Data-driven computation of minimal robust control invariant set

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***Abstract*— We propose a data-driven framework to compute an approximation of a minimal robust control invariant set (mRCI) for an uncertain dynamical system where the model of the system is also unknown and should be learned from data. First, the set of admissible models is characterized via a set of linear constraints extracted from the experimental data. Each model in the set of admissible models contains information about the nominal model, as well as the characterization of the model uncertainty, including additive and multiplicative uncertainties. Then an iterative algorithm based on robust optimization is proposed to simultaneously compute a minimal robust control invariant set while selecting an optimal model from the admissible set. The numerical results show that the proposed method greatly reduces the size of the invariant set compared to a benchmark method that sequentially selects a model with least squares and then computes the invariant set.**

I. INTRODUCTION

Correct-by-construction control synthesis has attracted increasing interest over the past decade with the promise that through rigorous reasoning of system behavior, the closed-loop system can be guaranteed to satisfy the design specifications. A fundamental concept related to safety specifications is robust control invariant sets (RCI). If an initial condition lies in an RCI, then there exist control inputs to guarantee that the trajectory of the system remains in the set indefinitely despite all possible uncertainties. In addition to providing a safety certificate, an RCI can be used in a supervisory control structure, which guarantees safety with minimal intervention on top of an existing controller [7], [15].

Existing methods for computing invariant sets include LMI-based Lyapunov type analysis [5], [9], [14], sum of squares programming [16], [25], Minkowski type methods [10], [13], [17], [18], polytopic projection [3], [15] and linear programming [26]. In this paper, inspired by the one-shot approach proposed in [26] for low-complexity invariant set computation for autonomous systems (i.e., systems without a control input), we propose an iterative algorithm to compute an RCI with constant representation complexity where one can leverage the available control authority for enforcing invariance.

Depending on the control problem in hand, either maximal or minimal control invariant sets can be relevant. A maximal control invariant set can describe the region of attraction with limited control authority. It is formally defined in [1], and the definition of the maximum is in the set inclusion sense, i.e., every control invariant set within a compact subset of the state space is a subset of the maximal control invariant set. For linear discrete-time systems, the maximal control invariant set can be computed via polytopic projection [15], [19]. On the other hand, a mRCI describes how small a robust invariant set can be under disturbance and uncertainty. This is useful when the objective is to limit the deviations from a desired operating point. However, in general there does not exist a unique mRCI that is a subset of every RCI.

The computation of invariant sets depends on a model of the system, which includes both the nominal model and the uncertainty characterization. In real engineering practice, the model is often the result of a system identification step. Even for physics-based models, the uncertainty is often characterized through experiments. The most classical system identification method is the least square regression, including many extensions that incorporate various filtering structures [12]. Set membership methods are another class of identification methods, which identify the set of admissible model parameters via set intersections [11]. Since the 1980s, control relevant identification has been studied, including $\mathcal{H}_\infty$ identification [6], generalized predictive control [21], and stochastic embedding [8]. However, the $\mathcal{H}_\infty$ identification and stochastic embedding approaches are for model identification in the frequency domain, and the generalized predictive control focuses on optimality rather than robustness. In terms of the identification of a model for an uncertain system that suits the need for correct-by-construction control synthesis, there is a gap to be filled. Sokolov proposed a framework of system identification with unknown bound for additive uncertainties [22]–[24]. Another approach with unknown uncertainty bound can be found in [20], where an identification method that uses mixed integer programming to identify a piecewise linear model with a bound on the disturbance. However, in the above mentioned papers, the system identification and control synthesis are done separately and the identified model is not necessarily "optimal" for the control synthesis task.

The main contribution of this paper is a data-driven framework for approximating a minimal robust control invariant set while simultaneously picking an optimal admissible uncertain model. An admissible model, which may not be unique, is defined as a model that explains a finite measurement history. The novelty of the proposed framework lies in two aspects. First, the proposed mRCI algorithm does not require an accurate model to begin with, but instead identifies the set of admissible models from data. Second, an optimal model is selected concurrently while computing

This work was supported by NSF grant #CNS-1239037.

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an mRCI. We demonstrate the proposed method with a lane keeping problem for road vehicles. The lateral dynamics of a vehicle is nonlinear, but typically it is approximated by a linear model, therefore modeling uncertainty is introduced. Moreover, the nominal model as well as the modeling uncertainty varies with road conditions, vehicle properties such as mass and tire properties, which might not be known exactly a priori. The objective of a lane keeping controller is to keep the vehicle inside the lane boundary, therefore an mRCI is useful to bound the lateral deviation.

The remainder of the paper is organized as follows. We first present the system identification framework that identifies the set of admissible models for systems with uncertainty in Section II and III. Then a robust LP algorithm that computes an approximation of an mRCI by selecting a model from the set of admissible models is presented in Section IV. The whole process of approximating a mRCI from data is demonstrated on a lane keeping problem in Section V and finally we conclude in Section VI.

