Structural detectability of faults in discrete-time affine systems

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***Abstract*— In this paper we propose a necessary and sufficient graphical condition for fault detection in structured affine systems where only the sparsity structure of system and fault matrices are known. The fault detection for such systems enables one to distinguish between the outputs of a nominal model and a faulty model within T time-steps. The resulting graphical condition involves checking for the existence of certain walks between pairs of vertices. Subsequently, we provide a simple algorithm to check for this condition and illustrate it via an example.**

I. Introduction

Fault detection has been a topic of interest for different engineering communities for quite some time now, see for e.g. [1] and the references therein. Several approaches exist for fault detection in dynamical systems including and not limited to spectrum analysis [2], pattern recognition [3], and residual generation [4]. A recent approach for fault detection proposed in [5], [6] provides a means to analyze if a fault can be detected in finite time and presents an algorithm to achieve finite-time detection whenever possible.

In several real-world scenarios the exact values of system parameters are unknown with the exception of presence or absence of interactions. Structural control which involves analysis of system properties based solely on their connectivity structure rather than actual values of system parameters, is a powerful tool in these situations. The premise of structural control is simple: the properties analyzed hold for almost all choices of system parameters except for possibly a set of measure zero [7], [8]. This makes structural control analyses robust to variations in parameters and suitable for large-scale systems [9], [10]. Furthermore, one typically uses graph-theory to obtain structural control results which often provides new insights about system behavior [11]. Several properties of systems such as controllability, observability, and left-invertability have been analyzed in the structural domain [12]–[15]. Motivated by these results, the goal of the current paper is to develop a structural theory of fault detectability, analogous to the unstructured results in [5], [6], for affine systems where only the structure of the system and the fault are known.

Results along the lines of structural fault detection has appeared in [15], [16]. In [15], the property of left-invertability of continuous-time structured linear descriptor systems in a power-network setting was studied which guarantees the non-existence of undetectable additive (affine) faults. The results were exact for the situations where the initial states were known *a priori* whereas they were sufficient with unknown initial states. On the other hand, in [16] the problem of structural detectability and isolability of faults was addressed for arbitrary nonlinear systems. The authors used a complete matching and residual generation technique that involved a bi-partite graph representation of system variables and constraints. Such a procedure turns out to be computationally very intensive where the time-complexity depends on number of variables (state and output variables) and the number of constraints (which is more than the number of variables). In addition, for discrete-time systems the fault detection procedure requires measurements of all present and future states, and is not applicable when the number of outputs is less than the number of states.

Our main contribution in this paper is a necessary and sufficient condition for (in)distinguishability of structured nominal and faulty models using system graphs for discrete-time autonomous affine systems. We assume that the sparsity structures of system and fault matrices are known *a priori*. For instance, this would encompass situations where the location of fault is known but not its type (e.g., additive, crosstalk, link addition, etc.) or magnitude. The resulting graphical condition involves existence of walks in the system graph containing edges from the faulty system. This conforms to the intuition that for a faulty interaction to be detectable, it must be connected to the output and be excited by the affine term which is acting as the auxiliary input. Next, we present an efficient algorithm to check for the presence of these walks in a graph. We finally illustrate our main result and the algorithm via the means of a suitable example.

The faulty system model in this paper is quite general in terms of location of the occurrence of faults. Our framework allows the faults to affect any interaction; including interactions between states (i.e., the internal dynamics), interactions from the affine term to the states (i.e., the drivers of the system), or the ones from states to outputs (i.e., measurement process). This encompasses the situations where we have additive or affine faults as in [15], though the nominal system class in [15] is more general (linear descriptor systems vs. affine systems). Moreover, it is possible to analyze distinguishability (detectability) for discrete-time systems only from the measurements of their outputs within the proposed framework as opposed to full state measurements.

Notation. The set of real numbers and non-negative integers are denoted by $\mathbb{R}$ and $\mathbb{N}_0$, respectively. The identity matrix of suitable dimensions is denoted by $\mathbf{I}$. Similarly, the zero matrix of suitable dimensions is denoted simply by $\mathbf{0}$. We will use $\mathbf{v}_i$ to denote the ith entry of a vector $\mathbf{v}$.

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II. Background

A. System Model

We are concerned with fault detection in affine systems described by the following:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{K}; \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{1b}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{K} \in \mathbb{R}^{n}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$. The vectors $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{y}(t) \in \mathbb{R}^p$ denote the state and the measured output of the system, respectively, at time $t \in \mathbb{N}_0$.

It is assumed that all the above matrices are structured; i.e., the exact parameter values are not known. However, the sparsity structure of the matrices (i.e., the location of zero and non-zero entries) is known. The non-zero entries are denoted by a $*$. A particular instance of a structured matrix $\mathbf{M}$ preserves the location of all the zero entries while the $*$ are all replaced by free independent parameters. Consequently, a real matrix $\tilde{\mathbf{M}}$ is said to be *structurally equivalent* to a structured matrix $\mathbf{M}$ if the location of zero and non-zero entries in both the matrices correspond to one another. A property (e.g., controllability, detectability) for a structured system is said to hold for a generic set of parameters if the set of independent values for which the property fails is a set of Lebesgue measure zero (more specifically a solution set of a nontrivial system of polynomials). We say that a structured variable m (resp. a structured matrix $\mathbf{M}$) is not constrained to be zero; i.e., $m \not\equiv 0$ (resp. $\mathbf{M} \not\equiv \mathbf{0}$) if m is a free parameter (resp. at least one entry of $\mathbf{M}$ is a free parameter). Similarly, $\mathbf{M} \equiv \mathbf{0}$ if all the entries of $\mathbf{M}$ are constrained to be zero.

B. Fault Detectability

Our main goal in this paper is to formulate a set of graphical conditions such that the nominal system model $\mathcal{S}$ described by (1) can be distinguished from the faulty system model $\mathcal{S}_f$ described below:

$$\mathbf{x}_f(t+1) = (\mathbf{A} + \mathbf{F}^A)\mathbf{x}_f(t) + (\mathbf{K} + \mathbf{F}^K); \tag{2a}$$

$$\mathbf{y}_f(t) = (\mathbf{C} + \mathbf{F}^C)\mathbf{x}_f(t). \tag{2b}$$

where $\mathbf{F}^A$, $\mathbf{F}^K$, and $\mathbf{F}^C$ are structured matrices of appropriate dimensions. The matrices $\mathbf{F}^A$, $\mathbf{F}^K$ and $\mathbf{F}^C$ capture the faults occurring within the system. For example, the (j,i)th entry of $\mathbf{F}^A$ is a free parameter if and only if a fault occurs between the interaction channel of state x_j to x_i. One key feature of (2) is the fact that since the matrices $\mathbf{F}^A$, $\mathbf{F}^K$ and $\mathbf{F}^C$ are structured, they can capture a wide variety of faults; for e.g., changes in parameter values and appearance of non-existing interaction between system parameters. We will define the faulty system matrices $\mathbf{A}_f$, $\mathbf{K}_f$, and $\mathbf{C}_f$ as $\mathbf{A}_f = \mathbf{A} + \mathbf{F}^A$, $\mathbf{K}_f = \mathbf{K} + \mathbf{F}^k$, and $\mathbf{C}_f = \mathbf{C} + \mathbf{F}^C$, respectively. We define the distinction between nominal and faulty models in terms of a distinguishability condition as follows:

