

## 2. Electromagnetic Waves I

In this and the following few lectures, we will review or extend your knowledge about electromagnetic waves with particular emphasis in near-field and nano-scale phenomena. This shall not be a replacement for your regular EM course. If you are serious in doing research in this area, a graduate-level EM theory is still highly recommended.

### 2.1. Interaction of EM waves with macroscopic media

We will start with the Maxwell equations and their extension to the macroscopic media. Last time we used 15 minutes to review (or remind you of) a few historical events that led to the quantitative description of electric and magnetic fields in the vacuum as follows.

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \frac{1}{\mu_0} \nabla \times \vec{B} &= \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{E} &= \rho / \epsilon_0 \\ \nabla \cdot \vec{B} &= 0\end{aligned}\tag{2.1}$$

The two terms  $\vec{J}$  ( $= \rho \vec{v}$ ) and  $\rho$  represent the current density and charge density, respectively. Once we know the distribution of these two source terms, we can calculate (most of the cases numerically) the electric field as well as the magnetic field at any coordinate and at any time according to (2.1).

In nano-photonics (in fact in most of the engineering scenarios), we deal with structures often comprising of at least 1000 atoms or molecules in spite of their “nano” dimensions. To treat this problem directly from (2.1) requires us to consider the interaction between the fields and the atoms/molecules as well as the interaction among the atoms/molecules. This is not impossible but simply not practical. In a more practical way, we will need to treat this problem from a macroscopic point of view, that is to consider the net effect from several atoms/molecules together by taking the spatial average over a length scale in which the details of the atom-atom interaction are not important anymore. How many atoms do we need to average over? From the facts that x-ray (with a wavelength of several Å) can probe the atomic structures and visible light (with a wavelength of several hundreds of nm) sees a continuous medium, a reasonable length scale for average is about 10 nm. We consider only the procedure of averaging over the charge density and leave the current density as a reading assignment<sup>1</sup>. The total charge density  $\rho$  can be divided into two contributions: one from the free charges  $\rho_{free}$  and one from the “bound” charges  $\rho_{bound}$ . The latter term includes all those charges (e.g. electrons and nuclei) from individual atoms/molecules in the media. At any instantaneous

<sup>1</sup> J. D. Jackson, *Classical Electrodynamics*, 2<sup>nd</sup> ed, section 6.7.

time  $t$ ,  $\rho_{bound}$  can be expressed as follows.

$$\rho_{bound} = \sum_n \rho_n(\vec{r}, t) = \sum_n \sum_j q_j \delta(\vec{r} - \vec{r}_j) \quad (2.2)$$

The (spatial) average of the charge density for the n-th molecule  $\rho_n(\vec{r}, t)$  is:

$$\langle \rho_n(\vec{r}, t) \rangle = \int d^3r' f(\vec{r}') \rho_n(\vec{r} - \vec{r}', t) \quad (2.3)$$

where  $f(\vec{r}')$  is the probability distribution function of the location of the n-th molecule.  $\vec{r}'$  is the coordinate system whose origin is chosen at the center of the n-th molecule. With this choice of coordinate system,  $f(\vec{r}')$  is only appreciable within an atomic scale. It enables us to take Taylor expansion around the origin of  $\vec{r}'$  and neglect higher order terms. Using (2.2) and writing  $\vec{r}_j$  as  $\vec{r}_j = \vec{r}_{jn} + \vec{r}_n$  where  $\vec{r}_{jn}$  is the location of the j-th charge in the  $\vec{r}'$  coordinate system, (2.3) becomes:

$$\begin{aligned} \langle \rho_n(\vec{r}, t) \rangle &= \sum_j q_j \int d^3r' f(\vec{r}') \delta(\vec{r} - \vec{r}' - \vec{r}_{jn} - \vec{r}_n) \\ &= \sum_j q_j f(\vec{r} - \vec{r}_{jn} - \vec{r}_n) \\ &= \sum_j q_j \left[ f(\vec{r} - \vec{r}_n) - \vec{r}_{jn} \cdot \nabla_{\vec{r}} f(\vec{r} - \vec{r}_n) + \dots \right] \\ &= q_n f(\vec{r} - \vec{r}_n) - \bar{p}_n \cdot \nabla_{\vec{r}} f(\vec{r} - \vec{r}_n) + \dots \\ &= \langle q_n \delta(\vec{r} - \vec{r}_n) \rangle - \nabla_{\vec{r}} \cdot \langle \bar{p}_n \delta(\vec{r} - \vec{r}_n) \rangle + \dots \end{aligned} \quad (2.4)$$