Nomenclature: The notation in this paper is fairly standard. $\mathbb{R}$ is the set of real number, $\mathbb{R}^n$ is the n dimensional Euclidean space, $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n_{\geq 0}$ are the open (and closed) positive orthant of $\mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, the inequality $\leq$ is defined element-wise: $x \leq y \Leftrightarrow y - x \in \mathbb{R}^n_{\geq 0}$. $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ represent the set of integers and nonnegative integers, respectively. $\mathbb{Z}_{1:T}$ denotes the sequence $1, 2, ..., T$ of natural numbers. For a matrix A, A_i denotes its i-th row, A^j denotes its j-th column, and A_{ij} denotes the entry on the i-th row, j-th column. $x(t_1 : t_2)$ denotes a sequence of vectors, indexed by time, starting from $t_1 \in \mathbb{Z}$ and ending at $t_2 \in \mathbb{Z}$. For simplicity, we use $\mathcal{P}(P, q)$ to denote the polyhedron $\{x \mid Px \leq q\}$.

II. LINEAR PARAMETRIZATION WITH UNCERTAINTY

We consider discrete-time linear models with uncertainty:

$$x^+ = \hat{A}x + \hat{B}u + \hat{E}d + \tilde{A}x + \tilde{B}u + \tilde{E}d + e, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state of the system, with x^+ being the state at the next sampling time, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input and $d \in \mathcal{D} \subseteq \mathbb{R}^l$ is the exogenous measured disturbance, $\hat{A}, \hat{B}, \hat{E}$ are the nominal model matrices, $\tilde{A}, \tilde{B}, \tilde{E}$ are the matrices for the multiplicative uncertainty, and $e \in \mathbb{R}^n$ is the additive uncertainty. In fact, this is simply n uncertain linear parametrizations, one per state dimension, stacked together. Taking the i-th dimension as an example and defining $z_i = x_i^+$, the linear parametrization appears as:

$$z_i = \varphi_i^\intercal \hat{\theta}_i + \varphi_i^\intercal \tilde{\theta}_i + e_i, \quad (2)$$

where

$$\varphi_i = [x^\intercal, u^\intercal, d^\intercal]^\intercal \in \mathbb{R}^{n+m+l},$$
$$\hat{\theta}_i = [\hat{A}_i, \hat{B}_i, \hat{E}_i]^\intercal \in \mathbb{R}^{n+m+l},$$
$$\tilde{\theta}_i = [\tilde{A}_i, \tilde{B}_i, \tilde{E}_i]^\intercal \in \mathbb{R}^{n+m+l}.$$

We assume that the uncertainty is bounded in hyper-boxes:

$$|\tilde{\theta}_i| \leq \Omega_M^i, \quad |e_i| \leq \Omega_A^i, \quad (3)$$

where $|\cdot|$ denote the entry-wise absolute value. However, unlike most set membership approaches which assume a fixed bound on the uncertainty, we aim to identify the bounds $\Omega_M^i \in \mathbb{R}^{n+m+l}_{\geq 0}$ and $\Omega_A^i \in \mathbb{R}_{\geq 0}$ as part of the identification process. It is assumed that u and d are bounded in polytopes and the bound is known a priori:

$$\begin{aligned} d &\in \mathcal{D} \doteq \mathcal{P}(G, g) \\ u &\in \mathcal{U} \doteq \mathcal{P}(R, r). \end{aligned} \quad (4)$$

This assumption is satisfied by many engineering problems since the bound for u and d are often determined by system specifications or physics (cf. the lane keeping example).

An uncertain linear model is determined by the value of $[\hat{\theta}, \Omega_M, \Omega_A]$, which contains the information of both the nominal model and the uncertainty characterization.

In many cases the model to be identified has additional structure. For example, due to underlying physics, some of the model parameters may be known to be zero or some entries of the system matrices may be linearly dependent. In order to incorporate such structure, we assume that the model parameters are affinely parameterized by a hyperparameter π. That is, we have, $\hat{\theta} = \hat{\mathbf{\Theta}}(\pi)$, $\Omega_M = \mathbf{\Omega_M}(\pi)$, and $\Omega_A = \mathbf{\Omega_A}(\pi)$, where we use **bold font** to denote the known affine mapping from π to the model parameters, e.g., $\hat{A} = \hat{\mathbf{A}}(\pi)$. Since these mappings are affine, the overall parametrization of the model is also affine in π. Moreover, when no structural information is available, these mappings can be taken to be the trivial ones.