Definition 1: A fault model $\mathcal{S}_f$ described by (2) and matrices $(\mathbf{A}_f, \mathbf{K}_f, \mathbf{C}_f)$ is said to be distinguishable from a system model $\mathcal{S}$ described by (1) and the triplet $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ if there exists a $T \in \mathbb{N}_0$ such that for all initial states $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ we have $\mathbf{y}(t) \neq \mathbf{y}_f(t)$ for some $t \in [0, T]$.

In other words, we say that $\mathcal{S}_f$ is distinguishable from $\mathcal{S}$ if their outputs can be distinguished within T time-steps for all initial states. Throughout the paper, we will use the word "distinguishable" to refer to system models and the word "detectable" while referring to faults ([17]). The quantity T is defined as the horizon for distinguishability, and if the systems are distinguishable for some T; they are distinguishable for any $\bar{T} \geq T$. It can be shown (from the results in [6]) that definition 1 for distinguishability is equivalent to the following for affine systems without noise: for all initial states $\mathbf{x}_0, \mathbf{x}_f$ there exists a $\tilde{T}(\mathbf{x}_0, \mathbf{x}_f) \in \mathbb{N}_0$ such that $\mathbf{y}(\tilde{T}(\mathbf{x}_0, \mathbf{x}_f)) \neq \mathbf{y}_f(\tilde{T}(\mathbf{x}_0, \mathbf{x}_f))$. Note that the time horizon T in definition 1 can be taken to be $\max_{\mathbf{x}_0, \mathbf{x}_f} \tilde{T}(\mathbf{x}_0, \mathbf{x}_f)$, where the maximum exists if and only if a finite T exists [6]. For simplicity of notation, in what follows, we write $\tilde{T}$ by dropping the dependence on the initial states whenever it is clear from the context. Proceeding along similar lines, we have the following definition for detectability of a structural fault.

Definition 2: A structured faulty system (2) is (structurally) distinguishable from a structured nominal system (1) if for almost every choice of system parameters (i.e., almost every instantiation of $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ and $(\mathbf{F}^A, \mathbf{F}^K, \mathbf{F}^C)$), there exists a T such that for all initial states $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ we have $\mathbf{y}(t) \neq \mathbf{y}_f(t)$ for some $t \in [0, T]$; or equivalently, for every initial states $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ there exists a $\tilde{T} \in \mathbb{N}_0$ such that $\mathbf{y}(\tilde{T}) \neq \mathbf{y}_f(\tilde{T})$.

On the other hand, systems described by structured triplets $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ and $(\mathbf{A}_f, \mathbf{K}_f, \mathbf{C}_f)$ are said to be structurally indistinguishable, if for almost every choice of parameters, there exist initial states $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ such that for every finite $T \in \mathbb{N}_0$, we have $\mathbf{y}(T) = \mathbf{y}_f(T)$; i.e., there exists a pair of initial conditions such that the outputs of the two systems will never be distinguishable. The negation of structured indistinguishability yields (just) a non-zero measure set in the parameter space such that every pair of system instantiations in this set is distinguishable. While this condition does not directly yield the structured distinguishability condition which involves distinguishability for *almost every choice* of parameters, we will show that this is indeed the case.

Lemma 1: The structured triplets $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ and $(\mathbf{A}_f, \mathbf{K}_f, \mathbf{C}_f)$ are structurally distinguishable; i.e., they are distinguishable for almost every choice of system parameters if and only if, there exists a non-zero measure set of system parameters such that every system pair instantiation in this set is distinguishable.

Proof: See Appendix. ■

The following result from [6] gives a complete characterization of a fault being undetectable for a system in the unstructured setting.

Lemma 2: A faulty model $\mathcal{S}_f$ of the form (2) is indistinguishable from a nominal system $\mathcal{S}$ of the form (1) in T-time steps for any finite T if and only if, there exist $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$

such that the following are satisfied:

$$\begin{bmatrix}\mathbf{C} & -\mathbf{C}_f\end{bmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_f\end{bmatrix} = \mathbf{0}, \tag{3a}$$

$$\begin{bmatrix}\mathbf{O}_G(2n) & -\mathbf{O}_{G_f}(2n)\end{bmatrix}\left(\begin{bmatrix}(\mathbf{A}-\mathbf{I})\mathbf{x}_0\\ (\mathbf{A}_f-\mathbf{I})\mathbf{x}_f\end{bmatrix} + \begin{bmatrix}\mathbf{K}\\ \mathbf{K}_f\end{bmatrix}\right) = \mathbf{0} \tag{3b}$$

where $\mathbf{O}_G(2n)$ and $\mathbf{O}_{G_f}(2n)$ are the extended observability matrices of the nominal and the faulty system of order $2n$, respectively; i.e., $\mathbf{O}_G(2n) = \begin{bmatrix}\mathbf{C}\\ \vdots\\ \mathbf{C}\mathbf{A}^{2n-1}\end{bmatrix}$ and $\mathbf{O}_{G_f}(2n) = \begin{bmatrix}\mathbf{C}_f\\ \vdots\\ \mathbf{C}_f\mathbf{A}_f^{2n-1}\end{bmatrix}$.

In essence, Lemma 2 states the fact that the faulty system model is indistinguishable from the nominal system model if and only if there exists a pair of initial conditions such that the outputs of the two systems for first $2n+1$ time-steps are equal where n denotes the number of states. Thus, for structured matrices we have the following result.

Corollary 1: The structured faulty system model (2) is indistinguishable from the structured nominal system model (1) in T-time steps for any finite T if and only if, for almost every choice of parameters, there exist $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ such that (3) is satisfied.

Note that the initial states $\mathbf{x}_0$ and $\mathbf{x}_f$ leading to indistinguishability depend on the choice of parameters describing the nominal (1) and faulty (2) system pair. Thus, even for two indistinguishable system pairs of same structure but different parameter choices, one would typically end up with different values of initial states $\mathbf{x}_0$ and $\mathbf{x}_f$ resulting in equal outputs. Furthermore, it can be seen that Lemma 1 implies that two structured systems are distinguishable if and only if there exist no initial states such that (3) is satisfied.

