# Dynamics-based Information Extraction: A Hybrid Systems Approach

Necmiye (N-edge-me-ye) Ozay
EECS, University of Michigan

Joint work with Mario Sznaier, Octavia Camps (Northeastern), Constantino Lagoa (Penn State), Farshad Harirchi (Michigan)

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# Outline

## Dynamics-based Information Extraction: A Hybrid Systems Approach

- Motivation
  - Control + Dynamics + Data + Information +Parsimony??
- Hybrid dynamical models (hybrid systems)
  - and their use in information extraction
- Tools from optimization
- Three concrete problems
  - Identification of hybrid models
  - Model (in)validation for hybrid models
  - Fault detection for hybrid models

# Motivation

Control + Dynamics + Data + Information +Parsimony??

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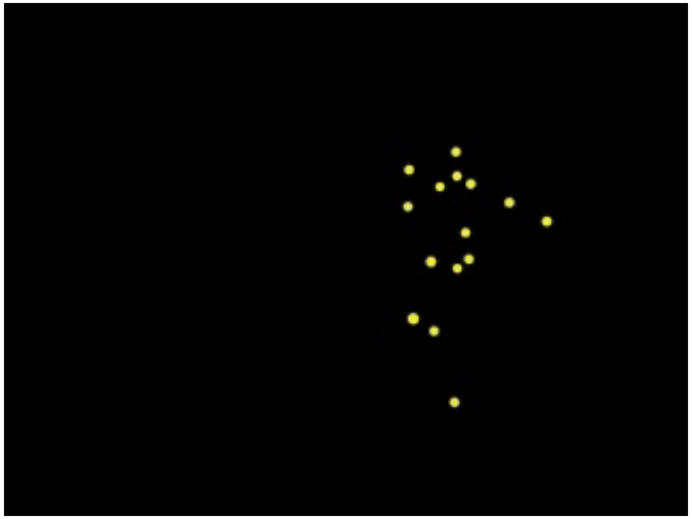
**Observation 1:** Dynamics enable parsimonious modeling

# Dynamics enable parsimony



A sparse set of features suffices for identifying and understanding dynamic events!

# Dynamics enable parsimony

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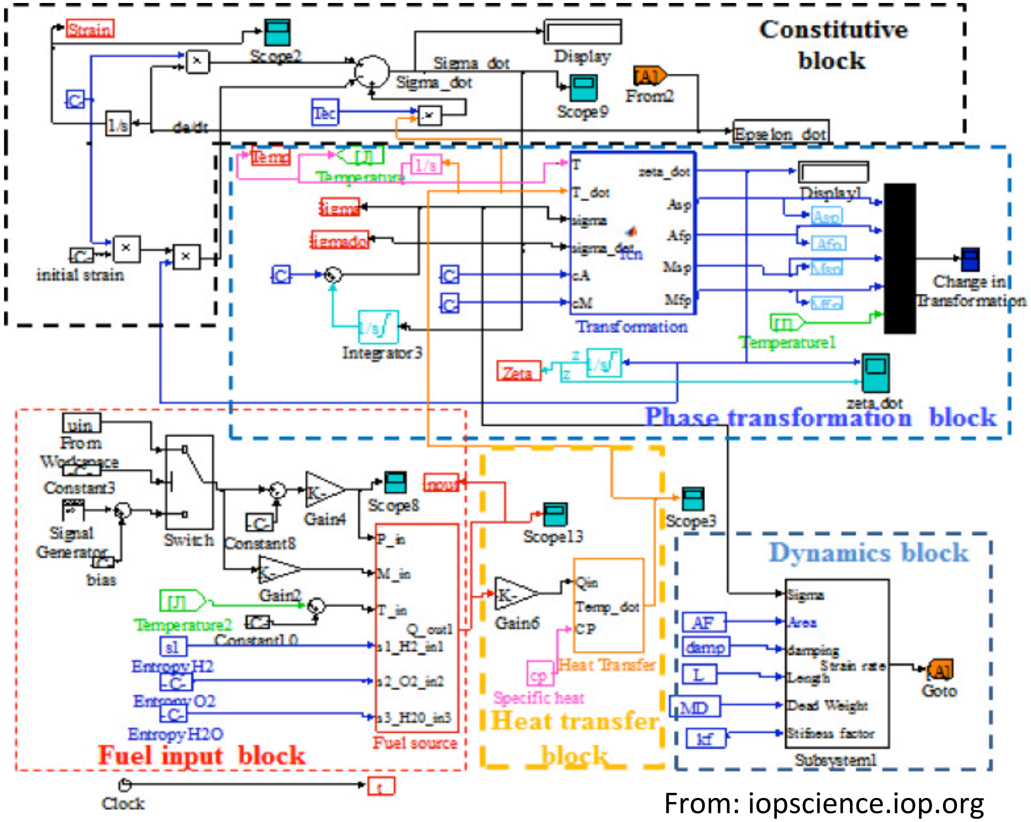
# Motivation

Control + Dynamics + Data + Information +Parsimony??

**Observation 1:** Dynamics enable parsimonious modeling

**Observation 2:** Control requires parsimonious modeling

# Control requires parsimony



From: iopscience.iop.org

From: www.rsrit.com

Complex models

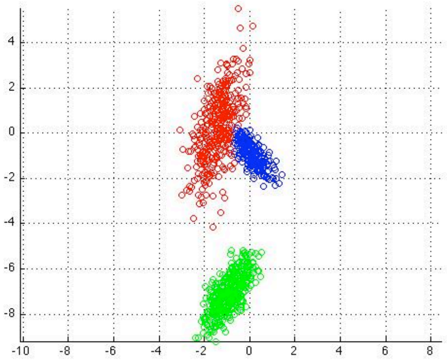
Big data

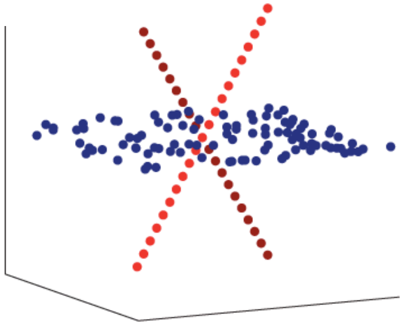
Useful/actionable models for (i) control design, (ii) fast simulations (iii) system monitoring, (iv) anomaly detection, etc.

