-mode waveguides.