III. ADMISSIBLE MODEL FOR MEASUREMENTS

Given a sequence of measurements $x(1 : T+1)$, the output and regressor for time step t is defined as

$$z_i(t) = x_i(t+1), \ \varphi_i(t) = [x(t)^\intercal, u(t)^\intercal, d(t)^\intercal]^\intercal. \quad (5)$$

A model $[\hat{\theta}, \Omega_M, \Omega_A]$ is called admissible if for $t = 1, 2, ..., T$,

$$\begin{aligned} &\exists e(t), \tilde{\theta}(t), \ s.t. \ |e(t)| \leq \Omega_A, |\tilde{\theta}(t)| \leq \Omega_M, \\ &z(t) = \left(\hat{\theta} + \tilde{\theta}(t)\right)^\intercal \varphi(t) + e(t). \end{aligned} \quad (6)$$

In fact, for each time step, the measurement induces a linear constraint on the model parameters:

$$\left| z(t) - \varphi(t)^\intercal \hat{\mathbf{\Theta}}(\pi) \right| \leq |\varphi(t)|^\intercal \mathbf{\Omega_M}(\pi) + \mathbf{\Omega_A}(\pi). \quad (7)$$

Since the π-parametrization of $[\hat{\theta}, \Omega_M, \Omega_A]$ is affine, the above condition is a linear inequality constraint on π. The set of admissible models is then parameterized by π constrained inside a polyhedron, with the following representation:

$$\Sigma = \left\{ \begin{matrix} \pi \mid \forall t \in \mathbb{Z}_{1:T}, |z(t) - \varphi(t)^\intercal \hat{\mathbf{\Theta}}(\pi)| \\ \leq |\varphi(t)|^\intercal \mathbf{\Omega_M}(\pi) + \mathbf{\Omega_A}(\pi) \end{matrix} \right\} \quad (8)$$

Fig. 1 shows the comparison between the proposed uncertain model structure and linear regression. The dots represent the measurement data and the center purple line represents the nominal model. In addition to the nominal model, the uncertain model on the right introduces the bound on additive

uncertainty, represented as the parallel red dashed lines, and the bound on multiplicative uncertainty, represented as the green radiating dashed lines. With additive and multiplicative uncertainty, the model on the right covers all data points and therefore is an admissible model.

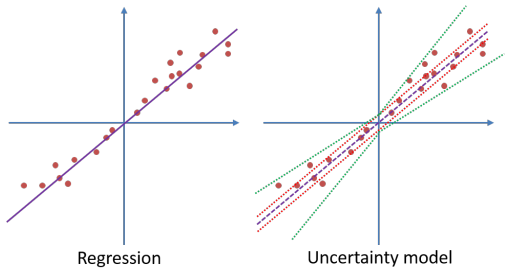

Fig. 1: Comparison of regression and uncertainty models

If Σ is nonempty, then all models in Σ explain the measurement data. In fact, under mild assumptions, Σ is guaranteed to be nonempty. In addition, there is a trade-off between different types of uncertainty, as shown in Fig. 2. When the additive uncertainty bound is large, the bound on multiplicative uncertainty can be smaller, and vice versa. This is the direct result of (7).

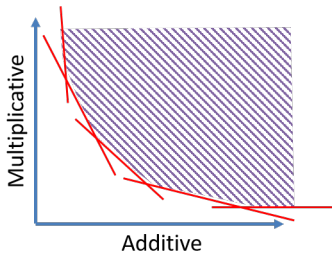

Fig. 2: Tradeoff between uncertainty bounds

The set of admissible models Σ gives the domain from which the model should be selected. Among the admissible models, which one is "the best" depends on how this model is to be used. If the goal is to find a model with the least squared error, then the least squares regression gives the best model, with corresponding uncertainty characterization. But since our goal is to compute an mRCI, the incorporation of model selection process into the mRCI computation, as shown in the next section, may result in more desirable invariant set.

IV. Robust LP for mRCI

A. One-step propagation

In this section, a robust LP algorithm that simultaneously picks an optimal model and approximates an mRCI is proposed. We first present the definition of an RCI.

Definition 1. *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called robust control invariant for the system described by Eqs.* (1), (3), (4) *if there exists a control strategy $\mu : \mathbb{R}^n \times \mathcal{D} \to \mathcal{U}$ such that for all $d \in \mathcal{D}$ and for all $x \in \mathcal{S}$, we have $x^+ \in \mathcal{S}$ with $u = \mu(x, d)$ under all possible uncertainty given by* (3).

As mentioned in Section I, the uniqueness of a minimal control invariant set in the set inclusion sense is not guaranteed. Though one can define an mRCI with respect to the set inclusion partial order as follows:

Definition 2. *A robust control invariant set $\mathcal{S}$ is a minimal robust control invariant set (mRCI) if there does not exist an $\mathcal{S}' \subsetneq \mathcal{S}, s.t.\ \mathcal{S}'$ is a robust control invariant set.*

However, even with this definition, finding an mRCI is non-trivial. Typically, one tries to find an (approximate) mRCI by minimizing a certain measure of size, such as volume [4], [17]. We propose a method that computes a polytopic RCI that minimizes a linear objective function.

We draw inspiration from [26], where the author proposed a one-step LP approach to compute a robust invariant set for an autonomous system with only additive uncertainty. The key idea is to fix the orientation of the separating hyperplanes defining a polytopic invariant set. We borrow the idea of fixing the hyperplane orientation, and propose an iterative approach that can deal with systems with control inputs, while allowing additive and multiplicative uncertainties.

The method starts by choosing a set of L hyperplanes with fixed orientation P_i and varying offset q_i, $i = 1, ..., L$. Let $P = [P_1^\intercal, P_2^\intercal, ..., P_L^\intercal]^\intercal$, $q = [q_1, ..., q_L]^\intercal$. Without loss of generality, assume $\|P_i\| = 1$. If $\mathcal{S} = \mathcal{P}(P, q)$ has nonempty interior, then P_i is the normalized normal vector pointing outwards the corresponding separating hyperplane.