# Overview of mixture models

Common in many fields:

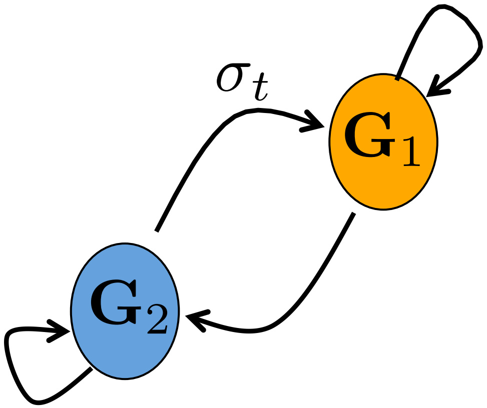
- Gaussian mixtures
- Subspace arrangements
- Hybrid systems





**Collection of simple models that can explain complex objects!**

- Two fold difficulty in learning such models:
  - Data association
  - Parameter estimation



# Hybrid systems

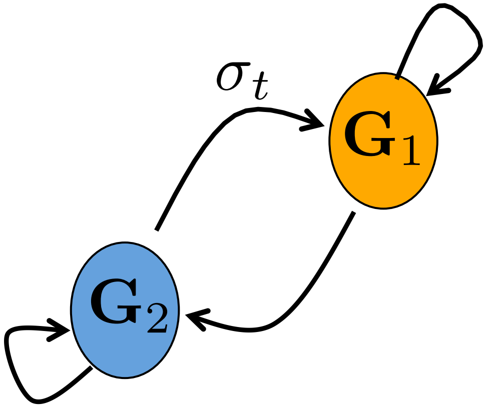
- “mixture models” for dynamical systems
- Switched systems

$$y(t+1) = G_{\sigma_t}(y(t:t-n_a), u(t:t-n_c))$$

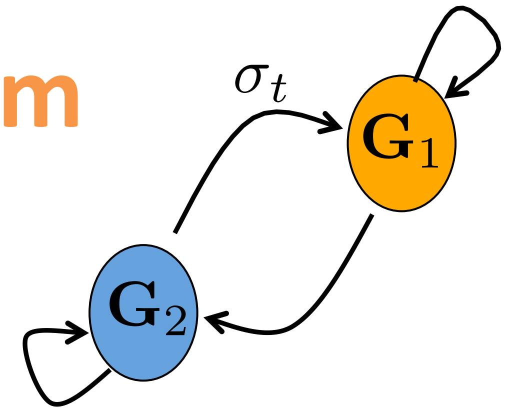
where mode signal $\sigma_t \in \{1, \ldots, s\}$

- For this talk, $G_i$‘s are polynomial (or affine)

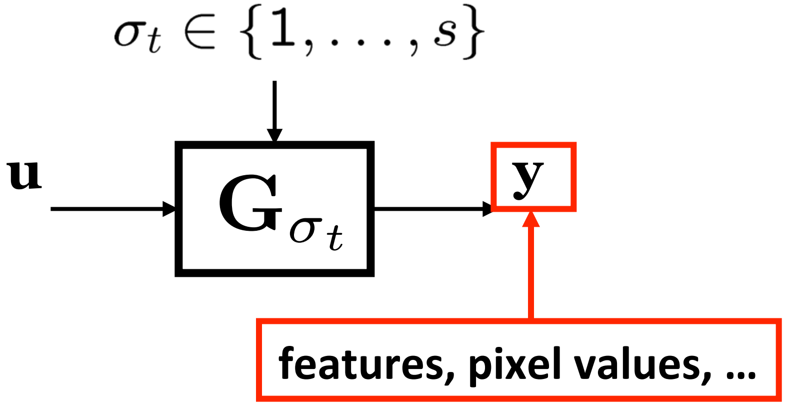
- **Global approximators even when G is affine!**
- **Identification from data is not easy!**
- Two fold difficulty:
  - Estimation of the mode signal (data association)
  - Estimation of the parameters (identification)



# Information extraction as an Identification Problem



- Hybrid Dynamical Models
  - Simple models for complex phenomena
- Two-fold difficulty
  - Estimation of mode signal (data association)
  - Estimation of parameters (identification)

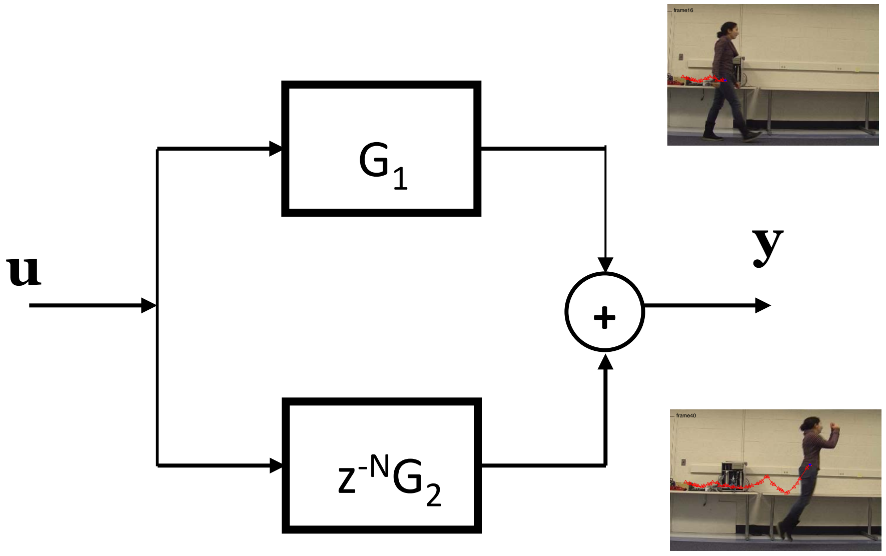


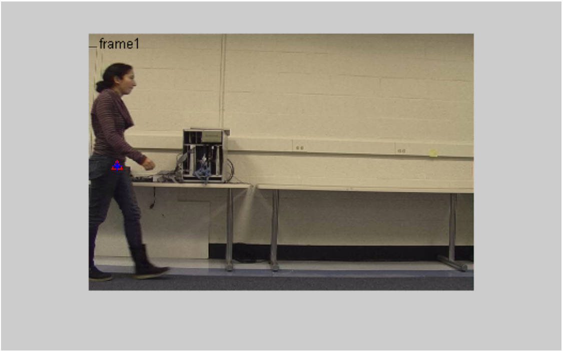
**features, pixel values, …**

- Model data streams as outputs of switched linear systems
- “Interesting” events ⟷ Changes in model invariants
- “Homogenous” segments ⟷ Output of a single submodel

# A Simple Problem: Event Detection

- Key observation: as new modes get excited, complexity (order) of the system increases





Look for changes in model complexity.