C. Graphical Model

One of the main advantages of using structured systems for analysis in control theory is the fact that we are able to leverage tools and results in graph theory to describe system properties. The structured system models (1) and (2) can be represented alternatively using directed graphs. Let us first consider the nominal system model (1) described by a structured triplet $(\mathbf{A}, \mathbf{K}, \mathbf{C})$. Such a system can be described by a directed graph $G = (V, E)$, where V denotes the vertex set and E denotes the edge set. The set V comprises of three types of nodes: n state nodes represented by $\mathcal{X} = \{x_1, \dots, x_n\}$, one auxiliary input node $\mathcal{K} = \{k\}$, and p output nodes $\mathcal{Y} = \{y_1, \dots, y_p\}$, so that we have $V = \mathcal{X} \cup \mathcal{K} \cup \mathcal{Y}$. Correspondingly, the edge set E also comprises of three sets of edges; i.e., $E = E_\mathcal{X} \cup E_\mathcal{K} \cup E_\mathcal{Y}$. We say that $(x_i, x_j) \in E_\mathcal{X}$ if and only if the (j, i)th entry of $\mathbf{A}$ (denoted by $\mathbf{A}_{ji}$) is a free nonzero parameter. Similarly, $(k, x_i) \in E_\mathcal{K}$ and $(x_j, y_l) \in E_\mathcal{Y}$ if and only if $\mathbf{K}_i$ and $\mathbf{C}_{lj}$ are not constrained to be zero, respectively. Here, $\mathbf{K}_i$ and $\mathbf{C}_{lj}$ denote the ith and (l, j)th parameter of $\mathbf{K}$ and $\mathbf{C}$, respectively.

The faulty system model (2) is described by a multi-graph $G_f = (V, E \cup E_f)$ such that the set E_f consists of *faulty edges* not present in the nominal system graph G; i.e., $E \cap E_f = \emptyset$. The set E_f consists of three types of faulty edges corresponding to the location of faults; i.e., location of free nonzero parameters in $\mathbf{F}^A$, $\mathbf{F}^K$, or $\mathbf{F}^C$. As one can observe, it is possible to have both a nominal system edge and a faulty edge between a pair of vertices, corresponding to a fault modifying an existing interaction, thus leading to a multi-graph G_f.

Standard definitions of walks, paths, and cycles are used throughout the paper [18]. A *walk* W in G is a sequence of edges such that the start vertex of the next edge is the end vertex of the preceding one. The number of edges in a walk is also known as the length of the walk. A walk where none of the edges and vertices are repeated is known as a *path* (or in some references, as a simple path). A path whose begin and end vertices are the same is known as a cycle. One important feature of G as well as G_f is that there is no direct edge from $\mathcal{K}$ to any of the nodes in $\mathcal{Y}$; i.e., every path from $\{k\}$ to any of $y_i \in \mathcal{Y}$ must go through one of the nodes in $\mathcal{X}$. Furthermore, we say that a faulty walk exists from k (or equivalently, from $\mathcal{K}$) to $\mathcal{Y}$, if and only if, there exists a walk from k to at least one of the vertices $y_1, \dots, y_p \in \mathcal{Y}$ containing at least one edge from E_f.

III. Main Result

In the following section, we will present an equivalent graphical characterization of Lemma 2 for structured affine systems.

Theorem 1: The structured faulty model $\mathcal{S}_f$ described by (2) and the structured triplet $(\mathbf{A}_f, \mathbf{K}_f, \mathbf{C}_f)$ is indistinguishable from the nominal system model $\mathcal{S}$ described by (1) and the structured triplet $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ for any finite $T \in \mathbb{N}_0$ if and only if there exists no faulty walk from $\mathcal{K}$ to $\mathcal{Y}$ in G_f (i.e., all the walks from k to $\mathcal{Y}$ are free of faulty edges).

Theorem 1 and Lemma 1 state that as long as there is at least one faulty walk from $\mathcal{K}$ to $\mathcal{Y}$, the two systems must be distinguishable for some finite T. This fits with the intuition that for a fault to be detectable, it must have two properties: it should be *excitable* from the affine term (serving as the auxiliary input) $\mathbf{K}$ and it should be *detectable* using a measured output in $\mathcal{Y}$. Furthermore, one should note that at least one of the affine terms $\mathbf{K}$ or $\mathbf{K}_f$ must be nonzero. Otherwise, a trivial choice of initial states with $\mathbf{x}_0 = \mathbf{x}_f = \mathbf{0}$ would make the faulty system indistinguishable for any choice of parameters.

The faulty walks under consideration in Theorem 1 can be of any length. However, as we will show in Lemma 5 later, a walk from k to $\mathcal{Y}$ of length $2n+2$ or greater exists only if there is a corresponding walk of length less than or equal to $2n+1$. Thus, while checking for indistinguishability certificates, one needs to focus on walks from k to $\mathcal{Y}$ of lengths up to $2n+1$.

Remark 1: One can construct a network where the shortest faulty walk from $\mathcal{K}$ to $\mathcal{Y}$ is of length $2n+1$. Consider a chain network with a single output such that only the following edges are present: (k, x_1), (x_s, x_{s+1}) with $s = 1, \ldots, n-1$ and (x_n, y). Assume that only a single faulty edge is present which starts at x_n and ends at x_1. Then the shortest faulty walk from k to y involves going through the edges (k, x_1) and (x_s, x_{s+1}) for $s = 1, \ldots, n-1$ followed by the faulty edge (x_n, x_1), and then again following the chain from x_1 all the way up to y. Thus, the overall length of the shortest faulty walk is $2n+1$.

Remark 2: We will call a single fault (i.e., a faulty edge in E_f) detectable if there exists a walk from k to one of the nodes in $\mathcal{Y}$ containing the fault such that all the other edges in the walk are nominal system edges (i.e., edges in E). While the existence of a detectable fault is sufficient for the distinguishability of the two systems, it is not a necessity. In other words, the faulty walk required in Theorem 1 can comprise of two or more faults i.e., two or more edges from E_f. Thus, one may have a situation such that none of the individual faults are detectable in isolation but the faulty system as a whole is distinguishable from the nominal system.

Remark 3: We finally conclude this section with a comment on the time horizon T required for the distinguishability of faulty and nominal systems. The length ℓ of the shortest faulty walk from k to $\mathcal{Y}$ serves as the infimum for all time horizons T required for distinguishability as defined in definition 1. Intuitively, one can see that this is due to the fact that the difference in effect of the affine term (which is acting as the auxiliary input here) will not show up at the output in less than $\ell+1$ time steps in the faulty graph. Thus, if one sets the initial states equal to zero, the outputs of both the systems will be the same for $t = 0, \ldots, \ell$.