Assumption 1. *The hyperplanes are chosen such that $\{x \mid Px \leq \mathbf{1}_L\}$ is a compact set, where $\mathbf{1}_L \in \mathbb{R}^L$ denotes the column vector consisting of all ones.*

Given a polytope $\mathcal{S} = \mathcal{P}(P, q)$, we consider the following one-step propagation which searches for $\mathcal{S}^+ = \mathcal{P}(P, q^+)$ that contains all possible x^+:

$$\begin{aligned} &\min_{q^+} \ c^\intercal q^+ \ s.t.\ \forall x \in \mathcal{P}(P, q), \forall d \in \mathcal{D}, \\ &\exists u \in \mathcal{U}, s.t. \forall |e| \leq \Omega_A, \forall |\tilde{\theta}| \leq \Omega_M, \\ &x^+ \in \mathcal{P}(P, q^+). \end{aligned} \tag{9}$$

The set $S^+ = \mathcal{P}(P, q^+)$ satisfies the following condition: for any $x \in \mathcal{S}$, $d \in \mathcal{D}$, there exists $u \in \mathcal{U}$ such that all possible x^+ under u is contained in $\mathcal{S}^+$. It is clear that if $\mathcal{S}^+ \subseteq \mathcal{S}$, $\mathcal{S}$ is control invariant. By minimizing $c^\intercal q^+$, we seek an $\mathcal{S}^+$ that is as small as possible.

Next we discuss a few simplifications so that the one-step propagation is solvable by convex optimization. First, as mentioned at the beginning of this section, there is no minimum RCI that is a subset of every RCI. Therefore, the RCI obtained depends on a specific control strategy. For the linear discrete-time system discussed in this paper, we impose the following control structure:

$$u = K_{ff}^\intercal d + K_{fb}^\intercal x, \tag{10}$$

where K_{ff} and K_{fb} are constant matrices, representing the feedforward and feedback gain respectively.

Second, we restrict $\tilde{B} = 0$, and fix $\hat{B}$ to get rid of the cross terms between K_{ff}, K_{fb} and $\tilde{B}$, $\hat{B}$. Such simplification is

possible when there is no actuation uncertainty or when the actuation uncertainty is lumped into other uncertainty terms.

Third, the one-step propagation should be robust against the model uncertainty, i.e., $\mathcal{S}^+$ should contain all possible x^+ under the uncertain model. This is enforced by considering the worst case uncertainty, captured by the "for all" quantifiers for e and $\tilde{\theta}$ in (9). Since the uncertainty bounds are assumed to be hyper-boxes, these quantifiers can be eliminated by observing that:

$$\begin{aligned}
&\max_{|\tilde{A}|\leq\Omega_{\tilde{A}}} P_i\tilde{A}x = \max_{|\tilde{A}|\leq\Omega_{\tilde{A}}} Tr\left(|xP_i|\,|\tilde{A}|\right) = |P_i\Omega_{\tilde{A}}|\,|x| \\
&\max_{|\tilde{E}|\leq\Omega_{\tilde{E}}} P_i\tilde{E}d = \max_{|\tilde{E}|\leq\Omega_{\tilde{E}}} Tr\left(|dP_i|\,|\tilde{E}|\right) = |P_i\Omega_{\tilde{E}}|\,|d| \\
&\max_{|e|\leq\Omega_A} P_i e = |P_i|\,\Omega_A,
\end{aligned} \tag{11}$$

where $\Omega_{\tilde{A}} \in \mathbb{R}^{n\times n}$ and $\Omega_{\tilde{E}} \in \mathbb{R}^{n\times l}$ are the bounds on $|\tilde{A}|$ and $|\tilde{E}|$ induced from Ω_M respectively. The absolute value constraints can be converted to linear constraints using standard LP techniques but, for the sake of keeping the notation simple, we will keep the absolute value form for the remainder of the paper.

With these simplifications, the one-step propagation problem takes the following robust optimization form:

$$\begin{aligned}
&\min_{K_{ff},K_{fb},\pi,q^+} \; c^\intercal q^+ \; s.t.\; \pi\in\Sigma, \\
&\forall x\in\mathcal{P}(P,q), \forall d\in\mathcal{D}, \\
&P\left(\hat{\mathbf{A}}(\pi)\,x + \hat{B}\left(K_{ff}^\intercal d + K_{fb}^\intercal x\right) + \hat{\mathbf{E}}(\pi)\,d\right) \\
&+ \left|P\mathbf{\Omega}_{\tilde{\mathbf{A}}}(\pi)\right|\,|x| + \left|P\mathbf{\Omega}_{\tilde{\mathbf{E}}}(\pi)\right|\,|d| + |P|\,\mathbf{\Omega}_{\mathbf{A}}(\pi) \leq q^+ \\
&K_{ff}^\intercal d + K_{fb}^\intercal x \in \mathcal{U}.
\end{aligned} \tag{12}$$

The above optimization simultaneously searches for 1) an admissible model, 2) a linear controller that satisfies the input bound constraint, and 3) $\mathcal{S}^+ = \mathcal{P}(P, q^+)$, the polytopic set containing all possible x^+ under the selected model and controller. It is a robust LP in the sense that the constraints have to be satisfied for all $x \in \mathcal{S}$ and all $d \in \mathcal{D}$. The next lemma, proof of which follows from [2], shows an equivalent LP that is obtained via duality.