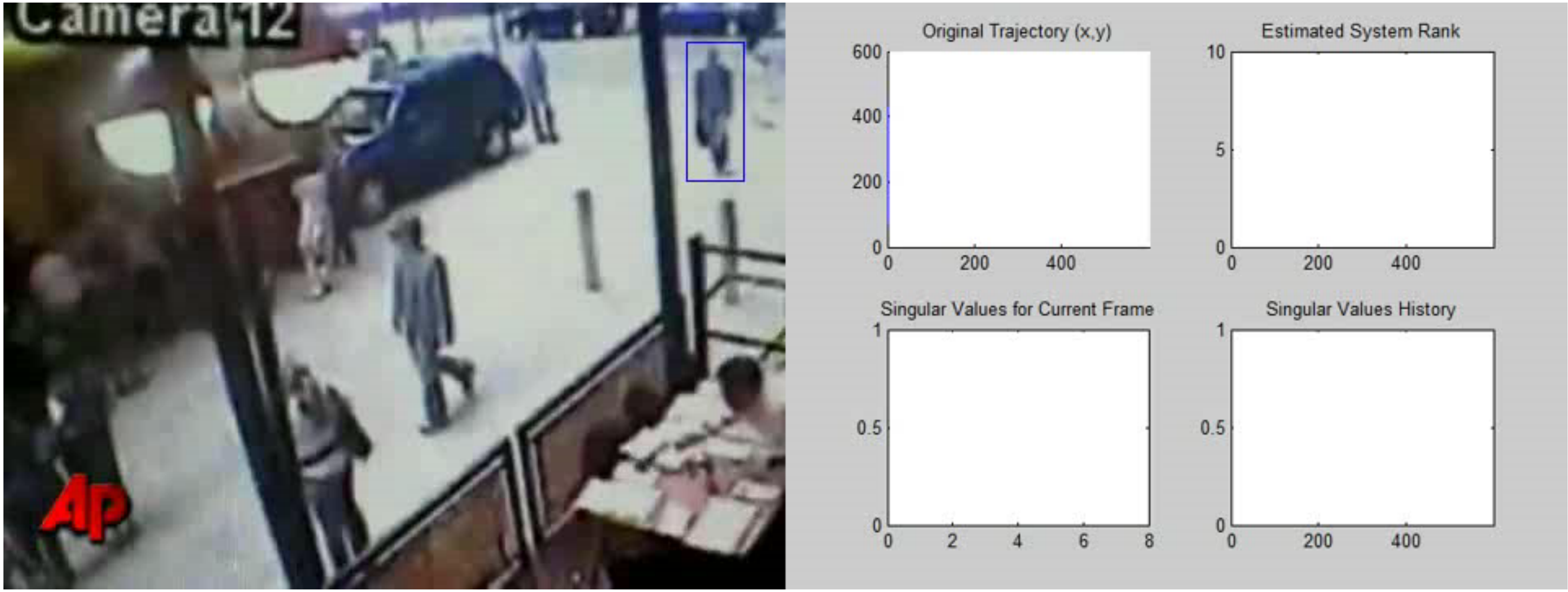
# A Simple Problem: Event Detection

- Key observation: as new modes get excited, complexity (order) of the system increases
- Order of the system is given by the rank of the Hankel matrix

$$H_y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_m & y_{m+1} & \cdots & y_{m+n-1} \end{bmatrix}$$

Look for changes in the rank of the Hankel matrix.
**(no need to explicitly find the model!)**

# Fast Event Detection

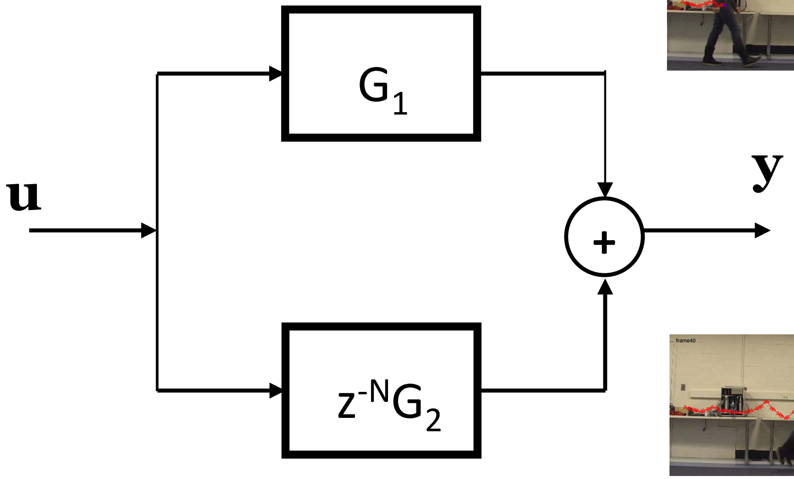


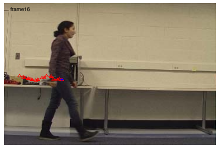
Use SVD to estimate the rank of the Hankel Matrix.
(five lines of Matlab code, runs on a laptop)

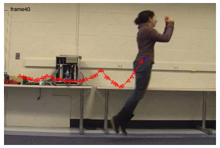
# A Simple Problem: Event Detection

- A few issues: delays, fast switching, noise and outliers

$$H_y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_m & y_{m+1} & \cdots & y_{m+n-1} \end{bmatrix}$$



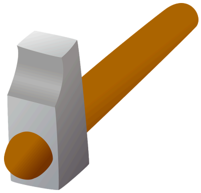




- How to more rigorously reason about noisy data?
- What if we want to learn individual dynamics?

# The tool (my big hammer )

Moments-based convex relaxations to polynomial optimization (Lasserre's hierarchy).

# The tool (my big hammer  )

## Moments-based convex relaxations to polynomial optimization (Lasserre's hierarchy).

Multivariate polynomial optimization problem on a compact basic semi algebraic set $K$

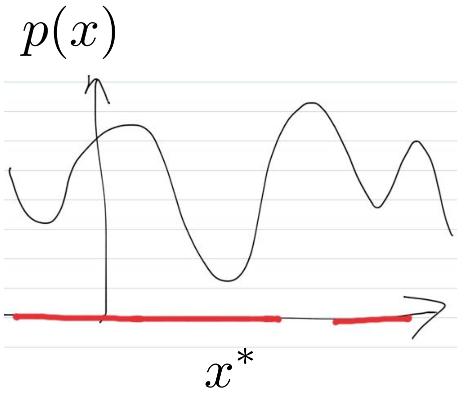
$$p_K^* := \min_{\mathbf{x} \in K} p(\mathbf{x}) \qquad \text{(P1)}$$

A functional optimization problem over the set of probability measures $\mu$ with support $K$

$$\tilde{p}_K^* := \min_{\mu \in \mathcal{P}(K)} \mathbb{E}_\mu [p(x)] \qquad \text{(P2)}$$

P1 is equivalent to P2

$$\tilde{p}_K^* = p_K^*$$



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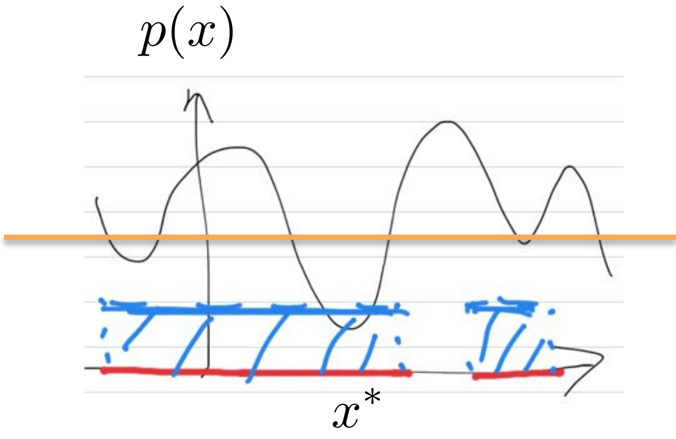

