SimCalc: Accelerating Students' Engagement with the Mathematics of Change

Jeremy Roschelle

SRI International

James J. Kaput, and Walter Stroup

University of Massachusetts, Dartmouth

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A central phenomenon of the twenty-first century will be change: economic, social, and technological change. Indeed, engaging students in analysis of change and variation is a central element of nearly every chapter in this book. Today, however, the mathematics of change and variation (MCV), despite its importance in understanding and controlling this ubiquitous phenomenon, is packed away in a course, Calculus, that sits at the end of a long series of prerequisites that filter out 90% of the population. This is especially true for students from economically poorer neighborhoods and families. And even the 10% who do have nominal access to MCV in calculus courses develop mostly symbol manipulation skill but little understanding (Tucker, 1990). The traditional curriculum thus excludes most children from the concepts of rate of change, accumulation, approximation, continuity, and limit (among others). These are the very concepts children most need not only to participate in the physical, social, and life sciences of the twenty-first century, but also to make informed decisions in their personal and political lives. Even though MCV concepts were at the heart of mathematics and science historically (Bochner, 1966), in education the opposite is more nearly true. Conventional curricula neglect, delay, or deny students' access to MCV.

The mission of our SimCalc project is to give ordinary children the opportunities, experiences, and resources they need to develop extraordinary understanding and skill with MCV. Using a combination of advanced technology and carefully reformulated curricula, we aim to democratize access to the mathematics of change. This chapter discusses the research findings and design principles guiding our approach, with specific attention to our first software product, "MathWorlds." MathWorlds provides dynamic, direct manipulation graphs, piecewise definable functions, and animated cartoon worlds to engage elementary, middle, and high school students in qualitative and quantitative reasoning about the relationships among position, velocity, and acceleration in complex contexts. Formative evaluation experiments with diverse inner city students (the large majority of whom were in the lowest quartile of both academic achievement and socio-economic status) show that MathWorlds, coupled with an appropriate curriculum and teaching practice, can enable students to construct viable MCV concepts.

DEMOCRATIZING ACCESS TO KNOWLEDGE

While focusing on MathWorlds, we explore the more general issue of democratizing access to knowledge through advanced technology (Kaput, 1994). Along with the burgeoning international excitement about the Internet and World Wide Web (WWW) comes a temptation to substitute the problem of democratic access to knowledge and skill with the problem of network access -- a superficial problem of wires, bandwidth, and transport protocols (Hardin & Ziebarth, 1996). If such a substitution were valid, our mission would be fulfilled, for soon every elementary school student will have “access” to any number of university calculus courses through the WWW. Alas, neither conduits nor conduit metaphors capture the conditions for learning (Reddy, 1979); learning requires more than delivering encoded knowledge across a wire. Indeed, the encoding of calculus in the formal algebraic language of university calculus courses creates barriers to learning that true democratic access must overcome (Kaput & Roschelle, 1996).

Similarly, the availability of multimedia on every personal computer suggests another superficial role for technology conflated with educational power--the delivery of exciting sounds and movies to motivate students and capture their interest. An inadequate analysis of video games may contribute to the confusion (Norman, 1993). Arcade and computer games do captivate young boys’ attention at length, and as many have pointed out, it would be wonderful to translate such intense engagement to academic subject matter. However, as any game designer will explain, achieving a constant flow of quarters into a video kiosk is not a simple matter of choosing the right media. As with other cases of deep motivation, children play games because of the constant incremental growth of challenge, skills, and success--a condition called “optimal flow” (Csikszentmihalyi, 1990). Democratic access, thus, is not simply a matter of choosing the right media, but rather creating the conditions in which students experience growth in their capability to solve and understand ever more challenging problems.

LINES OF INNOVATION

Fortunately, decades of research sponsored by the National Science Foundation and others points beyond a superficial understanding of the conditions necessary for true democratic access. Real opportunity for diverse children to understand the difficult concepts of twenty-first century science requires more than availability of a conduit to encoded knowledge, and more than pandering to their jaded media preferences. In many ways, this not news. Indeed, the roots of SimCalc’s approach can be found in Dewey’s seminal analysis of the conditions for democratic access to education:

> Abandon the notion of subject-matter as something fixed and ready-made in itself, outside the child’s experience, cease thinking of it as also something hard and fast; see it as something fluent, embryonic, vital... it is continuous reconstruction, moving the child’s present

> experience out into that represented by the organized bodies of truths we call studies. (John Dewey, in McDermott, 1981, p. 427)

This quote captures two of the three lines of innovation underlying our SimCalc approach. First, democratic access requires deep inquiry into the reconstruction of subject matter. Rather than teaching a “calculus course” to middle school students, SimCalc is seeking to collaboratively define a “mathematics of change and variation” strand that is appropriate to children’s development from elementary school through university (Kaput, 1994). Second, democratic access begins from a deep understanding of the genetic seeds of understanding within children’s experience. Hence, with our colleagues in the mathematics education community, SimCalc seeks to ground the design of learning activities in a thorough understanding of the experiences, resources, and skills students can bring to this subject matter (Kaput, 1992). Although it is not captured in this quote, Dewey also spoke to a third line of innovation: the role of technology in mediating the process of inquiry. Inquiry allows incremental, continual growth of understanding from the child’s experience to the core subject matter concepts (Hickman, 1990). SimCalc is exploiting the capability of novel dynamic, graphical notations and representations (Kaput, 1992) to provide tools that engage students’ conceptual resources, enable mathematical conversation (and hence exploits students' linguistic resources), and support growth towards more sophisticated understandings (including more formal notations and forms of reasoning).

