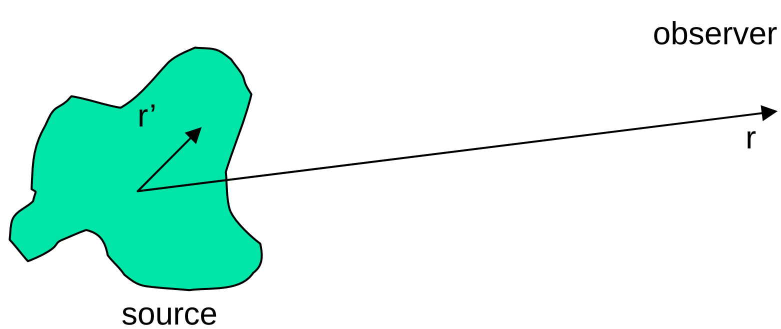
3. Electromagnetic Waves II

Last time, we discussed the following.

1. The propagation of an EM wave through a macroscopic media: We discussed how the wave interacts with the media and how all of the details of inter-atomic and atom-EM wave interactions can be described by a constitutive relation. Examples we discussed last time included the change of the speed of the propagation, reflection and refraction, gain and loss, and tunneling through a thin slab.
2. Macroscopic Maxwell's equations: We derived the macroscopic Maxwell's equations. These equations can describe all the EM interaction with macroscopic media with a linear dimension > 10 nm. The constitutive relation can be measured from experiments. It can also be calculated by taking into account all the microscopic interactions. For example, to the first order, the dielectric constant of a homogeneous medium can be determined by the polarization vector *P* which is the total dipole moment in the medium. We will show you how this can be done later in the class when we discuss microscopic interaction between light and matters.

$$\varepsilon = \varepsilon_0 + P / |E| \tag{3.1}$$

3.1. Generation of EM waves



3.1.1. Antenna basics

In the discussions so far, we have only studied the behavior of a given EM wave (e.g. a plane wave) and its interaction with macroscopic media but we have not discussed how the EM wave is generated. We will see in this section how a time-variant current source can generate an EM wave. We will solve the wave equation with the inclusion of the current source as follows.

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = i\omega\mu\vec{J} \tag{3.2}$$

Once the electric field is obtained, we can calculate the magnetic field by:

$$\vec{H} = \frac{\nabla \times \vec{E}}{i\omega\mu} \tag{3.3}$$

The solution to (3.2) can be written as follows for any observer at $\vec{r}$ outside the source distribution region.

$$\vec{E}(\vec{r}) = i\omega\mu \int_{source} \bar{\bar{G}}(\vec{r},\vec{r}\,')\cdot J(\vec{r}\,')d\vec{r}\,' \tag{3.4}$$

where $\bar{\bar{G}}$ is a dyadic[2] Green's function which satisfies

$$\nabla\times\nabla\times\bar{\bar{G}} - k^2\bar{\bar{G}} = \bar{\bar{I}}\delta(\vec{r}-\vec{r}\,') \tag{3.5}$$

$\bar{\bar{I}}$ is an identity matrix ($\bar{\bar{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} = \hat{r}\hat{r} + \hat{\theta}\hat{\theta} + \hat{\varphi}\hat{\varphi}$). You can easily verify by inserting (3.4) into the LHS of (3.2) and using (3.5), you get the RHS of (3.2). Note also that because $\vec{r}$ is outside the source, we can interchange $\nabla\times\nabla\times$ and the volume integral. Please note all the differential operators below including (3.5) act on $\vec{r}$ unless otherwise specified.