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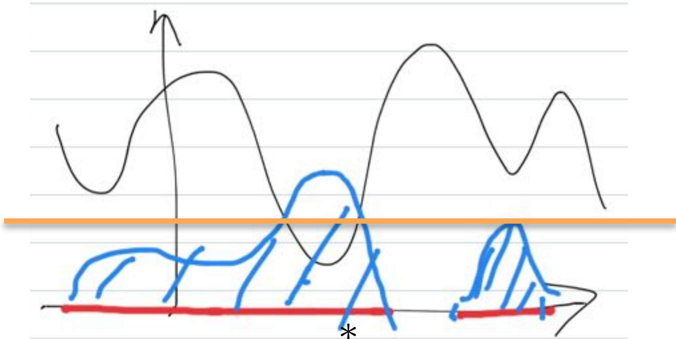

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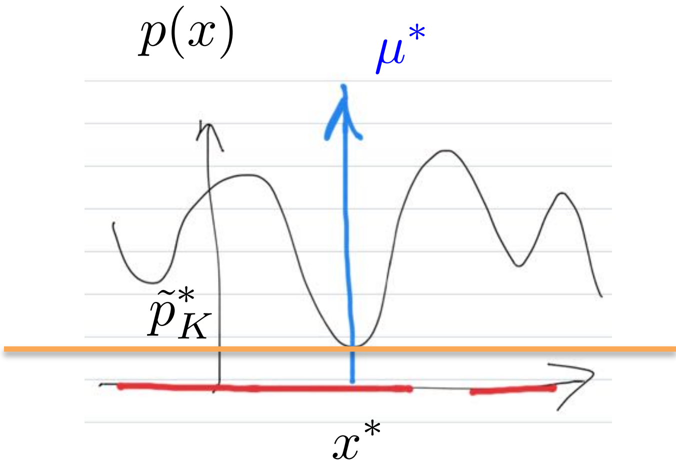

# Moment-based relaxations for polynomial optimization

A functional optimization problem over the set of probability measures $\mu$ with support $K$

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Equivalent to an SDP with countably infinite variables, where variables are moments of the distribution $\mu$.

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If $K = [a, b]$ univariate polynomial on an interval, there is a finite exact SDP (Hausdorf moment problem).

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If $K = [a, b]$ univariate polynomial on an interval, there is a finite exact SDP (Hausdorf moment problem).

If there is a sparse structure in the polynomial and constraints defining K, possible to get a hierarchy where we have smaller LMIs at each relaxation order (Lasserre, Nie, Waki, Kojima).

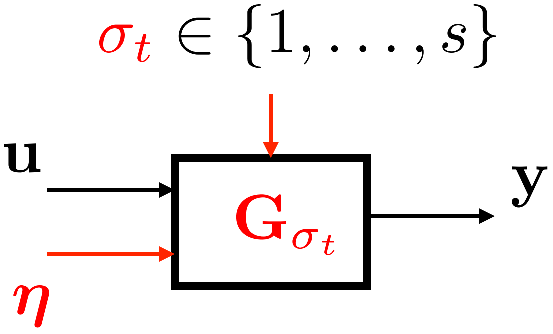
# Three problems

- Identification of switched affine systems (SARX Id)
- Model (in)validation for switched affine systems (SARX invalidation)
- Fault/anomaly detection for systems with polynomial state-space models

Given that we will be using polynomial optimization affine or polynomial or switched or non-switched makes a little difference.

# Switched System Identification

- Particular interest to switched linear models in control and system identification communities.
- **Problem Formulation:**
    - Given experimental input/output data, and bounds on noise and submodel orders $(n_a, n_c)$
    - Find a switched linear autoregressive model with exogenous inputs (ARX) of the form:



$$y(t) = \sum_{i=1}^{n_a} a_i(\sigma_t) y(t-i) + \sum_{i=1}^{n_c} c_i(\sigma_t) u(t-i) + \eta(t)$$

$$y(t) = \mathbf{p}(\sigma_t)^T \mathbf{r}(t) + \eta(t) \qquad ||\eta(t)|| \leq \epsilon$$

**Ill-posed, always have a trivial solution!**

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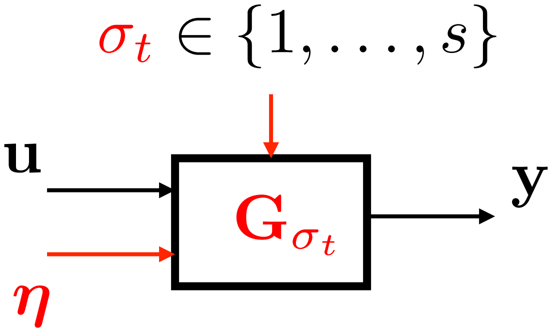

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Possible objectives:

- Minimum # of switches
- Minimum # of submodels
- Fixed # of submodels

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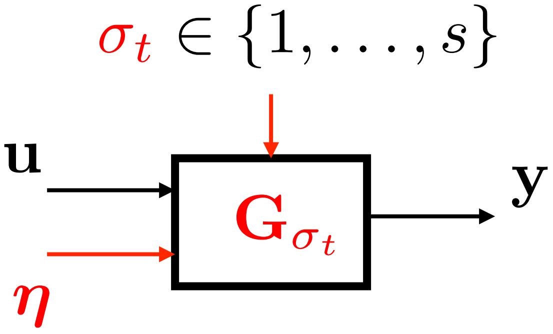

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$$y(t) = \mathbf{p}(\sigma_t)^T \mathbf{r}(t) + \eta(t) \quad ||\eta(t)|| \leq \epsilon$$

Possible objectives:

- Minimum # of switches (non-convex – polytime exact algorithms exist)
- Minimum # of submodels
- **Fixed # of submodels**

# SARX Id in noise-free case

- GPCA: an algebraic geometric method due to Vidal *et al.*
- Main Idea:

**hybrid decoupling constraint**

$$\mathbf{b}(\sigma_t)^T \mathbf{r}_t = 0, \quad \sigma_t \in \{1, \dots, s\}$$

**Neither the mode signal nor the parameters, b, are known!**

↓

$$p_s(\mathbf{r}) = \prod_{i=1}^{s} (\mathbf{b}_i^T \mathbf{r}_t) = \mathbf{c}_s^T \nu_s(\mathbf{r}_t) = 0$$