Lemma 1. *Consider the following robust LP problem:*

$$\begin{aligned}
&\min_{\alpha}\; c^\intercal\alpha \; s.t. \\
&\forall\beta\in\mathcal{P}(F,f), \\
&H_1^i\beta + \alpha^\intercal H_2^i\beta + H_3^i\alpha \leq h^i, i=1,...,M
\end{aligned} \tag{13}$$

where α is the decision variable, β is the uncertain variable, $\mathcal{P}(F,f)$ is the bound for uncertainty, and H_1^i, H_2^i, H_2^i, and h^i are constant matrices and vectors of appropriate dimensions. This problem is equivalent to a standard LP of the form:

$$\begin{aligned}
&\min_{\alpha,\lambda}\; c^\intercal\alpha \; s.t. \\
&H_3^i\alpha + \left(\lambda^i\right)^\intercal f \leq h^i \\
&H_1^i + \alpha^\intercal H_2^i = \left(\lambda^i\right)^\intercal F \\
&\lambda^i \geq 0, i=1,...M,
\end{aligned} \tag{14}$$

where the uncertain variable β is eliminated and the dual variable λ is introduced.

Observe that the one-step propagation in (12) is in the robust LP form of (13), by taking $\alpha = [K_{ff}, K_{fb}, q^+, \pi]$ and $\beta = [x, d]$. Therefore, by Lemma 1, it can be transformed to a standard LP and can be solved efficiently.

B. Iterative algorithm

As mentioned previously, if $\mathcal{S}^+ \subseteq \mathcal{S}$, $\mathcal{S}$ is a robust control invariant set. With the one-step propagation solvable as an LP, it is possible to devise two different iterative methods that solve for an invariant set, the outside-in method and the inside-out method [15], [17], [19]. In our implementation, the inside-out algorithm is used to solve for an RCI, and the outside-in algorithm is used to shrink a known RCI to a smaller size.

1) Inside-out algorithm: The inside-out algorithm starts with a small initial $\mathcal{S}$, iteratively solves for $\mathcal{S}^+$ with the one-step propagation and replaces $\mathcal{S}$ with $\mathcal{S}^+$, until $\mathcal{S}^+ \subseteq \mathcal{S}$ is satisfied.

Algorithm 1 Inside-out algorithm for mRCI

procedure RCI-IO(Σ, P, q^0, $\mathcal{D}$, $\mathcal{U}$, ϵ)
  $q \leftarrow q^0$
  do
    Find $\left[q^+, \pi, K_{ff}, K_{fb}\right]$ s.t. $\pi\in\Sigma$,
    $\forall x\in\mathcal{P}(P,q), \forall d\in\mathcal{D}, K_{ff}d + K_{fb}x \in \mathcal{U}$,
    $x^+ \in \mathcal{P}(P, q^+ - \epsilon\mathbf{1}_L)$
    $q \leftarrow q^+$
  while $q^+ \leq q + \epsilon\mathbf{1}_L$
  return $[q, \pi, K_{ff}, K_{fb}]$
end procedure

The algorithm is shown in **Algorithm** 1, where $0 < \epsilon << 1$ is a small constant that helps accelerate the convergence.

Proposition 1. *If **Algorithm** 1 terminates, $\mathcal{S} = \mathcal{P}(P,q)$ is an RCI.*

Proof. By construction, $\mathcal{S}^+ = \mathcal{P}(P, q^+ - \epsilon\mathbf{1}_L)$ contains all possible x^+ with $x \in \mathcal{S}$, $d \in \mathcal{D}$, and since $q^+ \leq q + \epsilon\mathbf{1}_L$, $\mathcal{S}^+ \subseteq \mathcal{S}$, therefore $\mathcal{S}$ is an RCI. ■

Remark 1. *With $\epsilon > 0$, the algorithm searches for an $\mathcal{S}^+$ slightly bigger than that in* (12)*, so that we can allow ϵ tolerance for the termination condition $\mathcal{S}^+ \subseteq \mathcal{S}$, which accelerates the convergence of the algorithm.*

2) Outside-in algorithm: On the other hand, if an initial RCI is known for some admissible model, the outside-in algorithm can further shrink the initial RCI with a convergence guarantee. The algorithm iteratively solves for $\mathcal{S}^+ \subseteq \mathcal{S}$, and replaces $\mathcal{S}$ with $\mathcal{S}^+$ until $\mathcal{S}$ can no longer be shrunk.