These three perspectives on innovation—subject matter reconstruction, grounding in children's conceptual and linguistic resources, and technological mediation—are recurrent themes of mathematics and science educational research throughout foundational writers such as Dewey, Piaget, and Vygotsky, as well as more recent educational research (Kaput, 1992; Roschelle & Jackiw, in press). In this chapter, we present SimCalc's work with MathWorlds as an example of a deep interpenetration of these three perspectives. In so doing, we illustrate the kind of deep inquiry in grounded classroom context which we believe will be necessary to provide democratic access to twenty-first century sciences.

Children's Conceptual Resources

Exploiting the students' existing knowledge and resources can lead to major, scaleable improvements in learning as Carpenter and colleagues have shown in their research on arithmetic learning (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Carpenter, Fennema, & Franke, 1996). Activities both technology mediated and non-technology mediated must engage the learners' best efforts, and technologies must draw upon their strongest cognitive capabilities.

Colleagues at TERC have studied children's conceptual resources, and their work has informed our design of MathWorlds. First, they found that children spontaneously engage in interval analysis to understand the behavior of a complex mathematical function. For example, students split a graph into intervals based on their understanding of the events that the graph represents (Nemirovsky, 1994; Monk & Nemirovsky, 1994). Operations include the construction of a graphical derivative, or an integral of a rate function, or a comparison of two functions. Students performed interval analysis without being explicitly taught, and readily constructed more flexible and richer schemes as they made sense of increasingly complex situations. Within this framework, students understood curved pieces of graphs as signifying behaviors of objects or properties of events, rather than as ordered pairs of points. Moreover, they readily constructed mathematical narratives that told a story of a graph over time (Nemirovsky, 1996). The density of students mathematical resources around interval analysis directly influenced our focus on piecewise linear functions in MathWorlds.

Second, research at TERC and elsewhere has uncovered the important roles of physical motion in understanding the meaning of mathematical representations (Nemirovsky et al., 1998; Nemirovsky & Noble, 1997; Noble et al. , 1995). In examining their own movement, students confront subtle relations among their kinesthetic sense of motion, interpretations of other objects' motions, and graphical, tabular, and even algebraic notations. Moreover, in a reversal and complement to Microcomputer-based labs (MBL), TERC developed the concept of Lines Become Motion (LBM) in which graphical representations on a computer control physical devices. Their studies of functions and derivatives in MBL (with body motion, air, and water flow) led us to realize the need for students to use symbols to control phenomena, not just to interpret them. These findings support the inclusion of MBL capabilities in MathWorlds, and also the use of manipulable graphs to control animated motion.

Reconstructing Subject Matter

As our introduction indicated, university calculus courses based in formal algebraic symbols *tacitly assume* rather than *actively develop* students' understanding of core concepts of change and variation that the formal symbolic calculus refers to. Thompson and Thompson's (1995) research, for example, shows that most university calculus students cannot correctly answer and explain simple qualitative problems, such as this (Figure 1): "Two cars leave from a bridge toll gate at the same time, with speeds as shown in these curves. Which car is ahead at the end of the duration of time shown on the graphs?" Note that the text of this problem can be re-phrased to be about technical, social, or economic change. For example, instead of asking about moving cars, we can invert the graphs, and ask: "Congress has two plans to balance the budget, bringing the rate of deficit spending to zero over seven years. Which plan is more desirable to the taxpayer and which is more desirable to the politicians in power?"

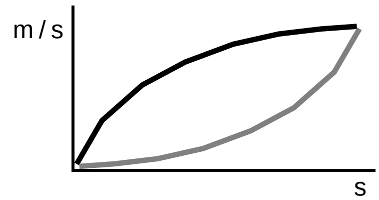

Figure 1. A qualitative integration problem, “Which car travels farther?”