A. Special Cases

In this section, we list a few of the special cases of graphs G and G_f where the application of Theorem 1 becomes easier. Our first situation involves G where every state node $x_i \in \mathcal{X}$ lies on at least one of the paths from k to $\mathcal{Y}$. In such a case, it turns out, every fault in the system is detectable.

Proposition 1: Suppose G is such that every node $x_i \in \mathcal{X}$ lies on some path from k to one of nodes in $\mathcal{Y}$. Then, the faulty system (2) is structurally distinguishable from the nominal system (1) whenever $E_f \neq \emptyset$; i.e., every fault occurring in such a system is detectable.

Proof: We will present a sketch of the proof here. First, consider a fault in $\mathbf{F}^k$ of the form (k, x_i). Since there exists a path from x_i to one of the nodes in $\mathcal{Y}$, one can construct a walk beginning with the faulty edge (k, x_i) and appending it to the path from x_i to $\mathcal{Y}$ which clearly shows that fault is detectable. Now consider, fault in $\mathbf{F}^A$ of the form (x_i, x_j). Since both x_i and x_j lie on at least one of the paths from k to $\mathcal{Y}$ (say P_i and P_j, respectively), one can construct a walk as follows: begin from k to x_i using the edges in P_i, follow it up with the faulty edge (x_i, x_j), and finally take the path from x_j to $\mathcal{Y}$ from P_j. Thus, this walk makes the faulty edge (x_i, x_j) detectable. Finally, a similar proof can be constructed for a fault occurring in $\mathbf{F}^C$. ∎

One of the immediate applications of Proposition 1 is in the case of strongly connected directed graphs.

Corollary 2: Suppose $\tilde{G} = (\mathcal{X}, E_{\mathcal{X}})$ is strongly connected; i.e., the state vertices form a strongly connected network among themselves. Such a network is distinguishable whenever $E_f \neq \emptyset$.

We conclude this section with a discussion on acyclic graphs.

Proposition 2: If G_f is acylic with the longest path length between any two vertices in $\mathcal{X}$ equal to ℓ_1 and there exists a faulty path from k to $\mathcal{Y}$ of length $\ell_2 + 2$, then the following hold true:

(a) The length $\ell_2 \leq \ell_1$, and
(b) The nominal and faulty systems are structurally distinguishable with $T = \ell_1 + 1$.

Proof: Part (a) is clear from the fact that G_f is acyclic and ℓ_1 is the longest path between any two vertices in $\mathcal{X}$. Furthermore, since the edge E is a subset of edges in G_f, one can observe that $\mathbf{A}^{\ell_1+1} = \mathbf{A}_f^{\ell_1+1} = \mathbf{0}$. Thus,

$$\mathbf{y}(\ell_1+1) - \mathbf{y}_f(\ell+1) = \mathbf{C}\mathbf{A}^{\ell_1+1}\mathbf{x}(0) - \mathbf{C}_f\mathbf{A}_f^{\ell_1+1}\mathbf{x}_f(0) + \sum_{t=0}^{\ell_1}\left(\mathbf{C}\mathbf{A}^t\mathbf{K} - \mathbf{C}_f\mathbf{A}_f^t\mathbf{K}_f\right)$$

is not equal to $\mathbf{0}$ for almost all choices of parameters since $\mathbf{C}\mathbf{A}^{\ell_2}\mathbf{K} - \mathbf{C}_f\mathbf{A}_f^{\ell_2}\mathbf{K}_f \not\equiv \mathbf{0}$ (see Lemma 4 for details). Thus, the systems are distinguishable for $T = \ell_1$. ∎

IV. Proof of Main Result

In this section, we present a detailed proof of Theorem 1. We proceed by first stating the Lemmas required to simplify the main proof. We will use the following notation throughout the section: we will denote $\mathbf{M}_{ji}$ to be the (j, i)th entry of $\mathbf{M}$. Furthermore, $\mathbf{M}_{ji} \not\equiv 0 \Leftrightarrow -\mathbf{M}_{ji} \not\equiv 0$.

A. Lemmas

We first recall the Cayley-Hamilton Theorem (CHT) for matrices and state it here for the sake of completeness.

Lemma 3 (CHT): Let $\mathbf{M}$ be a square matrix with entries $\mathbf{M}_{ij}$ and let $p_{\mathbf{M}}(\beta) = \det(\beta\mathbf{I} - \mathbf{M})$ denote its characteristic polynomial. Then, $p(\mathbf{M}) = \mathbf{0}$.

Note that when $\mathbf{M}$ is a structured matrix; i.e., $\mathbf{M}_{ij}$ are indeterminate quantities or 0, the coefficients of $p_{\mathbf{M}}(\beta) = \det(\beta\mathbf{I} - \mathbf{M})$ are simply either 0 or polynomials without constant terms in entries $\mathbf{M}_{ij}$ (except for the leading coefficient which is 1).

For the rest of the section, we will define augmented matrices $\bar{\mathbf{A}} \triangleq \begin{bmatrix} \mathbf{A} & \\ & \mathbf{A}_f \end{bmatrix}$, $\bar{\mathbf{C}} \triangleq \begin{bmatrix} \mathbf{C} & -\mathbf{C}_f \end{bmatrix}$, and $\bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K} \\ \mathbf{K}_f \end{bmatrix}$. Our final two lemmas in this section relate the existence of faulty walks from k to in $\mathcal{Y}$ to the sparsity of matrix products.

Lemma 4: A faulty walk of length $\ell+2$ (for $\ell \in \mathbb{N}_0$) from k to one of the vertices $y_1, \ldots, y_p \in \mathcal{Y}$ exists if and only if $\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}} = -\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f + \mathbf{C}\mathbf{A}^\ell\mathbf{K} \not\equiv \mathbf{0}$; i.e., not all the entries in $\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f - \mathbf{C}\mathbf{A}^\ell\mathbf{K}$ are identically zero.

Proof: Recall that G_f is a supergraph of G since the edge set of G is a subset of the edge set of G_f. The jth entry of the matrix product $\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f$ (resp. $\mathbf{C}\mathbf{A}^\ell\mathbf{K}$) not being identically zero is equivalent to saying that there exists a walk of length $\ell+2$ from k to $y_j \in \mathcal{Y}$ in the graph G_f (resp. G) through ℓ vertices in $\mathcal{X}$. Let $\mathcal{W}_1$ (resp. $\mathcal{W}_2$) denote the collection of all walks from k to the vertices in $\mathcal{Y}$ in the graph G_f (resp. G) via exactly ℓ state nodes in $\mathcal{X}$. Note that some of the state nodes may be repeated. Since the edge set of G is a subset of the edge of G_f, we have $\mathcal{W}_2 \subset \mathcal{W}_1$. It can be seen that the nonzero entries of the matrix product $\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f$ (resp. $\mathbf{C}\mathbf{A}^\ell\mathbf{K}$) correspond to the walks in $\mathcal{W}_1$ (resp. $\mathcal{W}_2$). By subtracting the quantity $\mathbf{C}\mathbf{A}^\ell\mathbf{K}$ from $\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f$, we are only considering the walks in $\mathcal{W}_1 \setminus \mathcal{W}_2$; i.e., the walks from k to $\mathcal{Y}$ that are present in G_f but not present in G. Thus, if entry j of $\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}} = -\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f + \mathbf{C}\mathbf{A}^\ell\mathbf{K}$ is not identically zero, there exists a walk $W \in \mathcal{W}_1 \setminus \mathcal{W}_2$; i.e., this walk W must contain at least one edge from E_f; or a faulty edge.