where  $\bar{p}_n \equiv \sum_j q_j \vec{r}_{jn}$  is the dipole moment of the n-th molecule. The 1<sup>st</sup> term in (2.4) vanishes due to charge neutrality. Hence the total charge density including both the free charges and the sum of contributions from n molecules is:

$$\langle \rho(\vec{r}, t) \rangle = \rho_{free}(\vec{r}, t) - \nabla \cdot \bar{P}(\vec{r}, t) + \dots \quad (2.5)$$

Now the Gauss's Law in the average form becomes:

$$\epsilon_0 \nabla \cdot \langle \vec{E} \rangle = \langle \rho \rangle = \rho_{free}(\vec{r}, t) - \nabla \cdot \bar{P}(\vec{r}, t) + \dots \quad (2.6)$$

Define  $\bar{D}$  (called the displacement) as:

$$\langle \bar{D} \rangle = \langle \epsilon_0 \vec{E} \rangle + \bar{P}(\vec{r}, t) + \dots \equiv \epsilon \langle \vec{E} \rangle \quad (2.7)$$

where  $\bar{P}$  is the polarization. We can rewrite the Gauss's Law as:

$$\nabla \cdot \langle \bar{D} \rangle = \rho_{free}(\vec{r}, t) \quad (2.8)$$

With the understanding of the quantity  $\bar{D}$  being an average quantity over 10 nm and  $\rho$  being the contribution from only the free charges, we can drop the signs for averages and simply write the Gauss's Law as:

$$\nabla \cdot \bar{D} = \rho \quad (2.9)$$

Similarly we can define the magnetic field strength  $\vec{H}$  as (again with the understanding of the proper

average being carried out):

$$\vec{H} = \vec{B} / \mu_0 - \vec{M} \quad (2.10)$$

where  $\vec{M}$  is the magnetization.

With the above discussions, we see that the interaction of the electromagnetic fields with a macroscopic medium (>10 nm in linear dimension) can be treated by a “macroscopic” version of Maxwell's equations as follows.

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0 \end{aligned} \quad (2.11)$$

where  $\rho$  and  $\vec{J}$  only consist of the “free” parts. Again, all the field quantities in (2.11) are macroscopically (~ 10 nm) averaged quantities. In general, the relationships between  $(\vec{B}, \vec{E})$  and  $(\vec{H}, \vec{D})$  can be expressed in general by the constitutive relations as follows.

$$\begin{aligned} \vec{D} &= \mathbf{f}(\vec{E}, \vec{H}) \\ \vec{B} &= \mathbf{g}(\vec{H}, \vec{E}) \end{aligned} \quad (2.12)$$

Different materials possess different details in their atomic scale interactions and therefore are governed by different constitutive relations. Examples of different constitutive relations are given in the Powerpoint slides.

## 2.2. Boundary conditions

From (2.11), we can study the behaviors of fields across the boundary between the vacuum and a macroscopic medium or the boundary between two macroscopic media. We consider an infinitesimal loop or volume across the interface. Applying the Gauss's theorem and Stoke's theorem as follows.

$$\begin{aligned} \int \nabla \cdot \vec{A} dv &= \oint \vec{A} \cdot d\hat{s} \\ \int \nabla \times \vec{A} \cdot d\hat{s} &= \oint \vec{A} \cdot d\hat{l} \end{aligned} \quad (2.13)$$