Because formal symbol-based university courses fail to develop the kind of understanding that students (and adults) need, teaching simplified versions of those courses to younger students gains nothing. Further, conventional curricula for introducing rates to younger students have serious problems. Most commonly, children encounter rates in the context of simple linear functions. Research suggests that simplified mathematics problems embody insufficient complexity to enable students to develop adequate generalizations (Duckworth, 1991). More specifically with respect to the rates, and the usual simplification to the linear case, Stroup writes:

> The conjecture of this thesis is that in contrast with the richness and complexity of the earlier settings, the linear case is too simple. There is not enough 'there' in the linear case to 'hang one's understanding on'. More formally, the linear case is degenerate in a way that collapses the complexity.... A major recommendation of the thesis regarding learners' developing understanding of the interaction of how much and how fast ideas, is to start with complexity. Start with

> graphs of situations where the slope varies and only eventually deal with the linear case as a special 'collapsed' or degenerate case of this complexity. (Stroup, 1996, p. 223-224)

Thus SimCalc seeks to construct a curricular strand that is neither a simplified symbolic calculus course, nor a typical exploration of linear functions and the related notions of rate and ratio. As we will discuss shortly, this strand builds upon piecewise linear functions. Piecewise linear functions, like linear functions, are fairly easy for students to conceptualize, but also allow discussion of considerably more complex (and familiar) motions. Furthermore, we will argue that piecewise linear functions bridge nicely to more abstract and general MCV concepts.

Technological Mediation of Mathematics and Science Learning

With respect to MCV subject matter, we build upon extensive research on the importance of visualization in math and science reasoning (Gordin & Pea, 1995; Larkin & Simon, 1987; Reiber, 1995). The history of science demonstrates that visualization and imagery have played a key role in the development of scientific thinking (Miller, 1986), and recent sociology of science has further emphasized the importance of visual displays (e.g. Kozma, this volume; Latour 1986; Lynch, 1985) to the everyday work of scientists. In education, simulations and animations that display conceptual objects have proven particularly valuable in advancing children’s thinking (Horwitz & Barowy, 1994; Snir, Smith, &

Grosslight, 1993). On one hand, artificial animations have proven exceptionally effective in provoking genuine inquiry involving difficult concepts (White, 1993; White & Frederickson, this volume; diSessa, 1986). On the other hand, microcomputer-based labs (Thornton, 1987; Mokris & Tinker, 1987) and physical output devices (Monk & Nemirovsky, in preparation) complement simulations by connecting to real phenomena. An important research topic within SimCalc is exploring the complementary advantages of cybernetic (i.e., simulated) and physical data, when both are available.

Our own perspective on utilizing the power of visualization and simulation has been shaped by microgenetic studies that examine how these tools affect learning. Contrary to the popular adage that "seeing is believing," these studies show that learning is not as simple as seeing, even with the best constructed visual depictions. In particular, students do not always "register" the features of a visual depiction that an expert would see, may not interpret the features they do see as an expert would, and experience visualizations as problematic (Roschelle, 1991; Meira, 1991). Instead, the power of visualization and simulation arises from the role of computer displays as sites for interaction among students and with teachers (Roschelle, 1996; Roth, 1997). In particular, manipulable visualizations mediate students' construction of shared meanings (Moschovich, 1996; Roschelle, 1992; Laurillad, 1992). Thus, we advocate the design of visualizations and simulations specifically to leverage their role as media for collaborative inquiry (see Kaput, 1992; Roschelle, 1996):

- extending students' engagement with the aspects of concepts that they find problematic
- supporting shared focus of attention and part-whole analysis
- enabling gestural and physical communication to effectively supplement verbal communication
- engaging students in actively doing experiments, and providing meaningful feedback through an interface that is appropriately suggestive and constraining.

THE DESIGN OF MATHWORLDS

SimCalc's first software product, entitled "MathWorlds," enables students to use the context of motion to explore MCV concepts such as relations among position, velocity and acceleration, connections between variable rates and accumulation, mean values, and approximations all in the context of motion. MathWorlds provides a collection of software components including a set of animation worlds and a variety of graphs. Actors in the worlds (such as a clown, or a duck) move according to mathematical functions. Graphs display these mathematical functions and allow students to directly edit the functions. (MathWorlds can be downloaded from the SimCalc Web site, http://www.simcalc.umassd.edu/, along with other articles and materials.)

MathWorlds provides a very rich set of tools in a flexible environment, in accordances with our component software architecture (Roschelle & Kaput, 1996). For example, we support AppleGuide for providing help, as well as drag-and-drop configuration and scripting to allow teachers and others to customize the environment and build new activities (Roschelle, Kaput, & DeLaura, 1996). Teachers and students can also draw upon tools such as masking tape (which temporarily hides a portion of the screen), hiliting pens, and the ability to mark points and lines in graphs and in the world.

In this chapter, our goal is to elucidate the connection between design and democratic access to scientific and mathematical concepts. In line with this goal, we will not go into further depth about general pedagogical features of the interface, as these could apply to any kind of subject matter. Instead, we focus on aspects of our design that relate directly to MCV concept learning. We should also point out that MathWorlds will not be SimCalc's only software product, and is not intended to implement the full extent of our vision or mission. Nonetheless, MathWorlds does illustrate how we interpret the principles of building upon children's strengths, reconstructing subject matter, and providing technological mediation.