Conversely, suppose there exists a walk W through the nodes $k \to a_0 \to \cdots a_\ell \to y_j$ in G_f from k to $y_j \in \mathcal{Y}$ containing at least one faulty edge from E_f. Then the jth entry of $\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f$ is not identically zero, and hence $(\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}})_j = (-\mathbf{C}_f\mathbf{A}_f^\ell\mathbf{K}_f + \mathbf{C}\mathbf{A}^\ell\mathbf{K})_j \not\equiv \mathbf{0}$ since W would not be present in G because $E \cap E_f = \emptyset$. Here $\mathbf{v}_j$ denotes the jth entry of the vector $\mathbf{v}$. ∎

Lemma 5: A walk of length greater than or equal to $2n+2$ from k to $\mathcal{Y}$ exists in G_f only if there exists a walk from k to $\mathcal{Y}$ of length less than or equal to $2n+1$.

Proof: This lemma is a consequence of Lemmas 4 and 3. We have $\bar{\mathbf{C}}\bar{\mathbf{A}}^l\bar{\mathbf{K}} = \sum_{s=0}^{2n-1}\alpha_s\bar{\mathbf{C}}\bar{\mathbf{A}}^s\bar{\mathbf{K}}$ for any $l \geq 2n$ where α_s are either 0 or polynomials without constant terms in the entries of structured matrix $\bar{\mathbf{A}}$. It is clear that $\bar{\mathbf{C}}\bar{\mathbf{A}}^l\bar{\mathbf{K}} \not\equiv \mathbf{0}$ for $l \geq 2n$ only if $\bar{\mathbf{C}}\bar{\mathbf{A}}^s\bar{\mathbf{K}} \not\equiv \mathbf{0}$ for some $s = 0, \ldots, 2n-1$. ∎

B. Proof of Theorem 1

1) Structural Indistinguishability: ⇒No faulty walk from k to $\mathcal{Y}$

By contradiction, assume that there exists a faulty walk of length $\ell+2$ from k to $\mathcal{Y}$ but the faulty system is structurally indistinguishable from the nominal system. Without loss of generality assume that the quantity ℓ is the smallest one; i.e., $\ell+2$ is the length of the shortest walk from k to one of the vertices in $\mathcal{Y}$ containing a faulty edge from E_f. Let N denote the total number of non-zero free parameters in $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ and $(\mathbf{F}^A, \mathbf{F}^K, \mathbf{F}^C)$. Then, every $\boldsymbol{\lambda} \in \mathbb{R}^N$ admits a realization of the nominal and faulty systems defined in (1) and (2), respectively. Since the faulty system is structurally indistinguishable from the nominal system, this implies that for almost every choice of $\boldsymbol{\lambda} \in \mathbb{R}^N$, there exist $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ such that (3) is satisfied (corollary 1). Let us define by $\bar{\mathbf{x}}$, the augmented vector $\bar{\mathbf{x}} = \begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_f\end{bmatrix}$.

Structural indistinguishability using (3) implies that $\mathbf{C}\mathbf{x}_0 - \mathbf{C}_f\mathbf{x}_f \equiv \mathbf{0} \Rightarrow \bar{\mathbf{C}}\mathbf{I}\bar{\mathbf{x}} = \mathbf{0}$. Similarly, expanding the first block row of the second equation in (3) gives us

$$\mathbf{C}[(\mathbf{A}-\mathbf{I})\mathbf{x}_0 + \mathbf{K}] - \mathbf{C}_f[(\mathbf{A}_f - \mathbf{I})\mathbf{x}_f + \mathbf{K}_f] = \mathbf{0}$$

which implies $\bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{x}} = -\bar{\mathbf{C}}\bar{\mathbf{K}}$. Moving on, expanding the second block row yields

$$\bar{\mathbf{C}}\bar{\mathbf{A}}^2\bar{\mathbf{x}} = -\bar{\mathbf{C}}\bar{\mathbf{K}} - \bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{K}}$$

Expanding each block row iteratively, we obtain $\bar{\mathbf{C}}\bar{\mathbf{A}}^r\bar{\mathbf{x}} = -\sum_{j=0}^{r-1}\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}}$ for $r = 1, \ldots, 2n$.

Let us denote the characteristic polynomial of $\bar{\mathbf{A}}$ by $p_{\bar{\mathbf{A}}}(\beta) \triangleq \det(\beta\mathbf{I} - \bar{\mathbf{A}})$. It can be seen that $p_{\bar{\mathbf{A}}}(\beta)$ has the following expanded form:

$$p_{\bar{\mathbf{A}}}(\beta) = \det(\beta\mathbf{I} - \bar{\mathbf{A}}) = \beta^{2n} - \gamma_{2n-1}\beta^{2n-1} - \cdots - \gamma_0. \quad (4)$$

where $\gamma_0, \ldots, \gamma_{2n-1}$ are either 0 or polynomials without constant terms in the entries of $\bar{\mathbf{A}}$ (equivalently, in the entries of $\mathbf{A}$ and $\mathbf{F}^A$). Lemma 3 (Cayley-Hamilton Theorem) implies that $\bar{\mathbf{A}}$ satisfies its own characteristic polynomial; i.e., $p_{\bar{\mathbf{A}}}(\bar{\mathbf{A}}) = \mathbf{0}$ and thus, $\bar{\mathbf{A}}^{2n} = \sum_{j=0}^{2n-1}\gamma_j\bar{\mathbf{A}}^j$.Thus, we have using (4)

$$\begin{aligned}\bar{\mathbf{C}}\bar{\mathbf{A}}^{2n}\bar{\mathbf{x}} &= \gamma_0\bar{\mathbf{C}}\mathbf{I}\bar{\mathbf{x}} + \gamma_1\bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{x}} + \cdots + \gamma_{2n-1}\bar{\mathbf{C}}\bar{\mathbf{A}}^{2n-1}\bar{\mathbf{x}}\\ &= \gamma_1\bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{x}} + \cdots + \gamma_{2n-1}\bar{\mathbf{C}}\bar{\mathbf{A}}^{2n-1}\bar{\mathbf{x}}\\ &= \gamma_1(-\bar{\mathbf{C}}\bar{\mathbf{K}}) + \cdots + \gamma_{2n-1}\Big(-\sum_{j=0}^{2n-2}\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}}\Big)\\ &= -\sum_{j=0}^{2n-2}\Big(\sum_{r=j+1}^{2n-1}\gamma_r\Big)\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}}\end{aligned}$$

Now, last block row of (3) implies $\bar{\mathbf{C}}\bar{\mathbf{A}}^{2n}\bar{\mathbf{x}} = -\bar{\mathbf{C}}\bar{\mathbf{K}} - \cdots - \bar{\mathbf{C}}\mathbf{A}^{2n-1}\bar{\mathbf{K}}$. Therefore, the last line of the above equation can be re-written as

$$\sum_{j=0}^{2n-1}\Big(\sum_{r=j+1}^{2n}\gamma_r - 1\Big)\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}} = \mathbf{0} \quad (5)$$

where γ_{2n} is introduced for notational convenience and it is equal to 0.