**Independent of mode signal, linear in parameters, c!**

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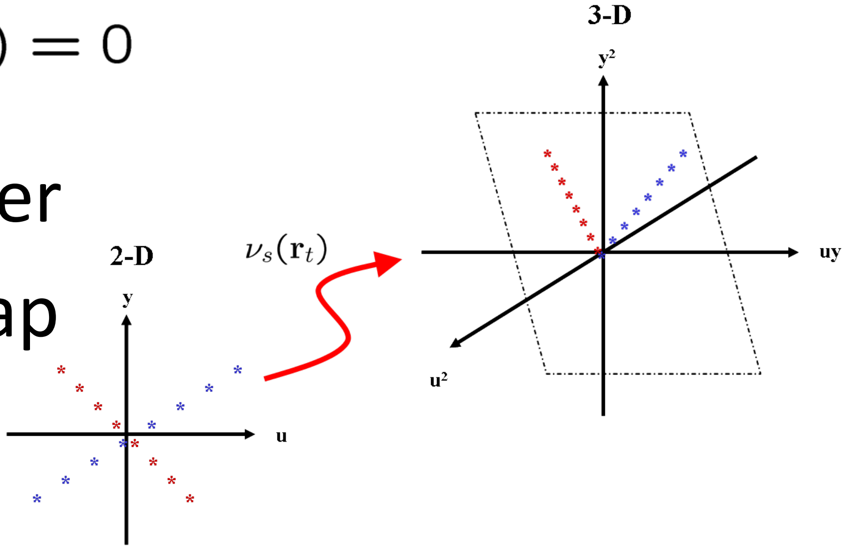
$$p_s(\mathbf{r}) = \prod_{i=1}^{s} (\mathbf{b_i}^T \mathbf{r}_t) = \mathbf{c_s}^T \nu_s(\mathbf{r}_t) = 0$$

- Embed the data in a higher dim. space via Veronese map

$$\nu_s([x_1, \dots, x_n]^T) = [\dots, \xi^s, \dots]^T$$

where

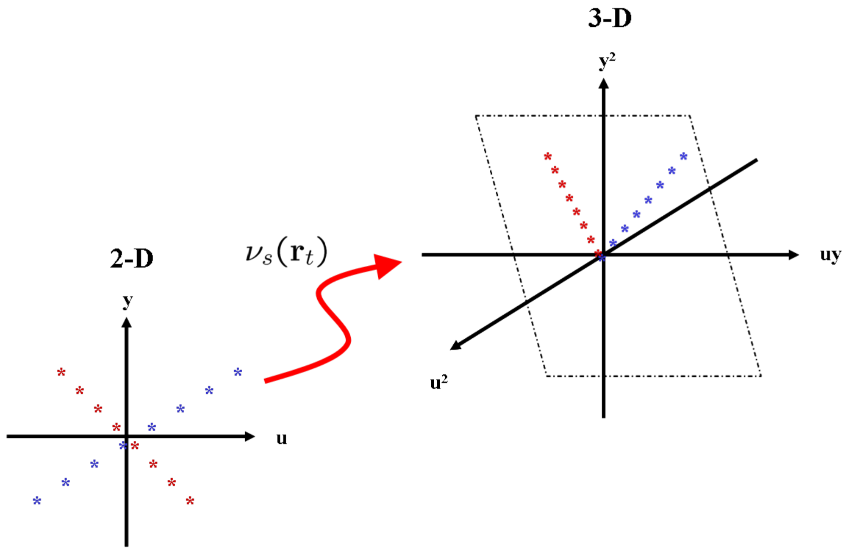
$$\xi^s \doteq x_1^{s_1} x_2^{s_2} \dots x_n^{s_n}, \ \sum s_i = s$$



# What happens when data is noisy?

- Veronese map $\nu_s(\mathbf{r}_t)$ :
  - Polynomial mapping
  - Lifts the data to higher dimensional space where parameter vector is in the nullspace of embedded data matrix
- Noisy case $\nu_s(\mathbf{r}_t, \eta_t)$ :
  - Lifted data depends on noise polynomially!
  - Need to find an admissible noise sequence to estimate the nullspace

$$\mathbf{V_s c_s} \doteq \begin{bmatrix} \nu_s(\mathbf{r}_{t_0})^T \\ \vdots \\ \nu_s(\mathbf{r}_T)^T \end{bmatrix} \mathbf{c_s} = \mathbf{0}$$



# Noisy embedded data matrix $V_s$

- 1st order system: $n_a = n_c = 1$,

with 2 modes: s=2

$$\mathbf{r}_t = [-y_t, y_{t-1}, u_{t-1}]^T$$

$$\nu_2(\mathbf{r}_t, \eta_t)^T = \begin{bmatrix} y_t^2 - 2y_t\eta_t + \boxed{\eta_t^2} \\ -y_t y_{t-1} + y_{t-1}\eta_t \\ -y_t u_{t-1} + u_{t-1}\eta_t \\ y_{t-1}^2 \\ y_{t-1}u_{t-1} \\ u_{t-1}^2 \end{bmatrix}^T$$

$V_s$ is polynomial in noise

Need to find a rank deficient $V_s$

$$\mathbf{V}_s(\mathbf{r}_t, \eta_t) \doteq \begin{bmatrix} \nu_s(\mathbf{r}_{t_0}, \eta_{t_0})^T \\ \vdots \\ \nu_s(\mathbf{r}_T, \eta_t)^T \end{bmatrix}$$

**Optimization Problem 1:**

$$\begin{array}{ll} \text{minimize}_{\eta_t} & \text{rank}\,\mathbf{V}_s(\mathbf{r}_t, \eta_t) \\ \text{subject to} & \|\eta_t\|_\infty \leq \epsilon \end{array}$$

# Optimization Problem

- Rank is not a polynomial function. Can we use ideas from polynomial optimization?
  - YES.
- Can we utilize the problem structure to find an efficient formulation?
  - YES. Main Idea: Noise is independent. Define one dimensional distributions for each noise term.

**Optimization Problem 1:**

$$\begin{array}{ll} \text{minimize}_{\eta_t} & \text{rank}\mathbf{V}_{\mathbf{s}}(\mathbf{r}_t, \eta_t) \\ \text{subject to} & \|\eta_t\|_\infty \leq \epsilon \end{array}$$

# Noisy embedded data matrix $V_s$

$$\nu_2(\mathbf{r}_t, \eta_t)^T = \begin{bmatrix} y_t^2 - 2y_t\eta_t + \eta_t^2 \\ -y_t y_{t-1} + y_{t-1}\eta_t \\ -y_t u_{t-1} + u_{t-1}\eta_t \\ y_{t-1}^2 \\ y_{t-1}u_{t-1} \\ u_{t-1}^2 \end{bmatrix}^T$$

$$\mathbf{m}^{(t)} = [m_1^{(t)}, \dots, m_s^{(t)}]$$
$$m_i^{(t)} = \mathbf{E}_{\mu^t}(\eta_t^i)$$

$$\mathbf{E}_\mu\left[\nu_2(\mathbf{r}_t, \eta_t)^T\right] = \begin{bmatrix} y_t^2 - 2y_t m_1^{(t)} + m_2^{(t)} \\ -y_t y_{t-1} + y_{t-1} m_1^{(t)} \\ -y_t u_{t-1} + u_{t-1} m_1^{(t)} \\ y_{t-1}^2 \\ y_{t-1}u_{t-1} \\ u_{t-1}^2 \end{bmatrix}^T$$