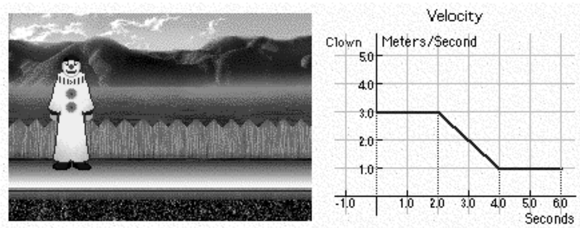

Figure 2. A piecewise linear graph of the velocity of a walking clown.

A central MathWorlds innovation is the use of piecewise linear functions to introduce and explore distance-rate-time concepts. (For brevity's sake, we will not distinguish between piecewise linear and the special case of piecewise constant functions, although the interface does.) In MathWorlds, the student or teacher can easily construct a function by concatenating segments of velocity or acceleration that are individually described as a linear rate of change over a specified duration. In a velocity graph, these functions appear as discrete steps (constant velocity) or rising or falling lines (constant acceleration). For example, Figure 2 shows a motion that begins fast, gradually slows down, and then continues at a slower rate. The first and last segments have constant velocity and the middle segment exhibits constant acceleration. In the corresponding Walking World, students can run the simulation and see the clown move according to this motion.

MathWorlds provides a range of other function types to complement piecewise linear functions. A “sampled” function type supports continuously varying positions, velocities, or accelerations. The varying data points can be entered directly with the mouse (by sketching the desired curve, ala Stroup, 1996), from Microcomputer-based Laboratory (MBL) data collection gear (Mokris & Tinker, 1987; Thornton, 1992), or by importing mathematical data from another software package such as FunctionProbe (Confrey, 1991). A linear or parabolic function can be constructed using a single piecewise linear segment (where, say, a velocity segment can have zero slope, yielding a linear position graph). In addition, MathWorlds can accept input of exponential and periodic functions.

In the sections below, we first discuss why our early design efforts converged on piecewise linear functions, and then how MathWorlds provides tools that enable students to learn fundamental MCV concepts by exploring piecewise linear functions. Before proceeding, we want to warn the reader that the following section is narrowly focused for rhetorical reasons. We are striving to illustrate how the design of MathWorlds integrates three design perspectives: children's resources, subject matter reconstruction, and technological mediation. However, due to space limitations, we cannot provide our full curricular vision, which reaches well beyond topics and skills addressable via piecewise linear functions. Thus, we restrict ourselves to an example of how design innovations can contribute to restructuring subject matter content.

Why Piecewise Linear Functions

Each of the three lines of innovation (children's resources, subject matter reconstruction, and technological mediation) informed our design perspective for piecewise linear functions in MathWorlds. In terms of children's resources, linear velocity segments provide a primitive object that can draw effectively upon pre-existing knowledge and skills. For example, middle school students can learn to predict position from a velocity graph by using two skills that they have already developed: counting and area multiplication. The velocity graph (see Figure 2) is drawn against a grid, which enables students to compute accumulated position by counting grid squares. (Note that the graph in this figure cuts across some squares. We will later present a student episode that illustrates how students readily extend their counting skills to deal with the linear velocity case by counting half squares.) Furthermore, students can integrate using the familiar area model of multiplication: height times width. Moreover, the TERC research cited earlier found that students spontaneously understand graphical representations of motions (and other phenomena) by performing interval analysis. Piecewise functions draw upon this natural inclination.

Our approach differs from the traditional algebraic approach in two ways: (a) in the way we respond to the need for computational tractability, and (b) the greater value we place upon experiencing phenomena (i.e., we put phenomena at the referential center of the learning environment). The starting point in the algebraic approach is governed by what is computationally simplest in that algebraic universe–the family of polynomial functions–which in turn leads to linear and quadratic functions as the inevitable starting point for computing derivatives and integrals symbolically. Hence, computational tractability drives the algebraic approach in the direction of simple, mathematical forms. In contrast, computational tractability in the graphically defined and manipulated universe pushes in a different direction, towards piecewise linearity that affords substantial semantic complexity without sacrificing computational tractability. This in turn allows richer relations with students' experience of motion, and a more appropriate conceptual foundation upon which students can build increasingly elaborate understandings of MVC ideas.

The second major difference–putting phenomena at the center of the enterprise–is partially served by the graphical approach to piecewise linear functions. Consider the problem of defining a function that represents the motion of an elevator that will pick up and drop off passengers in a building. Where as such a function is very difficult to formulate algebraically, it is relatively easy to directly drag hotspots on piecewise linear velocity segments to create an appropriate function. Similarly, defining motion-functions for two characters who are dancing would be extremely cumbersome to do algebraically, and be especially cumbersome for younger students in entirely unproductive ways. (The mathematically inclined reader might try to write out an algebraic description of the functions depicted in Figure 2, or Figure 3 below.)

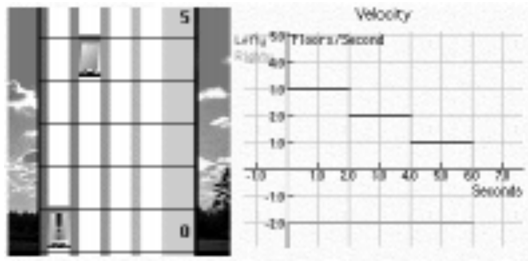

Figure 3. A decreasing staircase and a constant negative velocity.