To solve $\bar{\bar{G}}$, first we notice that $\nabla\times\nabla\times\bar{\bar{G}} = -\nabla^2\bar{\bar{G}} + \nabla\nabla\cdot\bar{\bar{G}}$. If we take the divergence of (3.5) and use the vector identity $\nabla\cdot\left(\nabla\times\vec{A}\right) = 0$, we get:

$$-k^2\nabla\cdot\bar{\bar{G}} = \nabla\delta(\vec{r}-\vec{r}\,') \tag{3.6}$$

Substituting (3.6) back into (3.5), we have:

$$\left(\nabla^2 + k^2\right)\bar{\bar{G}} = -\left(\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right)\delta(\vec{r}-\vec{r}\,') \tag{3.7}$$

We can verify that $\bar{\bar{G}}$ now can be written in terms of a scalar function:

$$\bar{\bar{G}} = \left(\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right)g \tag{3.8}$$

where the scalar function *g* satisfy:

$$\left(\nabla^2 + k^2\right)g(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,') \tag{3.9}$$

To solve *g* in (3.9), we notice that $g(\vec{r},\vec{r}\,')$ should depend only on $\vec{r}-\vec{r}\,'$ but not the absolute location of $\vec{r}\,'$. Therefore we can arbitrarily set $\vec{r}\,'$ at origin. After the choice of $\vec{r}\,'$, we can easily see the solution for *g* must be spherical symmetric around the origin. (3.9) becomes:

$$r^2\frac{d^2g(r)}{dr^2} + 2r\frac{dg(r)}{dr} + k^2r^2g(r) = -\delta(r) \tag{3.10}$$

The solution to (3.10) is:

$$g(r) = C\frac{e^{ikr}}{r} \tag{3.11}$$

[2] Dyadic $\bar{\bar{G}}$ is a direct product of two vectors. For example, if $\bar{\bar{G}} = \vec{A}\vec{B}$, its index notation becomes $G_{ij} = A_iB_j$. In matrix notation, the direct product of two vectors can be represented by a 3x3 matrix.

To determine the constant *C*, we integrate (3.9) over a volume including the origin and let the volume go to zero:

$$\begin{aligned}&\int_V \nabla^2 g d\bar{r} = -1\\ &\Rightarrow \int_S \nabla g \cdot \hat{n} dS = 4\pi r^2 \left.\frac{dg}{dr}\right|_{r=\delta\to 0} = -1\\ &\Rightarrow C = \frac{1}{4\pi}\end{aligned} \tag{3.12}$$

Combining (3.4), (3.8), (3.11) and (3.12), we have:

$$\vec{E}(\bar{r}) = i\omega\mu\left(\overline{\overline{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot \int_{source} \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} J(\bar{r}')d\bar{r}' \tag{3.13}$$

The magnetic field is (from (3.3) and (3.13)):

$$\begin{aligned}\vec{H} &= \frac{\nabla\times\vec{E}}{i\omega\mu}\\ &= \nabla\times\int_{source}\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}J(\bar{r}')d\bar{r}'\end{aligned} \tag{3.14}$$

Before we proceed, we have to remember that all the quantities in (3.13) and (3.14) are in the frequency domain. We have dropped their $e^{-i\omega t}$ dependence. If $J(\bar{r}')e^{-i\omega t}$ is a static source, $\omega = 0$ and $k = 0$.

3.1.2. General properties of near field and far field

Depending on the distance between the observer and the source, we can study two extreme cases, the near field ($k|\bar{r}-\bar{r}'| \ll 1$) and the far field ($k|\bar{r}-\bar{r}'| \gg 1$). In the far field, we have $|\bar{r}-\bar{r}'| \approx r - \hat{r}\cdot\bar{r}'$. The electric field becomes:

$$\vec{E}(\bar{r}) = i\omega\mu\left(\overline{\overline{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\frac{e^{ikr}}{4\pi r}\int_{source} J(\bar{r}')e^{i\bar{k}\cdot\bar{r}'}d\bar{r}' \tag{3.15}$$

The integral in (3.15) results in a function that depends only on θ and φ. We can define a vector current moment as:

$$\bar{f}(\theta,\varphi) = \int_{source} J(\bar{r}')e^{-i\bar{k}\cdot\bar{r}'}d\bar{r}' \tag{3.16}$$