# Optimization Problem

- "Theorem"
  - There exists a rank deficient solution for Problem 2 if and only if there exists a rank deficient solution for Problem 1.
  - If c belongs to the nullspace of the solution of Problem 2, there exists a noise value $\eta^*$ with $||\eta^*||_\infty \leq \epsilon$ $\mathbf{V}_s(\mathbf{r},\eta^*)$ such that c belongs to the nullspace of

**Optimization Problem 1:**

$$\begin{array}{ll} \text{minimize}_{\eta_t} & \text{rank}\mathbf{V_s}(\mathbf{r}_t,\eta_t) \\ \text{subject to} & \|\eta_t\|_\infty \leq \epsilon \end{array}$$

**Optimization Problem 2:**

$$\begin{array}{ll} \text{minimize}_{\mathbf{m}^{(t)}} & \text{rank}\tilde{\mathbf{V}}_\mathbf{s}(\mathbf{r}_t,\mathbf{m}^{(t)}) \\ \text{subject to} & \text{each } \mathbf{m}^{(t)} \text{ is a} \\ & \text{moment sequence} \end{array}$$

Convex constraint set:

- **Finite** Hankel matrix of moments should satisfy to LMIs

**no relaxation!!**

# Optimization Problem

Problem 2

- Matrix rank minimization
- Subject to LMI constraints

- Use a convex relaxation (e.g. log-det heuristic of Fazel *et al.*) to solve Problem 2
- Find a vector c in the nullspace
- Estimate noise by root finding ($V_s c = 0$ polynomials of one variable)
- Proceed as in noise-free case

**Optimization Problem 1:**

$$\begin{array}{ll} \text{minimize}_{\eta_t} & \text{rank}\mathbf{V}_\mathbf{s}(\mathbf{r}_t, \eta_t) \\ \text{subject to} & \|\eta_t\|_\infty \leq \epsilon \end{array}$$

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Can also “handle” missing data, outliers

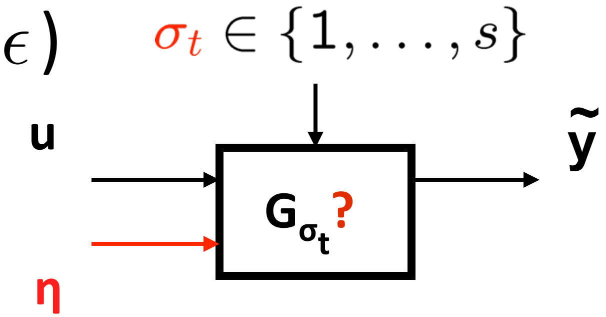
# SARX (In)validation Problem

**_Unknown switches: Consistency set is non-convex!_**

- Given:
  - A nominal hybrid model of the form:

$$\begin{array}{rcl} \mathbf{y}_t & = & \sum_{k=1}^{n_a} \mathbf{A}_k(\sigma_t)\mathbf{y}_{t-k} + \sum_{k=1}^{n_c} \mathbf{C}_k(\sigma_t)\mathbf{u}_{t-k} + \mathbf{f}(\sigma_t) \\ \tilde{\mathbf{y}}_t & = & \mathbf{y}_t + \boldsymbol{\eta}_t \end{array}$$

  - A bound on the noise ($||\eta||_\infty \leq \epsilon$)
  - Experimental input/output
  - data $\{\mathbf{u}_t, \tilde{\mathbf{y}}_t\}_{t=t_0}^{T}$



- Determine:
  - whether there exist noise and switching sequences consistent with a priori information and experimental data

# Semialgebraic Consistency Set

- If i$^{th}$ submodel is active at time t

$$\begin{array}{l}\mathbf{A}_1(i)(\tilde{\mathbf{y}}_{t-1}-\boldsymbol{\eta}_{t-1})+\ldots+\mathbf{A}_{n_a}(i)(\tilde{\mathbf{y}}_{t-n_a}-\boldsymbol{\eta}_{t-n_a})-(\tilde{\mathbf{y}}_t-\boldsymbol{\eta}_t)\\ \quad+\mathbf{C}_1(i)\mathbf{u}_{t-1}+\ldots+\mathbf{C}_{n_c}(i)\mathbf{u}_{t-n_c}+\mathbf{f}(i)=\mathbf{0}\end{array}$$

  - all components of the output evolve with i$^{th}$ submodel (logical AND)

$$[h_{t,i}^{(1)}(\boldsymbol{\eta}_{t:t-n_a})=0]\wedge\ldots\wedge[h_{t,i}^{(n_y)}(\boldsymbol{\eta}_{t:t-n_a})=0]$$

$$\Longleftrightarrow$$

$$g_{t,i}(\boldsymbol{\eta}_{t:t-n_a})\doteq\sum_{j=1}^{n_y}\left[h_{t,i}^{(j)}(\boldsymbol{\eta}_{t:t-n_a})\right]^2=0$$

- One of the submodels is active at time t (logical OR)

$$[g_{t,1}(\boldsymbol{\eta}_{t:t-n_a})=0]\vee\ldots\vee[g_{t,s}(\boldsymbol{\eta}_{t:t-n_a})=0]$$

$$\Longleftrightarrow$$

$$p_t(\boldsymbol{\eta}_{t:t-n_a})\doteq\prod_{i=1}^{s}g_{t,i}(\boldsymbol{\eta}_{t:t-n_a})=0$$

# Semialgebraic Consistency Set

- The model is invalid if and only if

$$\mathcal{T}'(\boldsymbol{\eta}) \doteq \Big\{\boldsymbol{\eta} \mid \epsilon^2 - [\eta_t^{(j)})]^2 \geq 0 \ \forall t \in [0, T], j \in \mathrm{N}_{n_y} \text{and} \\ p_t(\boldsymbol{\eta}_{t:t-n_a}) = 0 \ \forall t \in [n_a, T]\Big\}$$

  is empty.