Equally important to drawing upon children's resources is providing opportunities to make necessary distinctions in places where prior knowledge may be poorly differentiated. A classic example is the distinction between slowing down and moving downward (between "going down and slowing down"). The graph in Figure 3 shows how this distinction can be expressed in a MathWorlds graph that is connected to our Elevator World. In this world, the elevator moves up or down according to the specified (piecewise linear) function. The upper graph is a decreasing staircase. Many students will intuitively interpret this as “moving down” whereas a correct interpretation is “moving up with decreasing speed.” The lower graph shows a function that makes the elevator car go down with constant speed. Children have great difficulty distinguishing “how much” from “how fast” (Stroup, 1996).

MathWorlds uses piecewise linear functions as fundamental building blocks for understanding these and other core MCV concepts. Here, we briefly trace how piecewise linear velocity segments can support a conception of mean value, and how the notions of approximation and limit, can lead to a fairly classical treatment of integration as a calculation of area under a curve. (Indeed, the analysis of complex variation in terms of piecewise linear segments is a core practice in many engineering and scientific disciplines.)

In our exploratory curricula, we often introduce mean value first in a discrete case: finding a single (positive) constant velocity segment which will produce the same final position as a set of varying velocity segments occupying the same duration. In this case, students can easily compute the mean value by adding up the total area under the velocity segments and dividing by total time. In fact, students can use counting to show that the mean velocity conserves area under the graph. For example, in Figure 4, a “momma duck” swims at the mean value of the rather erratic motion of her baby duckling. Assuming they started at the same location, will they arrive at the beach at the same time? (Incidentally, the software’s name, “MathWorlds,” reflects the variety of animated “worlds” available to contextualize motion for different activities, age groups, and cultural situations.)

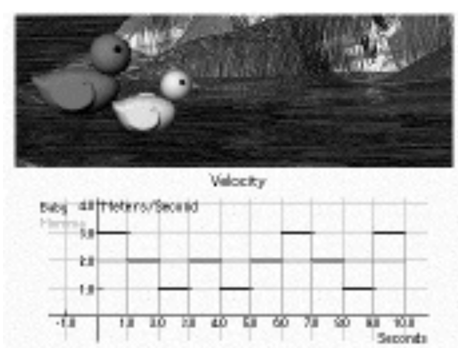

Figure 4: Momma duck swims at the mean value of baby duck

As mentioned earlier, MathWorlds also supports continuously varying motions. In particular, students can walk at a varying pace, and their body motion can be digitized (via MBL) and entered into a graph corresponding to an animated character. Students can then use a constant velocity graph to express the mean value of their motion, and compare the two motions in the animation.

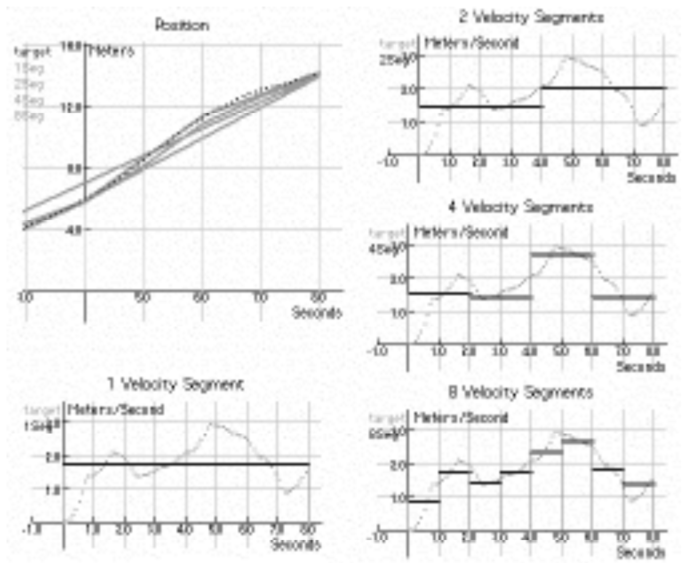

Figure 5: Approximating varying velocity with more and more constant velocity segments

Once students have gained an understanding of the mean value in the continuous variation case, they can use piecewise constant functions to find the mean value at a set of intermediate points. Figure 5 shows a progression in which the mean value is found once, twice, four times, and then eight times. The iterations suggest the process of finding the limit: using smaller time interval, and more segments to achieve a closer and closer approximation. Indeed, students can run the simulation to see their approximation improve; the characters will stay ever closer together throughout the motion as the number of segments increases. Moreover, the velocity graphs are each dynamically linked to the corresponding position graph, so the students can see the position graph achieve a better and better approximation. The calculus teachers note that the student here is creating a picture found in every calculus text, of rectangles under a varying curve. But in MathWorlds, the student builds this picture with deep prior understanding of the mean value theorem and the meaning of each constant velocity rectangle. (Incidentally, we have zoomed in on the position graph in Figure 5 so the approximations can be seen. The dark dotted line in the position graph is the varying motion, and it is very closely approximated by the graph with eight constant velocity segments.)