Since $\ell+2$ is the length of the shortest faulty walk from k to $\mathcal{Y}$, Lemma 4 implies that $\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}} \not\equiv \mathbf{0}$, but $\bar{\mathbf{C}}\bar{\mathbf{A}}^s\bar{\mathbf{K}} \equiv \mathbf{0}$ for all $s = 0, \ldots, \ell-1$. Now equation (5) can be re-written as

$$\boldsymbol{\psi}(\boldsymbol{\lambda}) \triangleq \sum_{j=\ell}^{2n-1}\Big(\sum_{s=j+1}^{2n}\gamma_s - 1\Big)\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}} = \mathbf{0} \quad (6)$$

for almost every $\boldsymbol{\lambda} \in \mathbb{R}^N$ with $\gamma_{2n} = 0$. Since, a polynomial can be either identically zero or non-zero almost everywhere in $\mathbb{R}^N$ [19], this implies that $\psi_i(\boldsymbol{\lambda}) \equiv 0$ for every $\boldsymbol{\lambda} \in \mathbb{R}^N$ and for every $i \in 1, \ldots, n$ where $\psi_i(\boldsymbol{\lambda})$ denotes the ith row of $\boldsymbol{\psi}(\boldsymbol{\lambda})$; i.e., ψ_i is the zero polynomial.

Note that since $\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}} \not\equiv \mathbf{0}$ there exists an index $l \in \{1, \ldots, p\}$ (where p denotes the number of outputs) such that $(\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}})_l \not\equiv 0$ for almost every choice of $\boldsymbol{\lambda} \in \mathbb{R}^N$. Let $\phi_l(\boldsymbol{\lambda})$ denote the lth row of $-\bar{\mathbf{C}}\bar{\mathbf{A}}^\ell\bar{\mathbf{K}}$. In other words, we have $\phi_l(\boldsymbol{\lambda}) \neq 0$ for almost every $\boldsymbol{\lambda} \in \mathbb{R}^N$; i.e., $\phi_l(\cdot)$ is not the zero polynomial.

Now one can observe that every monomial term in any entry of $\bar{\mathbf{C}}\bar{\mathbf{A}}^j\bar{\mathbf{K}}$ contains one entry from either $\mathbf{C}$ or $\mathbf{F}^C$, j entries of $\mathbf{A}$ and/or $\mathbf{F}^A$, and one entry of $\mathbf{K}$ or $\mathbf{F}^K$; i.e., has a total degree of $j+2$. That is, every monomial in $\phi_l(\boldsymbol{\lambda})$ has a degree of $\ell+2$.

Let $\eta_l(\cdot)$ be defined as $\eta_l(\boldsymbol{\lambda}) = \psi_l(\boldsymbol{\lambda}) - \phi_l(\boldsymbol{\lambda})$. We can see that

$$\eta_l(\boldsymbol{\lambda}) = \sum_{r=\ell+1}^{2n} \gamma_r(\bar{\mathbf{C}}\bar{\mathbf{A}}^{\ell}\bar{\mathbf{K}})_l + \sum_{j=\ell+1}^{2n-1}\bigg(\sum_{s=j+1}^{2n}\gamma_s - 1\bigg)(\bar{\mathbf{C}}\bar{\mathbf{A}}^{j}\bar{\mathbf{K}})_l$$

To achieve $\psi_l(\cdot) \equiv 0$, we must have $\eta_l(\cdot) = -\phi_l(\cdot)$. Since γ_i for $i = 1, \ldots, 2n$ is either 0 or a polynomial without constant terms in the entries of $\mathbf{A}$ and/or $\mathbf{F}^A$, we have that either $\gamma_i = 0$ or $\mathrm{degree}(\gamma_i) \geq 1$. Thus, the polynomial $\eta_l(\boldsymbol{\lambda})$ is either zero or has a degree greater than $\ell+2$. Since $\phi_l(\boldsymbol{\lambda})$ has a degree of $\ell+2$, we cannot have $\psi_l(\cdot) = \eta_l(\cdot) + \phi_l(\cdot) = 0$. Therefore, (6) cannot be satisfied (or $\boldsymbol{\psi}(\cdot) \not\equiv \mathbf{0}$) since $\psi_l(\cdot) \not\equiv 0$. Hence, we arrive at a contradiction which concludes the proof for this direction.

2) Structural Indistinguishability: $\Leftarrow$*No faulty walk from k to $\mathcal{Y}$*

Since there is no faulty walk from k to any of the output vertices $y_1, \ldots, y_p \in \mathcal{Y}$, one can deduce from Lemmas 4 and 5 that $\bar{\mathbf{C}}\bar{\mathbf{A}}^l\bar{\mathbf{K}} \equiv \mathbf{0}$ for all $l \in \{0, \ldots, 2n-1\}$. One can then satisfy the relations in (3) by choosing $\mathbf{x}_0 = \mathbf{x}_f = \mathbf{0}$. Therefore, if the initial states are chosen to be zero, the faulty system model is always indistinguishable from the nominal system model. ■

Remark 4: It is worth noting that while for the purpose of simplification of the proof, we considered the initial state $\mathbf{x}_0 = \mathbf{x}_f = \mathbf{0}$ in the above paragraph, several non-zero initial states also produce indistinguishable outputs whenever there exists no faulty walk from k to $\mathcal{Y}$ in G_f. To see this, first note that whenever this condition is satisfied; there always exists an $x_i \in \mathcal{X}$ such that there is no walk starting from x_i to $\mathcal{Y}$ containing a faulty edge. In other words, the $[\mathbf{C}\mathbf{A}^l]_i = [\mathbf{C}_f\mathbf{A}_f^l]_i$ for any $l \in \mathbb{N}_0$ where $\mathbf{M}_i$ denotes the ith column of matrix $\mathbf{M}$. Denoting by $\mathbf{x}_i$ and $\mathbf{x}_{f_i}$ the ith component of the vectors $\mathbf{x}_0$ and $\mathbf{x}_f$, respectively, we have

$$[\mathbf{C}\mathbf{A}^l]_i\mathbf{x}_i - [\mathbf{C}_f\mathbf{A}_f^l]_i\mathbf{x}_{f_i} = 0, \quad \forall \mathbf{x}_i = \mathbf{x}_{f_i} \in \mathbb{R} \tag{7}$$

In other words, if we consider any initial state which has the following properties:

1) $\mathbf{x}_i = \mathbf{x}_{f_i} \in \mathbb{R}$, whenever x_i does not have a faulty walk to $\mathcal{Y}$;
2) $\mathbf{x}_i = \mathbf{x}_{f_i} = 0$, otherwise;

equation (3) is still satisfied and the systems are indistinguishable.