- Structured polynomial optimization problem:

$$o^* = \min_{\eta} \sum_{t=n_a}^{T} p_t(\boldsymbol{\eta}_{t:t-n_a}) \\ \text{s.t.} \\ f_{t,j}(\eta_t^{(j)}) \geq 0 \ \ \forall t \in [0, T], j \in \mathrm{N}_{n_y}.$$

***Model is invalid iff o*>0***

# Polynomial Optimization

- Problem has a sparse structure (*running intersection property* holds)

$$
\begin{aligned}
o^* = \quad & \min_{\eta} \quad \textstyle\sum_{t=n_a}^{T} p_t(\boldsymbol{\eta}_{t:t-n_a}) \\
& \text{s.t.} \\
& \qquad\qquad f_{t,j}(\eta_t^{(j)}) \geq 0 \;\; \forall t \in [0, T], j \in \mathbb{N}_{n_y}.
\end{aligned}
$$

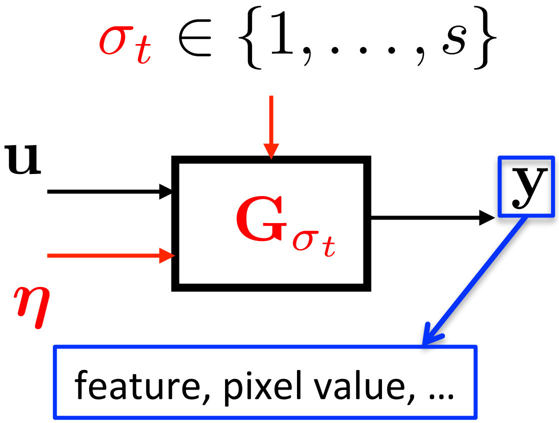
- We can create a convergent SDP hierarchy with $O((n_a n_y)^{2N})$ variables using structure (instead of $O((Tn_y)^{2N})$ variables), where N is the relaxation order.

- Theorem (O., Sznaier, Lagoa, TAC 14): The hierarchy converges latest at $N = s^{T-na+1}+1$.

  where s: # of submodels, $n_a$: regressor order, T: time horizon

# A fun example in information extraction

Normal behaviors: walking and waiting
Walking dynamics are learnt from training data using sys id, waiting dynamics are trivial



feature, pixel value, …

**Example:** Activity monitoring via model invalidation

A priori hybrid model: walking (learned from data) and waiting, 4% noise

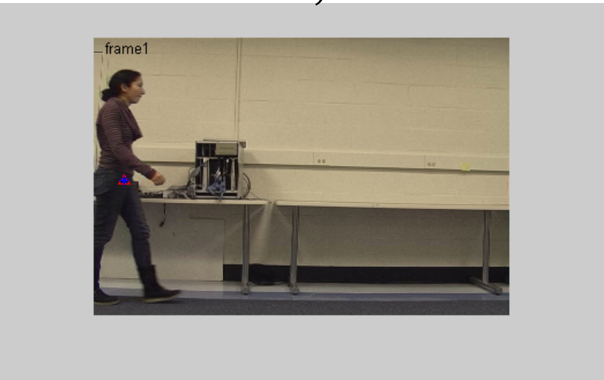
**WALK, WAIT**

**Not Invalidated**

**RUN**

**Invalidated**

**WALK, JUMP**

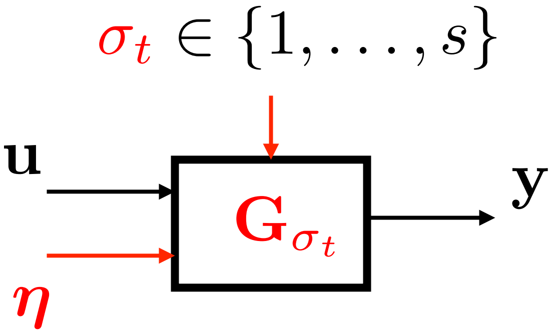


**Invalidated**

# Model Invalidation – fault detection?

- Can be easily extended to uncertain models:

$$\Sigma: \quad \begin{aligned} x(t+1) &= f_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \\ y(t) &= g_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \end{aligned}$$



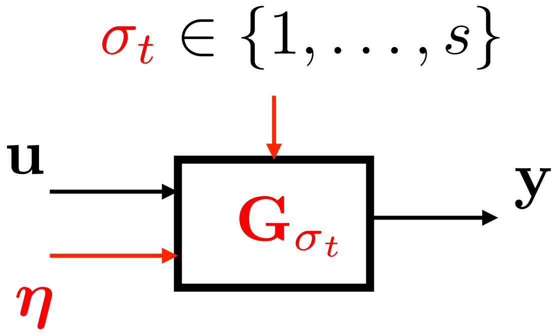
There is a basic semialgebraic consistency set.

- Can be used to:
  - **Run-time:** do anomaly detection (abnormal with respect to model and spec)
  - **Design-time:** find tight *provable error bounds* on uncertain parameters

No need to have explicit fault models (complex systems can fail infinitely many different ways!)
Can handle missing data!

# Fault detection

- Model invalidation directly applies but the problem size increases with time…
- What if we have fault models?



$$\Sigma : \begin{aligned} x(t+1) &= f_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \\ y(t) &= g_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \end{aligned}$$

$$\Sigma_F : \begin{aligned} x(t+1) &= f^F_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \\ y(t) &= g^F_{\sigma(t)}(x(t), u(t), \delta(t), \Delta) \end{aligned}$$

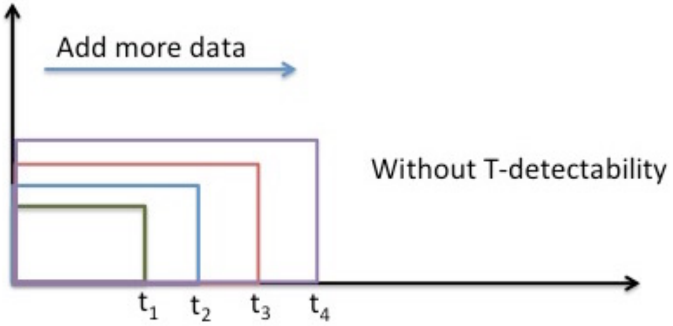
- Can we use the models to bound the amount of data needed to do fault detection?

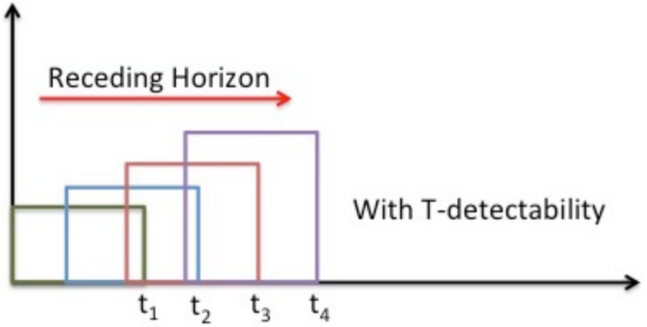
# T-detectability

- Given a system model and fault model with associated state, input and noise bounds, if there exists a T such that for any initial condition and any input/noise realization the "T-length behaviors" deviate, the fault is said to be T-detectable for the system.

➔ intersection of the consistency sets for the system and fault models for horizon T should be empty!