MathWorlds can also support the exploration of different algorithms for approximating the area under a curve. Students can build rectangular approximations that sample the varying quantity at the beginning, mid-point, or end of each segment. Moreover, they can use constant acceleration segments to explore a trapezoidal (rather than rectangular) approximation to area. This is in strong contrast with the notation and index-laden approaches that are required when one attempts such approximations for algebraically defined functions. Hence, MathWorlds readily builds from simple, comprehensible mathematical objects towards core concepts and reasoning processes in the MCV.

Finally, piecewise linear functions can be readily expressed in a format that supports technological mediation of learning. In particular, MathWorld's direct manipulation interface renders piecewise linear functions in a format that makes it easy for students to construct and operate on functions and to discuss their efforts with peers and teachers.

MathWorld provides direct "click-and-drag" editing of any segment. For example, a user can drag the top of a rectangular velocity segment higher to make a faster velocity. Or a user can drag the right edge of rectangular segment to the right to give the segment a longer duration. Students can also construct a function (or extend an existing one) by dragging additional segments into the graph. Thus, operations on the representation have clear and simple qualitative interpretations. And as students need more quantitative information, piecewise linear functions support a number of easy measurement operations, like counting and area multiplication.

Likewise, students and their teachers share a sufficient vocabulary to conversing about the meaning of piecewise linear graphs. It is easy to identify a segment of the graph (the "first rectangle") or its properties ("taller" or "wider"). Similarly, corresponding motions can easily be described in a narrative such as "it goes slow, then speeds up, and then continues at a fast speed." Piecewise linear functions thus provide a convenient conversational context for talking about motion without introducing an unfamiliar technical vocabulary.

To summarize, the design of MathWorlds to utilize graphically editable piecewise linear functions draws upon strength from each of the three perspectives. Children have ample resources to make sense of segmented graphs, and these graphs enable them to work on conceptually difficult and important distinctions. Piecewise linear functions lead naturally to core MCV concepts, such as mean value, approximation, and computing the integral via area under a curve. Further, piecewise linear functions can be realized in a technological interface that supports meaningful direct manipulation operations and sense-making conversations.

Tools for Learning

Choosing suitable conceptual primitives, such as piecewise linear functions, is a necessary but not sufficient basis for implementing learning technology. Thus MathWorlds contains a number of features and tools intended to contextualize and support the mathematical learning. Below we briefly describe some of the key features.

Like many modern learning technologies, MathWorlds supports dynamic linking among multiple representations of the same mathematical function (Kozma, this volume; Goldenberg, 1995). An activity document can contain any combination of position, velocity, and acceleration graphs. A mathematical object can be linked to a particular graph by dragging and dropping, and once linked, with all representations being updated simultaneously. Thus, a student can adjust a constant acceleration and watch simultaneous changes to the corresponding straight line in the velocity and parabola in the position and graph. Our approach to designing tools for multiple representations puts phenomena in the center. The multiple representations of MathWorlds always connect to simulated motion or real world motions, digitized via MBL hardware.

MathWorlds gets its name from the availability of different animated backgrounds and characters for contextualizing a motion activity (and versions of our software under development that present water flow, and other familiar phenomena that involve change over time). Each world supports different kinds of problems and challenges. For example, the elevator is used for vertical motion with a natural ordinality including both positive and negative numbers, whereas the walking characters move horizontally and invite complex motions of the sort that might occur in marching or dancing. A space world provides a UFO that can pick up rocks and drop them in a crusher, which is useful for setting up challenges that involve hitting targets in position and time. A water world provides a momma and baby ducks. The momma squawks if the babies get too far behind, which is useful for activities where the goal is to match a given motion, or approximate a varying motion with a mean value. MathWorlds also allows any (reasonable) number of moving actors, not just two or three. This makes it possible to create activities in which large numbers of actors move in patterned ways, such as a marching band (see Kaput & Roschelle, 1996 for a scenario that uses this feature).

With MathWorlds, SimCalc has also been exploring the relationships between cybernetic (simulated) and kinesthetic (physical) experiences. For example, students can import their own body motion into MathWorlds via an MBL motion probe. TERC is developing complementary hardware that generates physical motion in toy carts based on a directly edited graphs, hence we can bi-directionally link the real world to graphical representations. Our conjecture is that cybernetic and kinesthetic explorations have complementary pedagogical value: cybernetics allow replay and re-examination of more controlled experiments, whereas kinesthetic explorations directly involve bodily understanding and connect directly to familiar experience. And our research is presently exploring the best ways to use these complementary qualities.

