EECS 598-002 Nanophotonics and Nanofabrication
Winter 2006
Homework 1 Solution Set

Problem 1. Brewster's Angle

In the class, we derived the Snell's Law with the incident magnetic field polarized perpendicular to the plane (TM polarized). We noticed that there existed an angle at which the reflection vanished. We called it the Brewster's angle.

1. **Duality Principle:** Verify that the Maxwell's Equations remain the same with the following exchange of variables:

$$\vec{E} \to \vec{H}$$
$$\vec{H} \to -\vec{E}$$
$$\mu \to \varepsilon$$
$$\varepsilon \to \mu$$
$$\vec{J} \to \vec{M}$$
$$\vec{M} \to -\vec{J}$$

This is the duality principle of the electromagnetics. Applying the duality principle to the TM-polarized results given in the class and argue that these are nothing but the TE case. Does the Brewster's angle exist in the TE case?

Answer: Please note that the vector $\vec{M}$ here is NOT the magnetization as discussed in the class. Instead it is the "magnetic current" as a counterpart to the electric current. The magnetic current corresponds to a small current loop. If we restrict ourselves from looking into the details of the current distribution of the current loop, the current loop will act as a point-like magnetic dipole (just like the coil) and induce an electric field. With the introduction of the magnetic current, we modify the Faraday's Law as follows.

$$\nabla \times \vec{E} = i\omega\mu\vec{H} - \vec{M}$$

Note in the calculations, the part of the current distribution that contributes to the magnetic current should not be included in $\vec{J}$ to avoid double counting the source. With this modification, the duality principle can be easily verified.

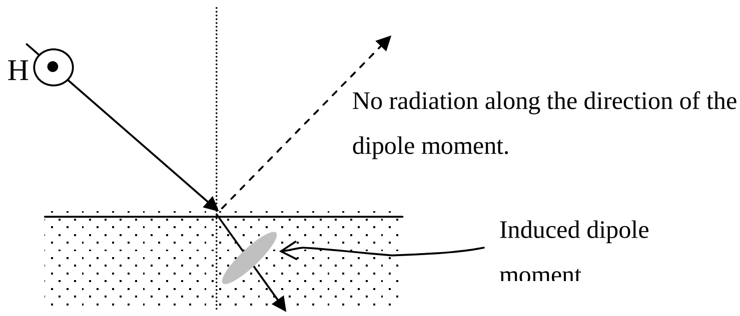
If we apply the duality principle to the TM solution, we get the TE solution since $\vec{H} \to -\vec{E}$ and now the it is the electric field that becomes perpendicular to the plane. For example, we can get the reflection coefficient simply by applying the duality principle as follows.

$$\frac{-E_r}{-E_i} = \frac{k_{1z}/\mu_1 - k_{2z}/\mu_2}{k_{1z}/\mu_1 + k_{2z}/\mu_2} \Rightarrow \frac{E_r}{E_i} = \frac{k_{1z}/\mu_0 - k_{2z}/\mu_0}{k_{1z}/\mu_0 + k_{2z}/\mu_0} = \frac{k_{1z} - k_{2z}}{k_{1z} + k_{2z}}$$

The reflection can only vanish if $k_{1z} - k_{2z}$. But that combining with the Snell's Law would mean $n_1 = n_2$, i.e. no interface exists. Therefore with the existence of an interface, the Brewster's angle does not exist for the TE case.

2. Please use the induced dipole moment picture we discussed in the class along with the radiation pattern of a dipole moment to explain qualitatively why Brewster's angle only exists in the TM case while not in the TE case.

Answer: In the TM case, the magnetic field is polarized perpendicular to the plane. The electric field is polarized parallel to the plane. The induced dipole moment in an isotropic medium has the same polarization as the incident electric field and therefore the induced dipole moment is polarized parallel to the plane. Since there is no radiation power along the direction of the dipole moment, it exists an incident angle such that the dipole radiation vanishes in the direction of reflection (see the following figure.)



In the case of TE, the induced dipole moment is always oriented perpendicular to the plane and therefore the reflection always exists.

3. List all applications you can think of from the Brewster's angle.

Answer: For example we can place the mirror in an optical system at the Brewster's angle to suppress the reflection.