We will only review one derivation as an example and leave the rest for your own exercise. Starting with the Gauss's Law and considering an infinitesimal area across the interface:

$$\begin{aligned} \int \nabla \cdot \vec{D} dv &= \oint \vec{D} \cdot d\hat{s} = \int \rho dv \\ \vec{D}_1 \cdot \hat{n}_1 \delta A + \vec{D}_2 \cdot \hat{n}_2 \delta A + (\dots) \delta h &= \int \rho dv \xrightarrow{\delta h \rightarrow 0} \rho_s \delta A \\ \hat{n}_{12} \cdot (\vec{D}_1 - \vec{D}_2) &= \rho_s \end{aligned} \quad (2.14)$$

All the boundary conditions are summarized as follows.

$$\begin{aligned}
 E_{1t} &= E_{2t} \\
 \vec{n}_2 \times (\vec{H}_1 - \vec{H}_2) &= \vec{J}_s \\
 D_{1n} - D_{2n} &= \rho_s \\
 B_{1n} - B_{2n} &= 0
 \end{aligned}
 \tag{2.15}$$

## 2.3. Basic properties of Maxwell's Equations

### 2.3.1. Conservation Laws

Taking the divergence of the Ampere's Law and using the Gauss's Law, we have

$$\begin{aligned}
 \nabla \cdot (\nabla \times \vec{H}) &= \nabla \cdot \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \\
 0 &= \nabla \cdot \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \\
 \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} &= 0 \\
 \nabla \cdot (\rho \vec{v}) + \frac{\partial \rho}{\partial t} &= 0
 \end{aligned}
 \tag{2.16}$$

This is the continuity equation. The physical meaning of (2.16) is obvious after integrating over a closed volume. The total charge is a conserved quantity.

We consider another quantity  $\vec{S} = \vec{E} \times \vec{H}$  (called the Poynting vector):

$$\begin{aligned}
 \nabla \cdot \vec{S} &= \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \\
 &= -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{E} \cdot \vec{J} \\
 \Rightarrow \nabla \cdot \vec{S} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} &= -\vec{E} \cdot \vec{J}
 \end{aligned}
 \tag{2.17}$$

By recognizing the second and the third terms as the time variance of the total stored energy in the fields:

$\frac{1}{2}(\vec{D} \cdot \vec{E} + \vec{B} \cdot \vec{H})$ , we can interpret the Poynting vector as the energy flow carried by the electromagnetic fields. The RHS of (2.17) is the total power dissipation to the source (i.e. the amount of work required to be done by the fields to generate the current source):

$$\vec{E} \cdot \vec{J} = \frac{V}{l} \frac{I}{A} = \frac{\text{power}}{\text{volume}}
 \tag{2.18}$$

### 2.3.2. Potentials

$\vec{E}$  and  $\vec{B}$  are basic physical quantities of the electromagnetic fields but mathematically may not be the most convenient forms in solving the equations. We may define another set of variables even themselves not being physical. For example, electromagnetic potentials are defined as follows.

$$\begin{aligned}\bar{B} &= \nabla \times \bar{A} \\ \bar{E} &= -\nabla\phi - \frac{\partial\bar{A}}{\partial t}\end{aligned}\quad (2.19)$$

The choice of  $\phi$  and  $\bar{A}$  is not unique as you can verify the Maxwell's equations remain the same through the following gauge transformation (or in plain English the coordinate transformation):

$$\begin{aligned}\bar{A} &\rightarrow \bar{A} + \nabla\Lambda \\ \phi &\rightarrow \phi - \frac{\partial\Lambda}{\partial t}\end{aligned}\quad (2.20)$$

where  $\Lambda$  is any scalar function. We have infinite choices of  $\Lambda$  but the most commonly used gauges are the Coulomb gauge and the Lorenz gauge:

$$\begin{aligned}\nabla \cdot \bar{A} + \frac{1}{c^2} \frac{\partial\phi}{\partial t} &= 0 && \text{Lorenz gauge} \\ \nabla \cdot \bar{A} &= 0 && \text{Coulomb gauge}\end{aligned}\quad (2.21)$$