In the far field region, we only keep terms on the order of $1/kr$ and neglect all the higher order terms. Using:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \tag{3.17}$$

(3.15) becomes:

$$\vec{E}(\bar{r}) = i\omega\mu\left(\overline{\overline{I}} - \hat{r}\hat{r}\right)\cdot\frac{e^{ikr}}{4\pi r}\bar{f} = i\omega\mu\frac{e^{ikr}}{4\pi r}\left(\hat{\theta}f_\theta + \hat{\varphi}f_\varphi\right) \tag{3.18}$$

This is an outgoing wave with a spherical wave front. The electric field is perpendicular to the propagation direction. At large distance, the wave can be approximately by a plane wave.

In the near field, we have $e^{ik|\bar{r}-\bar{r}'|} = 1 + ik|\bar{r}-\bar{r}'| + \cdots \approx 1$ and the observer is at a distance many times smaller than the wavelength from the source. (3.13) becomes:

$$\vec{E}(\bar{r}) = i\omega\mu\left(\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\int_{source}\frac{J(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|}d\bar{r}' \tag{3.19}$$

This is a quasi-static field because of the absence of the oscillating exponential term. Note the field is not truly static since there is an implicit time harmonic factor $e^{-i\omega t}$. Similarly, the magnetic field is:

$$\vec{H}(\bar{r}) = \nabla\times\int_{source}\frac{J(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|}d\bar{r}' \tag{3.20}$$

Because in near field region, we have $k|\bar{r}-\bar{r}'| \ll 1$, the contribution from the 2nd term in the parenthesis of (3.19) dominates and the magnetic field can usually be neglected (because magnetic field gets differentiation once while the electric field gets differentiation twice). We will see an example in the following when we discuss the dipole radiator. But we notice that if magnetic field can be neglected, the solution of fields satisfies the electrostatic equation or the Poisson equation:

$$\nabla\cdot\vec{E} = 0 \tag{3.21}$$

Or in terms of the potential ϕ:

$$\nabla^2\phi = 0 \tag{3.22}$$

3.1.3. Dipole radiation

The most fundamental antenna is a Hertzian dipole which consists of a current-carrying wire with an infinitesimal length *l*:

$$J(\bar{r}') = \hat{z}Il\delta(\bar{r}') \tag{3.23}$$

Substituting (3.23) into (3.13), we get:

$$\begin{aligned}\vec{E}(\bar{r}) &= i\omega\mu\left(\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\int_{source}\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\hat{z}Il\delta(\bar{r}')d\bar{r}' \\ &= i\omega\mu\left(\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\frac{e^{ikr}}{4\pi r}\hat{z}Il \\ &= i\omega\mu Il\left(\hat{z} + \frac{\nabla}{k^2}\frac{\partial}{\partial z}\right)\frac{e^{ikr}}{4\pi r}\end{aligned} \tag{3.24}$$

Now we make the coordinate transformation to the spherical coordinate by

$$\begin{aligned}&\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \\ &\frac{\partial}{\partial z}\frac{e^{ikr}}{4\pi r} = \left(ik - \frac{1}{r}\right)\cos\theta\frac{e^{ikr}}{4\pi r} \\ &\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta\end{aligned} \tag{3.25}$$

(3.24) becomes:

$$\vec{E}(\vec{r}) = -i\omega\mu Il\frac{e^{ikr}}{4\pi r}\left(\hat{r}2\cos\theta\left[\frac{i}{kr}+\left(\frac{i}{kr}\right)^2\right]+\hat{\theta}\sin\theta\left[1+\frac{i}{kr}+\left(\frac{i}{kr}\right)^2\right]\right) \tag{3.26}$$

3.1.4. Near and far fields for dipole radiators

In the far field, (3.26) reduces to:

$$\vec{E}(\vec{r}) = -i\omega\mu Il\frac{e^{ikr}}{4\pi r}\hat{\theta}\sin\theta \tag{3.27}$$