Algorithm 2 shows the outside-in algorithm, which is very similar to the inside-out algorithm except for two differences. First, the one-step propagation has an additional constraint $q^+ \leq q$, which ensures that $S^+ \subseteq S$. Second, the termination condition is on the norm of difference between q and q^+.

Although the Outside-in algorithm requires that an initial RCI is known, it is guaranteed to converge.

Algorithm 2 Outside-in algorithm for mRCI

```
procedure RCI-OI(Σ, P, q^0, D, U, ε)
    q ← q^0
    do
        Find [q^+, π, K_ff, K_fb] s.t. π ∈ Σ, q^+ ≤ q
        ∀x ∈ P(P, q), ∀d ∈ D, K_ff d + K_fb x ∈ U,
        x^+ ∈ P(P, q^+)
        q ← q^+
    while ‖q^+ − q‖ ≥ ε
    return [q^+, π, K_ff, K_fb]
end procedure
```

Theorem 1. *If for a certain admissible model π^0, a polytopic RCI $\mathcal{P}(P, q^0)$ is known, then the outside-in algorithm is guaranteed to converge.*

Before proving the above theorem, we present some required lemmas:

Lemma 2. *For a non-empty compact polytope $\mathcal{P}(P, q)$, suppose one moves the i-th hyperplane from q_i to $\bar{q}_i$, and leaves the rest unchanged, resulting in the following polytope $\mathcal{P}(P, q')$, where $q' = [q_1, ...\bar{q}_i, q_{i+1}, ...q_L]^\intercal$. Then there exists a constant c_i such that if $\bar{q}_i < c_i$, $\mathcal{P}(P, q') = \emptyset$.*

Proof. Since $\mathcal{P}(P, q)$ is compact, $f(x) = P_i x$ is a continuous function and it always achieves its minimum value on a compact set. Let

$$c_i = \min_{x \in \mathcal{P}(P,q)} P_i x. \tag{15}$$

Obviously $c_i \leq q_i$. Note that if $\bar{q}_i \leq q_i$, $\mathcal{P}(P, q') = \mathcal{P}(P, q) \cap \{x \mid P_i x \leq \bar{q}_i\}$. Therefore when $\bar{q}_i < c_i$, $\mathcal{P}(P, q') = \emptyset$. ■

Lemma 3. *Let $\mathcal{P}(P, q)$ be a non-empty compact polytope. Let c_i be defined as in* (15)*. For any $q' \leq q$ and $c = [c_1, c_2, \ldots, c_L]^\intercal$, if the set $\mathcal{P}(P, q')$ is nonempty, then $q' \geq c$.*

Proof. The proof follows from **Lemma** 2 and the fact that $c_i' = \min_{x \in \mathcal{P}(P,q')} P_i x \geq c_i$. ■

Now let's prove **Theorem** 1.

Proof of Theorem 1. First we show that the one-step propagation is always feasible, and every q^+ during the iteration leads to an RCI. For clarity, denote the offset q found in the i-th iteration as q^i. We show this by induction. For the first iteration, by assumption, $\mathcal{P}(P, q^0)$ is an RCI, so $q^1 = q^0$ is a feasible solution for the one-step propagation in the first iteration. Since it is an LP, thus convex, a feasible solution can be reliably found whenever one exists. Assume at the n-th iteration, $\mathcal{P}(P, q^{n-1})$ is an RCI. Then

$$\begin{aligned}
&\exists K_{ff}, K_{fb}, \pi, q^n \ s.t. \ \pi \in \Sigma, \\
&\forall x \in \mathcal{P}(P, q^{n-1}), \forall d \in \mathcal{D}, |\tilde{\theta}| \leq \mathbf{\Omega_M}(\pi), |e| \leq \mathbf{\Omega_A}(\pi) \\
&P \begin{pmatrix} \hat{\mathbf{A}}(\pi) x + \hat{B}\left(K_{ff}^\intercal d + K_{fb}^\intercal x\right) \\ +\hat{\mathbf{E}}(\pi) d + \tilde{E} d + \tilde{A} x + e \end{pmatrix} \leq q^n \\
&K_{ff}^\intercal d + K_{fb}^\intercal x \in \mathcal{U}, q^n \leq q^{n-1}.
\end{aligned} \tag{16}$$

This implies that the one-step propagation is feasible at the n-th iteration. Consider the one-step propagation in the $n+1$-th iteration. The only difference between the robust LP in the n-th and $n+1$-th iteration is that the uncertainty set of x changes from $\mathcal{P}(P, q^{n-1})$ to $\mathcal{P}(P, q^n)$. Since $\mathcal{P}(P, q^n) \subseteq \mathcal{P}(P, q^{n-1})$, the uncertainty set for the $n+1$-th iteration is a subset of that in the n-th iteration, therefore q^n is still a solution to the one-step propagation in the $n+1$-th iteration, and the one-step propagation is still feasible. By induction, the one-step propagation in the outside-in algorithm is always feasible, and for all q^n, $\mathcal{P}(P, q^n)$ is always an RCI.