### 2.3.3. Basic solutions for source-free, homogeneous, time-invariant and linear media

Taking the curl of the Faraday's Law:

$$\begin{aligned}\nabla \times \nabla \times \bar{E} &= -\mu_0\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} \\ \text{LHS} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} &= -\nabla(\epsilon^{-1} \bar{E} \cdot \nabla \epsilon) - \nabla^2 \bar{E} = \nabla(\bar{E} \cdot \epsilon \ln \epsilon \nabla \epsilon) - \nabla^2 \bar{E} \\ \nabla^2 \bar{E} - \mu_0\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} + \nabla(\epsilon^{-1} \bar{E} \cdot \nabla \epsilon) &= 0\end{aligned}\quad (2.22)$$

For homogeneous media ( $\nabla \epsilon = 0$ ), the last term vanishes. We have the wave equation as follows.

$$\nabla^2 \bar{E} - \mu_0\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \quad (2.23)$$

The wave equation is a second order differential equation that describes a wave propagating at a speed of light:  $c \equiv \sqrt{\frac{1}{\mu_0\epsilon}}$ . From the theory of differential equations, solutions to the wave equation can be expressed

as a sum of a complete set of basis functions. Examples of basis functions include plane waves, Hermite-Gaussian, Bessel, spherical harmonics, ... The choice of a basis is arbitrary but one basis may be much more convenient than the others in certain question. For example, in optical fiber (cylindrically symmetric), Bessel basis is the best choice; in a laser cavity, Hermite-Gaussian is the best choice; in a homogeneous media, plane-wave basis is the best choice.

The plane-wave basis can be expressed as:

$$\bar{E} = \bar{E}_0 e^{i(\bar{k} \cdot \bar{x} - \omega t + \phi)} + \text{c.c.} \quad \text{where } \bar{k} \cdot \bar{k} = \omega^2 \mu_0\epsilon = \left(\frac{\omega n}{c}\right)^2 \quad (2.24)$$

where  $\vec{k}$  is the wave vector and  $\varphi$  is an arbitrary phase. Any solution can be written as the linear superposition of above basis:

$$\text{solution} = \int d\omega d\vec{k} d\varphi \bar{E}(\omega, \vec{k}, \varphi) \exp[i(\vec{k} \cdot \vec{x} - \omega t + \varphi)] + \text{c.c.} \quad (2.25)$$

### 2.3.4. Frequency domain

In the plane-wave basis, the only time dependence of the solution is in the factor  $e^{-i\omega t}$ . We can redefine the field quantity via the Fourier transform as follows.

$$\bar{E}(\vec{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{E}(\vec{r}, \omega) e^{-i\omega t} d\omega \quad (2.26)$$

With (2.26), we can rewrite the Maxwell's equations (2.11) as follows.

$$\begin{aligned} \nabla \times \bar{E} &= i\omega \bar{B} \\ \nabla \times \bar{H} &= \bar{J} - i\omega \bar{D} \\ \nabla \cdot \bar{D} &= \rho \\ \nabla \cdot \bar{B} &= 0 \end{aligned} \quad (2.27)$$

where all the field quantities including the two source terms are in the frequency domain as defined in (2.26).

The wave equation (2.23) can be written as:

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0 \quad \text{where } k^2 = \vec{k} \cdot \vec{k} \quad (2.28)$$

You can verify yourself that the time average of Poynting vector in terms of the fields quantities in the frequency domain is:

$$\langle \bar{S} \rangle = \text{Re}(\bar{E} \times \bar{H}^*) / 2 \quad (2.29)$$

### 2.3.5. Gain and loss

The propagation of a plane wave in an isotropic, homogeneous, time-invariant and linear dielectric medium must satisfy the wave equation (2.28). We compare the change of the field over a distance  $d$ :