The wavefront is a spherical outgoing wave and radiation pattern consists of two side lobes with no electric field along the z axis. The polarization of the electric field is perpendicular to the direction of the propagation. The magnetic field can be calculated by (3.3) to be:

$$\vec{H}(\vec{r}) = -ikIl\frac{e^{ikr}}{4\pi r}\hat{\varphi}\sin\theta \tag{3.28}$$

In the near field, (3.26) reduces to:

$$\begin{aligned}\vec{E}(\vec{r}) &= -i\omega\mu Il\frac{1}{4\pi r}\left(\hat{r}2\cos\theta\left[\left(\frac{i}{kr}\right)^2\right]+\hat{\theta}\sin\theta\left[\left(\frac{i}{kr}\right)^2\right]\right)\\ &= \frac{iIl}{4\pi\omega\varepsilon}\left(\frac{1}{r}\right)^3\left[\hat{r}2\cos\theta+\hat{\theta}\sin\theta\right]\end{aligned} \tag{3.29}$$

The field fades away quickly with r^{-3} in contrast to far field dependence of r^{-1}. Note the field is quasi-static and has the $\hat{r}$ component.

3.1.5. Radiation from a Moving Charge

The dipole radiator is one special type of radiation sources. Since the dipole source usually consists of lots of oscillating (or moving) charges. It is interesting to study the radiation from a single moving charge. It has applications for example in particle detectors (using Cherenkov radiation) and synchrotron radiation. The charge density of a moving charge with a trajectory $\vec{r}_0\,'(t)$ is:

$$\rho(\vec{r}\,',t) = q\delta(\vec{r}\,'-\vec{r}_0\,'(t)) \tag{3.30}$$

The current density is therefore:

$$\vec{J}(\vec{r}\,',t) = q\frac{d\vec{r}_0\,'(t)}{dt}\delta(\vec{r}\,'-\vec{r}_0\,'(t)) \tag{3.31}$$

To solve the electric field with (3.13), we need to convert $\vec{J}(\vec{r}\,',t)$ to the frequency domain. To do that, we use the Fourier transform as follows.

$$\begin{aligned}\vec{J}(\vec{k},\omega) &= \int d\vec{r}\,'\int dtq\frac{d\vec{r}_0\,'(t)}{dt}\delta(\vec{r}\,'-\vec{r}_0\,'(t))e^{i(\omega t-\vec{k}\cdot\vec{r}\,')}\\ &= \int dtq\frac{d\vec{r}_0\,'(t)}{dt}e^{i(\omega t-\vec{k}\cdot\vec{r}_0\,'(t))}\end{aligned} \tag{3.32}$$

We consider two cases. In the first case, the particle is moving at a constant velocity along the z-axis (i.e. no acceleration.) (3.32) becomes:

$$\vec{J}(\vec{k},\omega) = 2\pi q\vec{v}_0\delta\left(\omega - \vec{k}\cdot\vec{v}_0\right) \tag{3.33}$$

From (3.33), the electric field will have terms with wavevectors given by

$$k = \omega/(v_0\cos\theta) \tag{3.34}$$

where θ is the angle from the z-axis to the propagation direction. Remember that the wavevector itself needs to satisfy the Maxwell's equations, i.e. $k = \omega n/c$. We have:

$$v_0 = \frac{c}{n\cos\theta} > \frac{c}{n} \tag{3.35}$$

That is in order to generate radiation from a charge moving at a constant velocity, the velocity has to be greater than the speed of light in the media being considered. The radiation generated in such a way is called the Cherenkov radiation. That's why usually a moving charge without any acceleration does not radiate.

The second case we will consider is a charge moving with acceleration. Because of the acceleration, the integral in (3.32) will have terms that can generate EM waves. It is generally hard to evaluate such an integral. But in summary an accelerated charge radiates.