➔ For fixed T, polynomial optimization problem (need to iterate on T)

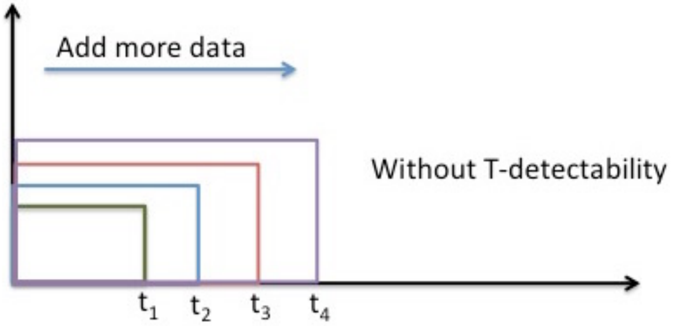


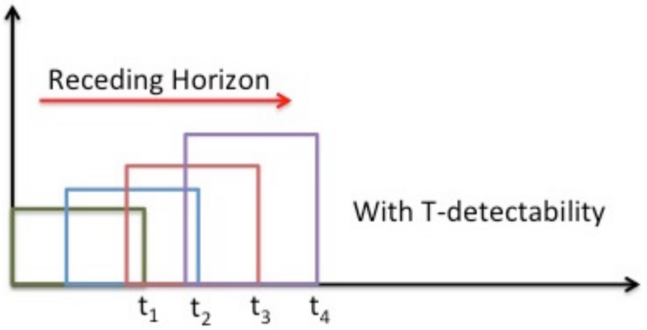


# T-detectability

➔ intersection of the consistency sets for the system and fault models for horizon T should be empty!

➔ For fixed T, polynomial optimization problem (need to iterate on T) – sufficient conditions for T-detectability

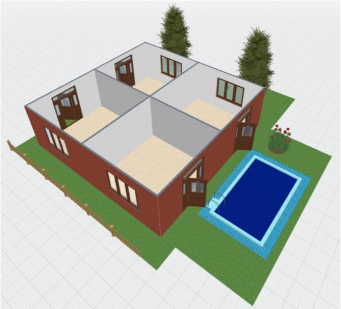


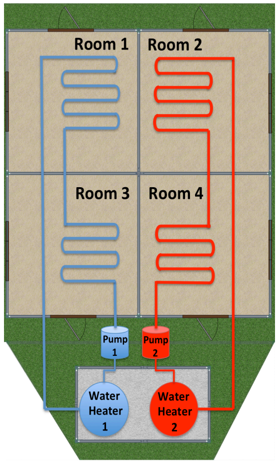


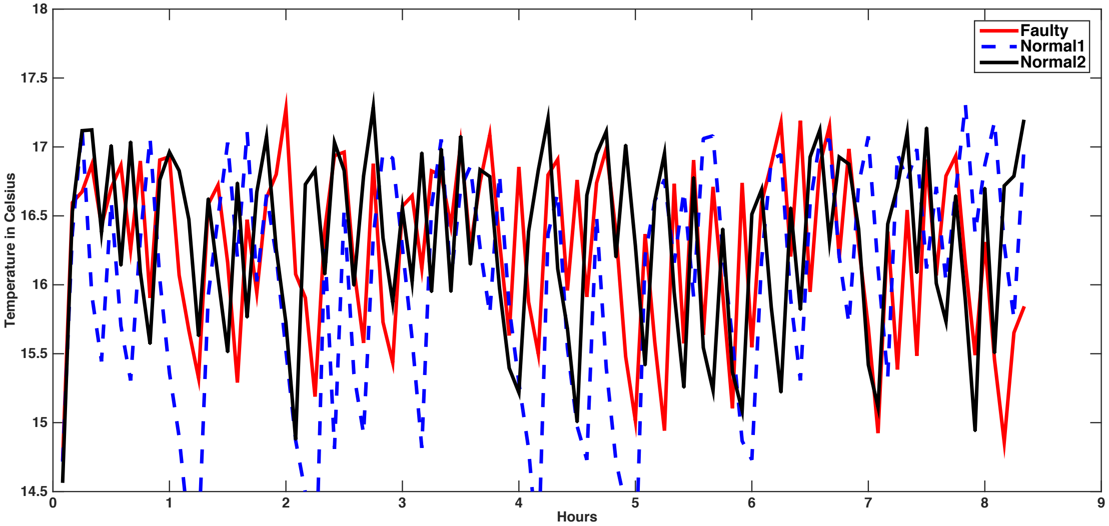
“Theorem”: If T-detectability certificate is obtained with a relaxation order N, then using the same relaxation order for model invalidation problem gives a N&S condition for online fault detection.

$$\begin{aligned} \epsilon^* := \min_{\{\mathbf{u}(k),\mathbf{y}(k)\}_{k=t}^{t+T}} \quad & \epsilon_o(\{\mathbf{u}(k),\mathbf{y}(k)\}_{k=t}^{t+T}) \\ \text{s.t.} \quad & \{\mathbf{u}(k),\mathbf{y}(k)\}_{k=t}^{t+T} \in \mathcal{B}_{poly}^{T}(G^f). \end{aligned}$$

# Anomaly detection in building control







Boiler fails at time 8:00 (supply temp drops)

Switched affine model:
-- switching due to control actions
-- six states (room temperatures, pipe temperatures)
-- only a sensor measuring pipe temperature
-- **noisy** sensor measurements

Invalidation algorithm detects the failure in 2 steps!

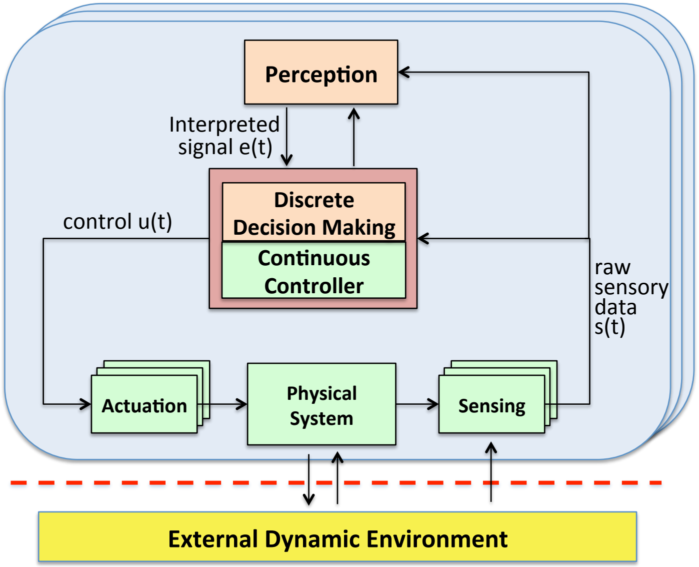
$$C_r\dot{T}_c = \sum_{i=1}^{2} K_{r,i}(T_i - T_c) + K_w(T_w - T_c),$$

$$C_i\dot{T}_i = K_{r,i}(T_c - T_i) + \sum_{j\neq i} K_ij(T_j - T_i)$$

# Summary

Goal: go from **data** to **information** to **control** in a rigorous way with correctness guarantees.

- **Dynamics based information extraction:**
  - Hybrid dynamical models as compact representation for complex data streams
  - Lots of structure in problems involving dynamics
  - Optimization is a good lens to look at these problems
  - ★ connections between system identification/invalidation and information extraction/machine learning



Computational efficiency through
- Convex Relaxations
- Structural decompositions