Problem 2. Left-Handed Materials

In a paper published in 1968 (Sov. Phys. Usp. **10** (1968) 509), Veselago investigated the propagation of plane waves in a medium that possesses both negative ε and μ.

1. From the Maxwell Equations, argue that the vectors **E**, **H**, and the wave vector **k** form a left-handed system, that is the direction of **ExH** is in opposite to the direction of **k**.

Answer: The Ampere's Law and the Faraday's Law in the plane-wave representation read as follows.

$$\vec{k} \times \vec{E} = \omega\mu\vec{H}$$
$$\vec{k} \times \vec{H} = -\omega\varepsilon\vec{E}$$

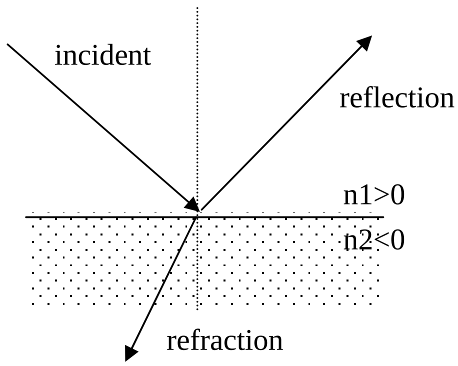
If both ε and μ are negative, we can easily verify that $-\vec{k} \parallel \vec{E} \times \vec{H}$ that is (E, H, k) forms a left-handed system.

2. Explain why in a left-handed medium, the refractive index $n = \sqrt{\varepsilon\mu}$ must be taken the negative sign. (Ref: Veselago's paper)

Answer: Since $\vec{k}$ is negative to the direction of the energy flow ($\parallel \vec{E}\times\vec{H}$), we have $k = -\omega\sqrt{\varepsilon\mu}/c = \omega n/c$, that is a negative refractive index.

3. Compare the refraction behavior between right-handed materials ($\varepsilon>0$ and $\mu>0$) and left-handed materials. You will need to re-derive the results in the class as $\mu_1 \neq \mu_2$ in this case.

Answer: By tracing the derivation of the refraction at the interface between two right-handed materials, we can see the same derivation holds true for the refraction between a right-handed material and a left-handed material. Therefore the refraction still satisfies the Snell's Law: $n_1 \sin\theta_i = n_2 \sin\theta_t$. But since now $n_1 > 0$ and $n_2 < 0$, we have $\theta_t < 0$. The refracted wave is deflected into the same side as the incident wave (see the following figure.)



4. What happens to the refracted wave if the total internal reflection condition is met? Describe the behaviors of the evanescent waves.

Answer: The critical angle is $\theta_0 = \sin^{-1}\frac{|n_2|}{n_1}$. When the incident angle is greater than the critical angle, $k_{2x} = k_2 \sin\theta_i / \sin\theta_0$. Note $k_2 = \omega n_2 / c$ is negative. Since energy flow of the refracted wave ($\parallel \vec{E}\times\vec{H}$) needs to be in the positive z direction and the boundary condition demands that $E_{1t} = E_{2t}$ and $H_{1t} = H_{2t}$, we can argue k_{2z} also needs to be negative. We then conclude that the evanescent wave has a z dependence as $\exp(+|k_2| z\sqrt{(\sin\theta_i/\sin\theta_0)^2 - 1})$ which grows exponentially along z.

Problem 3. Photon tunneling and applications in optical microscopes:

In the class, we outlined the results of EM wave tunneling through a narrow layer under the total internal reflection condition. This has been applied to a photon scanning tunneling microscope (PSTM; J. Vac. Sci. Technol. B **9** (1991) 525.)

1. Calculate the reflection and transmission coefficients for the TE case. The reflection and transmission coefficients are defined as the fraction of intensity ($\propto |E|^2$) that is reflected or transmitted through the narrow layer, respectively.

Answer: For the TE case, the electric and magnetic fields in *l*-th layer can be expressed as

$E_y^l = \left(A_l e^{ik_{lz}z} + B_l e^{-ik_{lz}z}\right) e^{ik_{lx}x}$. After matching the boundary conditions at both interfaces, the reflection and the transmission coefficients are given by the following (with the index of the incident layer being 0).