V. Algorithm for checking faulty walks

In this section, we describe an algorithm for determining whether or not there exists a faulty walk of a certain length from k to $\mathcal{Y}$ in G_f. The input to the algorithm are the edge sets E and E_f. The main steps of the algorithm are described briefly here:

(a) First step is to find all the nodes in $\mathcal{X}$ that are not lying on any walk connecting k to $\mathcal{Y}$, and remove them along with their associated edges from the graph. This can be done using a breadth-first search. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ denote the remaining graph.

(b) All the remaining edges; i.e., edges in $\tilde{E}$ make up all the walks that connect k to $\mathcal{Y}$. If any of these edges is faulty, then we have a faulty walk from k to $\mathcal{Y}$, and thus, the systems are distinguishable. This can be accomplished by checking $\tilde{E} \cap E_f$.

Algorithm 1 ($\{0,1\}$) = exist_walk(G, G_f)

```
1: breadth-first search to obtain state node set: X_α = {x_{i_1}, ..., x_{i_l}} such that x_{i_r} is not connected from k and to any node in Y for r = 1, ..., l.
2: G̃_f = (V \ X_α, (E ∪ E_f) \ E_α) where E_α is the set of edges associated with X_α.
3: Define Ṽ ≜ V\X_α, X̃ ≜ X\X_α, and Ẽ ≜ (E∪E_f)\E_α.
4: Ē = Ẽ ∩ E_f.
5: if |Ē| ≥ 1 then
6:    return True
7: else
8:    return False
```

By virtue of Lemma 5, it is evident that if there exists a faulty walk from k to $\mathcal{Y}$ then there must at least one such walk of length $\leq 2n+1$. Thus, just checking for the existence of a single faulty edge in $\tilde{E}$ or equivalently, checking for the emptiness of $\tilde{E} \cap E_f$ suffices to guarantee the distinguishability of the associated systems. In case, the set E_f is not available at the outset, but only E and $E \cup E_f$ are available, then one needs to perform a set difference operation as the first step to calculate E_f. These steps are captured as a formal algorithm in Algorithm 1.

VI. Illustrative Example

In this section, we illustrate the salient features of Theorem 1 using a 12-state example as shown in Fig. 1. The vertex set V consists of a node k, 12 state nodes denoted by $x_1, \ldots, x_{12}$ and 2 output nodes, namely y_1 and y_2. The nominal system edge set E consists of all the edges shown in solid lines whereas the faulty edge set E_f consists of the edges represented by dashed lines. As one can see, the graph G_f represented in Fig 1 is a multi-graph with two edges from the node x_4 to x_8. Out of these two edges, one belongs to E and the other one belongs to E_f. This shows that the interconnection present from x_4 to x_8 is vulnerable and "its strength" can change due to a fault, captured as an edge in the faulty model.

We consider several variations of G_f to discuss how the systems become distinguishable. For instance, consider the scenario where $E_f = (k, x_7)$, i.e., only the faulty edge from k to x_7 is present and there is no fault between the state nodes. Using a breadth-first search, one can observe that nodes x_4, x_7, x_8, x_{10}, x_{11} and x_{12} are not on any walk from k to y_1 or y_2 in G_f. Thus, they can be "removed" from the graph for the purpose of fault detection. In other words, the fault (k, x_7) is not detectable on its own. Since, there is no fault between the nodes in $\mathcal{X}$ or in the edges from $\mathcal{X}$ to $\mathcal{Y}$,

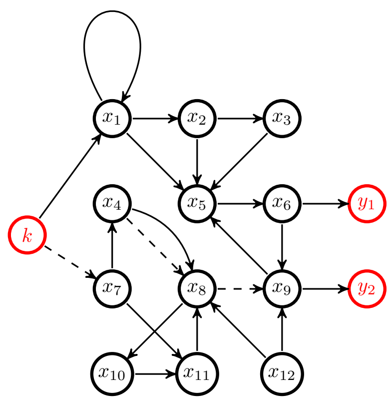

Fig. 1. A 12 node example used for illustrating the salient features of Theorem 1

the entity $\mathbf{C}\mathbf{A}^l - \mathbf{C}_f\mathbf{A}_f^l \equiv \mathbf{0}$ for every $l \in \mathbb{N}_0$. Thus, for any initial state $x_0, x_f \in \mathbb{R}^{12}$, the systems are indistinguishable.

We also face an indistinguishable situation in the scenario where E_f just consists of (x_4, x_8) and/or (x_8, x_9) but the faulty edge (k, x_7) is absent. Since, there would be no walks from k leading to y_1 or y_2 containing the faults (x_4, x_8) and/or (x_8, x_9) in the absence of (k, x_7), these faults will not be detectable for certain initial states. Moreover, one can see that there is no walk leading to $\mathcal{Y}$ starting from the nodes in set $\mathcal{N} = \{x_1, x_2, x_3, x_5, x_6, x_9\}$ that has a faulty edge. Therefore, as explained in Remark 4, any pair of initial states $\mathbf{x}_0, \mathbf{x}_f$ satisfying $\mathbf{x}_i = \mathbf{x}_{f_i} \in \mathbb{R}$ for $i \in \mathcal{N}$ and $\mathbf{x}_i = \mathbf{x}_{f_i} = 0$ for $i \in V \setminus \mathcal{N}$ would be indistinguishable.

The situation changes, however, when both (k, x_7) and (x_8, x_9) are present in E_f. One can construct a walk, namely, $k \to x_7 \to x_{11} \to x_8 \to x_9 \to y_2$ which satisfies the condition of Theorem 1; i.e., it goes from k to $\mathcal{Y}$ and contains faulty edges (both (k, x_7) and (x_8, x_9)). Thus, in such a situation, the faulty system described by G_f is distinguishable from G. In summary, while none of these faults are detectable in isolation, they are detectable in combination. Also, using the fact the length of the shortest walk from k to $\mathcal{Y}$ is containing a faulty edge is 5, we have $T \geq 5$ where T denotes the time horizon for distinguishability as discussed in definition 1.