Next, let $c_i = \min_{x \in \mathcal{P}(P,q^0)} P_i x$. Since for all $n \in \mathbb{Z}_{\geq 0}, q^n \leq q^0$, by **Assumption** 1, $\mathcal{P}(P, q^0)$ is compact, then by **Lemma** 3, for all $n \in \mathbb{Z}_{\geq 0}, q^n \geq c$. Since q is monotonically decreasing, and lower bounded by c, by bounded convergence theorem, $\{q^n\}$ eventually converges. ■

TABLE I: Comparison of iterative algorithms

	Initialization	Convergence
Inside-out	Arbitrary	Not guaranteed
Outside-in	Need an RCI to start with	Guaranteed

In conclusion, the inside-out algorithm solves for an RCI, and the outside-in algorithm shrinks the size of a known RCI, the comparison is shown in TABLE I

V. APPLICATION ON LANE KEEPING OF A GROUND VEHICLE

In this section, we present an application of the proposed method on the vehicle lane keeping problem. The measurement comes from simulation data from Carsim, a commercial software highly acknowledged by the auto industry that runs simulations with high-fidelity physics based vehicle models. The model we use has 113 states. A linear model with 4 states is used to approximate this detailed model.

A. Model structure

The model to be identified for the lateral dynamics of ground vehicle is called lateral-yaw model, or bicycle model, which has 4 states:

$$x = [y, v_y, \psi, r]^\intercal, \tag{17}$$

where y is the lateral displacement from the lane center, v_y is the sideslip velocity, ψ is the heading angle with respect to the lane direction and r is the yaw rate. The model is linear, yet the coefficient may change with forward speed v_x, road

condition and vehicle condition such as mass, inertia and tire properties. A linear discrete model with uncertainty is used to describe the dynamics. The input is the steering angle on the front axle δ_f; the measured disturbance is the road curvature r_d.

In order to reduce the complexity of the uncertainty characterization, certain structure is imposed on the model based on the properties of the dynamics. The dynamics for $\dot{y}$ and $\dot{\psi}$ are essentially integrators, therefore no multiplicative uncertainty is put on these two dimensions. Similarly, since $\dot{v}_y$ and $\dot{r}$ do not depend on y and ψ, no multiplicative uncertainty is put on the corresponding entries of $\tilde{A}$. The influence of r_d follows a simple kinetic equation, therefore $\tilde{E}$ is set to 0. Thus, the model is in the following form:

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \\ \dot{\psi} \\ \dot{r} \end{bmatrix} = \hat{\mathbf{A}}(\pi) \begin{bmatrix} y \\ v_y \\ \psi \\ r \end{bmatrix} + \hat{B}\delta_f + \hat{\mathbf{E}}(\pi) r_d + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{42} & 0 & \tilde{A}_{44} \end{bmatrix} \begin{bmatrix} y \\ v_y \\ \psi \\ r \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}. \tag{18}$$

The bounds on uncertainty are

$$\left[|\tilde{A}_{22}| \quad |\tilde{A}_{24}| \quad |\tilde{A}_{42}| \quad |\tilde{A}_{44}|\right]^\intercal \leq \mathbf{\Omega_M}(\pi), |e| \leq \mathbf{\Omega_A}(\pi). \tag{19}$$

If we were to fully parameterize the nominal model, 20 independent variables would be needed, which may not be necessary since some entries of $\hat{A}$ and $\hat{E}$ do not vary much. Instead, we assume that $\hat{A}$ and $\hat{E}$ are linearly parameterized by some bases:

$$\hat{A} = \sum_{i=1}^{n_1} \pi_1^i \bar{A}_i, \ \hat{E} = \sum_{i=n_1+1}^{n_1+n_2} \pi_1^i \bar{E}_i, \tag{20}$$

where $\{\bar{A}_i\}$ and $\{\bar{E}_i\}$ are the bases for $\hat{A}$, $\hat{E}$, with cardinality n_1 and n_2 respectively.

Remark 2. *We run multiple simulations under different scenarios to obtain multiple nominal models with least square regression, and then principle component analysis is applied on the models obtained to extract bases for $\hat{A}$ and $\hat{E}$.*

The overall π parametrization appears as

$$\begin{aligned} &\hat{\mathbf{A}}(\pi) = \sum_{i=1}^{n_1} \pi_1^i \bar{A}_i, \ \hat{\mathbf{E}}(\pi) = \sum_{i=n_1+1}^{n_1+n_2} \pi_1^i \bar{E}_i, \\ &\mathbf{\Omega_M}(\pi) = \left[\pi_2^1, \pi_2^2, \pi_2^3, \pi_2^4\right]^\intercal, \mathbf{\Omega_A}(\pi) = \left[\pi_3^1, \pi_3^2, \pi_3^3, \pi_3^4\right]^\intercal, \\ &\pi = \left[\pi_1^\intercal, \pi_2^\intercal, \pi_3^\intercal\right]^\intercal. \end{aligned} \tag{21}$$

The parametrization is indeed affine.