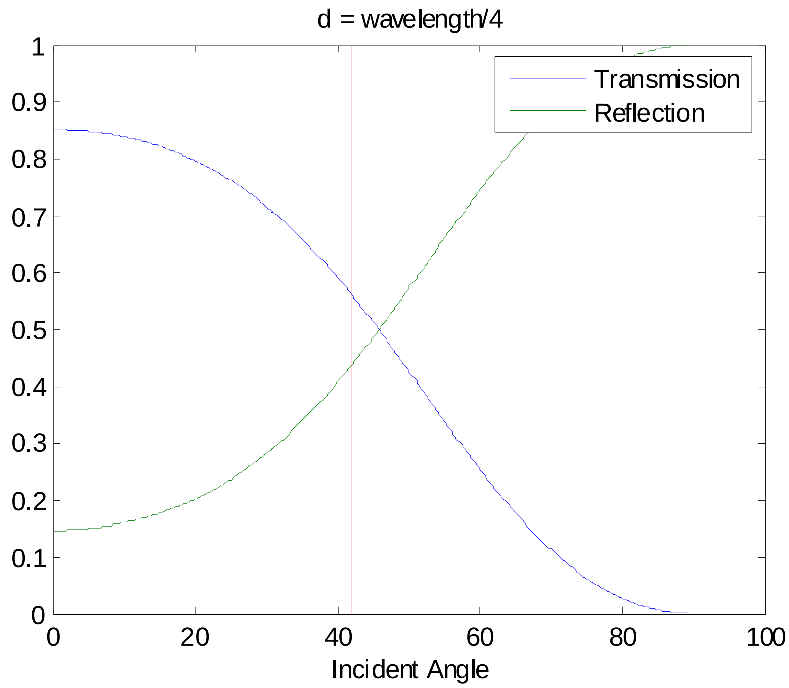
$$R = |r|^2 = \left| \frac{r_{01}(1 - e^{-i2k_{1z}d})}{1 - r_{01}^2 e^{-i2k_{1z}d}} \right|^2$$

$$T = |t|^2 = \left| \frac{4e^{-ik_{1z}d}}{(2 + k_{1z}/k_{0z} + k_{0z}/k_{1z})(1 - r_{01}^2 e^{-i2k_{1z}d})} \right|^2$$

where

$$r_{01} = \frac{k_{0z} - k_{1z}}{k_{0z} + k_{1z}}$$

is the reflection from the first interface and d is the thickness of the slab. We have assumed all three layers are dielectric and the first and the third layers are identical. The plots of R and T are as follows.



The red line represents the critical angle. We can see that by inserting the second interface, there is finite transmission through the thin slab even when the incident angle is greater than the critical angle.

2. Calculate the Poynting vector inside the narrow layer. Is it zero? If so, why? If not, why?

Answer: When the incident angle is greater than the critical angle, k_{1z} becomes imaginary. But because of the backward propagating terms B_1, the time-average Poynting vector is not necessarily zero along the *z* direction. First of all we have:

$$A_1 = \frac{1}{2}\left(1 + \frac{k_{0z}}{k_z}\right) t E_i$$

$$B_1 = \frac{1}{2}\left(1 - \frac{k_{0z}}{k_z}\right) t E_i$$

The Poynting vector along the *z* direction is:

$$\begin{aligned}\langle S \rangle_z &= \hat{z} \cdot \frac{1}{2} \operatorname{Re} E_y H_x = -\frac{\operatorname{Im} k_{1z}}{\omega \mu_0} \operatorname{Im}\left(A_1^* B_1\right) \\ &= \frac{k_{0x}}{2\omega\mu_0} T \left|E_i\right|^2\end{aligned}$$

We can see that the power flow along the *z* direction is proportional to the transmission coefficient.

3. Explain briefly how this tunneling effect can be applied to a scanning tunneling microscope that uses light to detect the surface topography.

Answer: Similar to atomic STM, since the transmission coefficient strongly depends on the thickness *d*, we can detect the surface topography by measuring the transmitted intensity.

4. Do your best to explain a physical picture of this tunneling effect.

Answer: The evanescent waves attributed to the two interfaces are out-of-phase and therefore contribute to a nonvanishing Poynting vector which leads to a finite transmission through the slab.