Another important set of MathWorlds features involves performing controlled experiments. A snap-to-grid option, Figures 6 and 7, constrains manipulations on graphs to integer values such as positions, times, or velocities. This can make it easier for students to produce graphs supporting direct measurement of area or slope by counting grid squares. This constraint can easily be removed to support free exploration of any values. Similarly, a flexible “step” command, Figures 8, 9 and 10, allows the student to control the clock, moving it forward in fixed ("delta-t") increments, which can make it easier to examine the correspondences among multiple representations at fixed time intervals. MathWorlds also supports a variety of ways in which students (or the simulation) can place marks in the animated world or in graph, Figures 11 and 12. "Marks" can provide a tool for making a prediction about behavior, or marks can be used to record the actor's position at uniform time intervals, thus leaving a trace of the actor's path through space that encodes velocity information, Figures 13 and 14.

Finally, MathWorlds also supports idealization by allowing a student to toggle between the visually rich world of actors and a visually bare or schematic view where the actors are replaced by dots color-coded to their respective graphs' colors moving along an easily scaleable one-dimensional coordinate system, Figures 13 and 14. This enables a move from qualitative examination of a situation to a distinctly quantitative examination. (Note, however, that a "world-ruler" can be invoked in any of the worlds to support quantitative analysis.)

STUDENT LEARNING WITH MATHWORLDS

In this section, we recount an episode from one of our early trial sessions with MathWorlds. The session featured a teacher, James Early, working after school with a middle school student in his inner city mathematics classroom. It illustrates how MathWorlds enabled a young student to learn how to integrate a velocity graph in order to determine the position of an elevator, and how the same student developed considerable facility with interpreting the distance, rate, and time represented in this graph. Compared to a typical calculus classroom, a striking feature of this episode is the lack of symbolic equations. Indeed, by using

MathWorlds, the student and teacher were able to explore the calculus concept of integration using only graphing and counting skills. This episode shows students can begin tackling significant concepts in the mathematics of change before taking algebra and years before they would satisfy the prerequisites for a normal calculus course.

Of particular interest here is the student's transition from reasoning about constant velocity graphs to reasoning about linear velocity graphs (constant acceleration). It was not easy for this student to grasp the relevant features of a linear velocity graph. Nonetheless, with the teacher's help, the student was able to see that the relevant area is the area under the velocity graph. The student then was able to quickly generalize to a variety of linear velocity situations, including one in which velocity goes from positive to negative.

Prior to the beginning of the episodes recounted below, the student had developed the ability to integrate a variety of piecewise constant velocity graphs, and was able to correctly distinguish time, rate, and distance. Work in class had centered on counting "blocks" of area under a velocity graph representing the vertical displacement of an elevator, and on interpreting the significance of different arrangements of the blocks. The student's confidence in his abilities was evident in his phrase: "You can't stop me." He uttered this phrase as a playful taunt to his teacher. The following exchanges are excerpts from a larger series of challenges and responses the teacher and student engaged in together. After working with a few piecewise constant velocity challenges, the teacher then began setting up challenges using linear velocity segments.

Integrating by Counting

The transcript below begins with a linear velocity segment with a slope of 1 m/sec^2 starting at zero velocity and extended for four seconds (to a velocity of 4 m/sec), as in Figure 15. As the transcript shows, the student was able to correctly integrate velocity to predict position. Moreover, the student was able to predict the change in speed of the elevator over time. Finally, the end of the transcript shows that the student was able to predict the motion of the elevator with a very challenging velocity graph — a decreasing linear velocity that continues below the axis.

<Insert Figure 15>

T: (Makes a linear velocity graph that extends over four seconds). What about this one?

S: It gonna go 8 floors.

T: 8 floors?

S: Yeah.

T: Why?

S: There's 6 floors right there (points to the six whole grid squares underneath the graph). That's a floor (places one finger on the half grid square at t=1, and the other finger at the grid square at t=2, indicating that these two halves make a whole floor) and that's another floor (fingers on the graph at t=3 and t=4).

T: So it's gonna go 8 floors.

S: Yep.

T: Try it and see. I don't know. Ooh, 8.

In excerpt above, we see the student attended to area under the graph, and used a counting procedure that correctly integrates the velocity function.

Identifying Changing Speed

Below, the teacher next focused on the change of speed, which the student correctly described but did not explain.

T: And then what else did we notice about the elevator?

S: First it went slow and then it goes faster and then it goes to another one and goes faster.

T: Ok.

S: And then the last one it goes real fast.

T: Ok, do you have any idea why? What makes it do that?

S: No.

T: No idea.

The teacher then made another, related problem with the same area, but decreasing speed.

T: Ok, what if I did this. (Flips the existing graph so it is decreasing over 4 seconds, instead of increasing.) How about if I did that?

S: It's still at 8.

T: It's still at 8?

S: It's the same problem.

T: Huh? Same problem.

S: Yeah.

T: The elevator going to go the same way.

S: Yeah.

T: So, how does it go up?

S: No, 1st it's going to go fast then slow.

T: Why?

S: Cuz it's a different way.

T: It's a different way.

S: That's high and that's low.

T: Ok. Try it. Let's see. It still wound up at 8.

With this contrast, the student was able to correctly describe both position and rate. In particular, the student correctly identifies speed with the height of the graph.