B. Results

We start by collecting data from Carsim, in which the vehicle is equipped with a simple LK controller with some input noise and follows a prescribed route. Then the data is used to formulate the set of admissible models Σ following the procedure in (7). Σ is a polyhedron and every π inside Σ corresponds to an admissible model. We pick $n_1 = 6, n_2 = 4$, so $\pi \in \mathbb{R}^{18}$. Fig. 3 shows the convergence of the inside-out algorithm and outside-in algorithm. In fact, the inside-out and outside-in algorithms converge to the same RCI in this case, but there is no guarantee that this will happen every time.

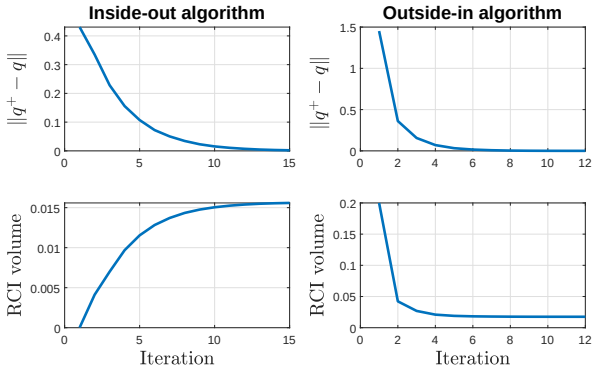

Fig. 3: Convergence of the iterative algorithms.

Fig. 4 shows the computed RCI. For benchmark, we use the least square regression result as the nominal model, and let the mRCI algorithm pick the optimal bound for multiplicative and additive uncertainty. In this case, π is simply $[\pi_2^\intercal, \pi_3^\intercal]^\intercal$ in (21), and $\pi \in \mathbb{R}^8$. The RCI with the least square model is shown in blue, and the RCI under optimal nominal model is shown in red. The reduction in volume is more than 50%. Since the RCI is in $\mathbb{R}^4$, the plot shows the slices of the polytope by setting one dimension to zero.

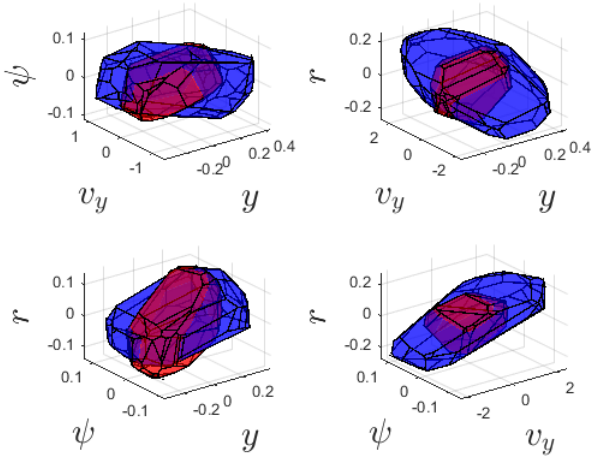

Fig. 4: Slices of the mRCIs obtained with the model identified via least squares (blue) and with the proposed algorithm (red).

Notice that the nominal model picked by the mRCI algorithm is only optimal for the purpose of computing the mRCI. In fact, under the least square model, the uncertainty

bound found by the mRCI algorithm is as follows:

$$\begin{aligned}\Omega_M &= \begin{bmatrix}0 & 0.0307 & 0 & 0.0022\end{bmatrix}^\intercal, \\ \Omega_A &= \begin{bmatrix}0.0046 & 0.0283 & 0.0088 & 0.0116\end{bmatrix}^\intercal.\end{aligned} \tag{22}$$

For the optimal nominal model, the uncertainty bounds become

$$\begin{aligned}\Omega_M &= \begin{bmatrix}0 & 0.129 & 0 & 0.0112\end{bmatrix}^\intercal, \\ \Omega_A &= \begin{bmatrix}0.0327 & 0.1643 & 0.0088 & 0.0236\end{bmatrix}^\intercal,\end{aligned} \tag{23}$$

which are significantly larger than the previous case. This shows that the least square model indeed fits the measurements better, but results in a larger mRCI. This again shows the necessity of performing model selection and mRCI computation simultaneously. Such co-optimization leads to a model that better suits the mRCI computation.

VI. CONCLUSION

This paper presents a novel data-driven algorithm to approximately compute a minimal robust invariant set by simultaneously selecting an admissible model and minimizing the size of the RCI. The algorithm has two steps: first the set of all admissible models with uncertainty characterization is identified from the measurement data, then a robust LP is formulated to iteratively search for an mRCI. The robust LP based algorithm is able to simultaneously pick an optimal model, finding a good tradeoff between the nominal model and different types of uncertainties and minimize the size of an mRCI. A vehicle lane keeping example is used to demonstrate the method, and the result shows more than 50% reduction in the volume of the RCI computed compared to the benchmark.

Although the current algorithm provides a means to co-optimize invariant set size, the selection of invariance-inducing controllers, and the selection of models among the models consistent with the experimentally observed data (i.e., the computed mRCI is unfalsifiable with the existing data), it does not necessarily provide invariance guarantees for unseen data in cases where the unseen data can reveal additional dynamics and can further shrink the admissible model set. Hence, another interesting direction is to extend the invariance guarantees to unseen data by incorporating a priori information on the dynamics as in [20].

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