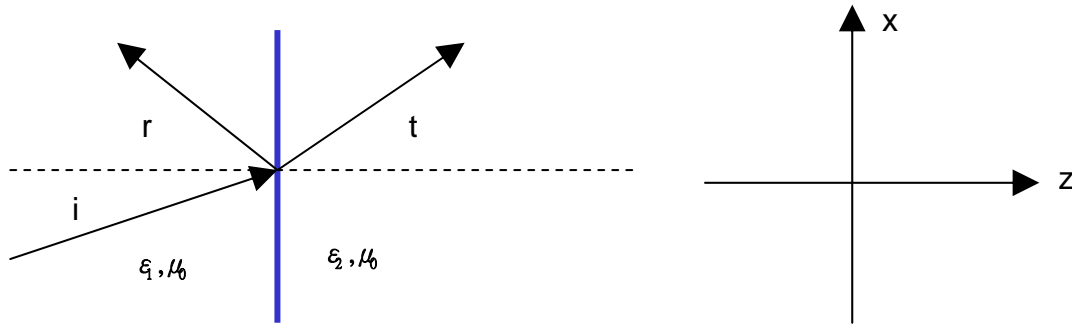
$$\frac{|E(z+d)|}{|E(z)|} = |e^{ikd}| = |e^{i\omega\mu_0\epsilon d}| = e^{-\omega\mu_0 d \text{Im}\epsilon} \quad (2.30)$$

We see that the field strength grows if  $\text{Im}\epsilon < 0$  and attenuates if  $\text{Im}\epsilon > 0$ . Thus a complex dielectric constant can represent the gain and loss of a medium.

## 2.4. Examples of interaction between EM waves and macroscopic media

### 2.4.1. Interface phenomena

Consider a plane wave incident upon a straight interface between two dielectric media with different dielectric constants.



Consider the incident wave is a TM (or s-) polarized plane wave.

$$\begin{aligned}\vec{H}_i &= \hat{y}H_i e^{i(k_{ix}x + k_{iz}z)} \\ \vec{H}_r &= \hat{y}H_r e^{i(k_{rx}x + k_{rz}z)} \\ \vec{H}_t &= \hat{y}H_t e^{i(k_{tx}x + k_{tz}z)}\end{aligned}\quad (2.31)$$

The electric field can be calculated from (2.31) according to:

$$\vec{E} = -\frac{1}{i\omega\epsilon} \nabla \times \vec{H} \quad (2.32)$$

From the boundary conditions, the tangential components of both the electric fields and the magnetic fields are continuous at the interface. We have:

$$\begin{aligned}k_{ix} &= k_{tx} = k_{rx} \\ \frac{H_t}{H_i} &= \frac{2k_{1z}/\epsilon_1}{k_{1z}/\epsilon_1 + k_{2z}/\epsilon_2} \\ \frac{H_r}{H_i} &= \frac{k_{1z}/\epsilon_1 - k_{2z}/\epsilon_2}{k_{1z}/\epsilon_1 + k_{2z}/\epsilon_2}\end{aligned}\quad (2.33)$$

The first equation in (2.33) is the Snell's Law, i.e.  $n_1 \sin \theta_i = n_2 \sin \theta_t$ . The second and the third equations represent the transmission and reflection, respectively. There are two special cases:

1.  $k_{1z}/\epsilon_1 = k_{2z}/\epsilon_2$ : In this case, there is no reflection. The incident angle that satisfies this condition is called the Brewster's angle.
2.  $\theta_i > \sin^{-1} \frac{n_2}{n_1}$ : In this case, the angle of refraction is 90 degrees. We have total internal reflection.  $k_{2x} = k_2 \sin \theta_i / \sin \theta_0 > k_2$  and  $k_{2z}$  is imaginary. The field in the 2<sup>nd</sup> medium has the following form:

$$\exp\left[ik_2 \frac{\sin \theta_i}{\sin \theta_0} x\right] \exp\left[-z \sqrt{\left(\frac{\sin \theta_i}{\sin \theta_0}\right)^2 - 1}\right] \quad (2.34)$$

In the event of total internal reflection, the field amplitude in the z direction is exponentially decayed. It only propagates along the interface. Although there is field penetration into the 2<sup>nd</sup> medium, there is no energy

flow along the  $z$  direction into the interface as you can verify that by evaluating the Poynting vector in the  $z$  direction.

#### **2.4.2. Thin slab**

See powerpoint slides for the extension of the above discussions to a thin slab.