A Linear Decrease from Positive to Negative

Following this correct prediction, the teacher and student tried to make a graph that would make the elevator travel four floors. However, by accident they made a graph instead that decreased linearly from two floors per second to negative two floors per second. This is a challenging graph for students to interpret because, as we noted above, many students interpret a linearly decreasing velocity graph as a motion that continually goes down.

T: What do you think is going to happen here (a linearly decreasing graph that starts at two floors per second and decreases to negative two floors per second)? Let it go. What do you thinks gonna happen here?

S: Gonna go back to zero after it's finished.

T: You think so?

S: Yeah.

T: Try it. See. I don't know. Watch, wait, before you do that, why do you think it's going to go back to zero?

S: Cuz it's the same up here, it's the same right there. There both the same.

Note that here, the student correctly predicted the motion would return to its initial position. Following this episode, the student confidently began to assert "I can figure that out" as the teacher introduced additional graphical problems.

DISCUSSION

This episode demonstrates how a student, working with a teacher, can learn to correctly interpret the motion described by a velocity graph. As should be clear from the difficulties described in part two, this is not an easy concept to learn.

Indeed, it would be extremely rare to find students integrating graphs outside of a physics or calculus classroom. But this middle school student was able to make rapid progress, and eventually achieved a fairly robust and flexible interpretation of linear velocity graphs. Of course, there is much more to learn. Our point is not that this student is finished learning, but rather that is possible to begin making progress towards learning the core conceptual facets of the mathematics of change at a much younger age than traditional approaches to teaching calculus typically attempt.

We would attribute the progress evident in this episode to the interweaving of the three perspectives introduced earlier. From a student resources perspective, it is clear that this episode built upon the student's well-established skills in counting and computing with whole numbers and simple fractions, without the need of any algebraic symbols. MathWorlds enable the student to build an understanding of the mathematics of change using conceptual tools that were already firmly established by middle school. From a reconstructing subject matter perspective, this episode moved from piecewise constant velocity segments and then introduced linear velocity segments. Because of the student's prior experience with the very simple piecewise constant cases, meanings for a single block (grid square) were already well established. Moreover, the student already understood the meaning of height in a velocity graph, and the interpretation of the horizontal (time) axis. Although it was initially difficult for the student to interpret a grid square that was "cut in half," with the teacher's help the student soon overcame this difficulty, and was able to interpret quite difficult graphs. Beginning with piecewise, constant velocity may be a radical change to the conventional calculus curriculum, but such an approach seems to provide a powerful route into this difficult conceptual space. Finally, technological mediation clearly played a key role in allowing the rapid learning evidenced here. Directly editable graphs allowed the teacher to quickly construct and pose problem situations, and running the simulation allowed the student and teacher jointly to ground the meaning of those graphs in observable motions. The computer screen also served as an enabling conversation space in which the student and teacher could identify and discuss the interpretation of various features of the graph.

CONCLUSION

In choosing the title for this chapter, we drew inspiration from the concept of Accelerated Schools (Levin & Hopfenberg, 1991). Levin so clearly captured the paradox of conventional school reform: given evidence that student learning outcomes are in trouble, a typical response is to slow everything down, which only ensures that the students become further and further behind, and more and more in trouble. To overcome this paradox, we must break outside the conventional wisdom and find ways to dramatically accelerate students' progress.

Without question, dramatic acceleration will be required. One hundred years ago, only 3.5% of all students needed to finish high school, and virtually no high school students took Calculus. Today all students must finish high school, and 3.5% take calculus (U.S. Department of Education, 1996). In even less that one hundred years into the future, we can comfortably predict that most students will need access to the mathematics of change and variation, and acquire skill with its core concepts such as rate, accumulation, limit, approximation, etc. Further, at least 3.5% of them will likely need to conquer more advanced topics such as dynamical systems modeling. Mathematics and science depend increasingly upon concepts of greater complexity and abstraction, and society requires ever-greater number of more diverse students to master these important and powerful concepts.

Routine applications of technology will not meet the order of magnitude challenges we face in bringing much more mathematics learning to many more diverse students. The problem of access is not as simple as a wire, a transport protocol, and a university willing to publish its courses on the Internet. Nor will simply encoding the same lesson in different media radically change the rate at which students master core concepts. We need to move beyond reforming university courses, and repackaging dead pedagogies in media sound bites.

Yet, we cannot achieve scientific mastery from ordinary children without technology either. Visualization, simulation, and modeling are increasingly important aspects of professional mathematics and science, and rely deeply upon technology. As we have argued, these technologies also can draw upon some of children’s powerful, well-developed resources. The opportunity such innovations present is more that the chance to teach an existing course better; technological innovation opens a window to dramatically restructuring school curricula so that we can accelerate student learning. Through iterative design that is mutually sensitive to the unique affordances of technology for learning, children’s resources, and the need to radically reconstruct the curriculum, we believe our society can provide democratic access for all children to the concepts most important to the next millennium. The concepts must include the power to understand and control physical phenomena through mathematical analysis of change and variation.

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