VII. Conclusions

In this work we presented a framework for representing systems subject to faults using ideas from structural control. Moreover, we developed a necessary and sufficient graph-theoretic condition for checking a fault is structurally detectable for a given affine system. The resulting condition turns out be a check for existence of walks containing faulty edges between the affine term (auxiliary input) and the output vertices. We then proposed an algorithm to check for the existence of such walks and illustrated it via an example.

The future extensions of this work are twofold. Our immediate goal is address a sensor placement problem which involves placing the minimum number of sensors to guarantee that the potential faults can be detected. This is particularly useful for guiding system design. Another direction of our research is to extend these results to switched affine systems, allowing time-varying connectivities. We also intend to apply these techniques for fault detection in large-scale complex networks arising in various fields such as power systems, transportation systems, and aerospace.

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Appendix

A. Proof of Lemma 1

Proof: Let N denote the total number of free parameters; i.e., the number of entries not constrained to be zero in $(\mathbf{A}, \mathbf{K}, \mathbf{C})$ and $(\mathbf{A}_f, \mathbf{K}_f, \mathbf{C}_f)$. Then any $\boldsymbol{\lambda} \in \mathbb{R}^N$ corresponds to a parameter instantiation (or choice) for the systems (1) and (2).

Suppose the system (2) is structurally distinguishable from (1); i.e., they are distinguishable for almost any choice of parameters $\boldsymbol{\lambda} \in \mathbb{R}^N$. Then the set of parameter choices for which the distinguishability condition fails forms a measure zero set in the parameter space $\mathbb{R}^N$. Thus, for any non-zero measure set $\mathcal{B} \subset \mathbb{R}^N$ of system parameters, the distinguishability condition fails for a zero measure subset $\tilde{\mathcal{B}} \subset \mathcal{B}$. Therefore, every $\boldsymbol{\lambda}$ in the non-zero measure set $\mathcal{B} \backslash \tilde{\mathcal{B}}$ yields a distinguishable system pair.

Conversely, suppose there exists a non-zero measure set $\mathcal{B} \subset \mathbb{R}^N$ such that every $\boldsymbol{\lambda} \in \mathcal{B}$ yields a distinguishable system pair instantiation. However, by contradiction assume that the faulty system is not structurally distinguishable from the nominal one. Thus, there exists a non-zero measure set $\mathcal{S} \subset \mathbb{R}^N$ such that for every $\boldsymbol{\lambda} \in \mathcal{S}$ the systems are indistinguishable. In other words for every $\boldsymbol{\lambda} \in \mathcal{S}$, there exists a pair of initial conditions $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ such that the following is satisfied ([6]):

$$\underbrace{\begin{bmatrix} \mathbf{O}_G(2n+1) & -\mathbf{O}_{G_f}(2n+1) \end{bmatrix}}_{\tilde{\mathbf{O}}} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_f \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0} \\ -\bar{\mathbf{C}}\bar{\mathbf{K}} \\ \vdots \\ -\sum_{s=0}^{2n-1} \bar{\mathbf{C}}\bar{\mathbf{A}}^s\bar{\mathbf{K}} \end{bmatrix}}_{\tilde{\mathbf{K}}} \tag{8}$$

where $\bar{\mathbf{C}}$, $\bar{\mathbf{A}}$, and $\bar{\mathbf{K}}$ are the augmented matrices as defined in Section IV. This is equivalent to saying that $\tilde{\mathbf{K}} \in \mathcal{R}(\tilde{\mathbf{O}})$ for every $\boldsymbol{\lambda} \in \mathcal{S}$ where $\mathcal{R}(\cdot)$ denotes the range space of a matrix. This is possible if and only if $(\tilde{\mathbf{O}}\tilde{\mathbf{O}}^\dagger - \mathbf{I})\tilde{\mathbf{K}} = \mathbf{0}$ where $\tilde{\mathbf{O}}^\dagger$ denotes the pseudo-inverse[1] of $\tilde{\mathbf{O}}$ satisfying $\tilde{\mathbf{O}}\tilde{\mathbf{O}}^\dagger\tilde{\mathbf{O}} = \tilde{\mathbf{O}}$ [20]. Since the entries of $\tilde{\mathbf{O}}$ are polynomials in the entries of $\boldsymbol{\lambda}$, the entries of $\tilde{\mathbf{O}}^\dagger$ and correspondingly $(\tilde{\mathbf{O}}\tilde{\mathbf{O}}^\dagger - \mathbf{I})\tilde{\mathbf{K}}$ are rational functions of entries of $\boldsymbol{\lambda}$. That is they are of the form $p_i(\boldsymbol{\lambda})/q_i(\boldsymbol{\lambda})$ where $p_i(\cdot)$ and $q_i(\cdot)$ are polynomials in entries of $\boldsymbol{\lambda}$. Now, $(\tilde{\mathbf{O}}\tilde{\mathbf{O}}^\dagger - \mathbf{I})\tilde{\mathbf{K}} = \mathbf{0}$ for every $\boldsymbol{\lambda} \in \mathcal{S}$ implies that $p_i(\boldsymbol{\lambda}) = 0$ and $q_i(\boldsymbol{\lambda}) \neq 0$ for every i.

A polynomial has the property that it is either identically zero or non-zero almost everywhere in $\mathbb{R}^N$ [19]. Since $\mathcal{S} \subset \mathbb{R}^N$ is a non-zero measure subset, we have that $p_i(\boldsymbol{\lambda}) = 0$ for every $\boldsymbol{\lambda} \in \mathbb{R}^N$ and $q_i(\boldsymbol{\lambda}) \neq 0$ for almost every $\boldsymbol{\lambda} \in \mathbb{R}^N$ (the zero set of $q_i(\boldsymbol{\lambda})$ is a measure-zero set). Using the fact that $\mathcal{B} \subset \mathbb{R}^N$ is also a non-zero measure set, this immediately implies that $p_i(\boldsymbol{\lambda}) = 0$ for every $\boldsymbol{\lambda} \in \mathcal{B}$ and there exists a non-zero measure subset of $\bar{\mathcal{B}} \subset \mathcal{B}$ such that $q_i(\boldsymbol{\lambda}) \neq 0$ for every $\boldsymbol{\lambda} \in \bar{\mathcal{B}}$. Therefore, (8) is satisfied for every $\boldsymbol{\lambda} \in \bar{\mathcal{B}}$ and every system pair instantiation in $\bar{\mathcal{B}}$ is indistinguishable. This contradicts with the choice of $\mathcal{B}$ which concludes the proof. ■

[1]Actually this result holds true for any generalized $\{1\}$-inverse of $\tilde{\mathbf{O}}$