Optimal Channel Probing and Transmission Scheduling for Opportunistic Spectrum Access

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ABSTRACT

In this study we consider optimal opportunistic spectrum access (OSA) policies for a transmitter in a multichannel wireless system, where a channel can be in one of multiple states. Each channel state is associated with either a probability of transmission success or a transmission rate. In such systems, the transmitter typically has partial information concerning the channel states, but can deduce more by probing individual channels, e.g. by sending control packets in the channels, at the expense of certain resources, e.g., energy and time. The main goal of this work is to derive optimal strategies for determining which channels to probe (in what sequence) and which channel to use for transmission. We consider two problems within this context, the constant data time (CDT) and the constant access time (CAT) problems. For both problems, we derive key structural properties of the corresponding optimal strategy. In particular, we show that it has a threshold structure and can be described by an index policy. We further show that the optimal CDT strategy can only take on one of three structural forms. Using these results we present a two-step lookahead CDT (CAT) strategy. This strategy is shown to be optimal for a number of cases of practical interest. We examine its performance under a class of practical channel models via numerical studies.

Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*Wireless Communication*;
G.3 [**Probability and Statistics**]: Distribution Functions

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General Terms

Algorithms, Design, Performance, Theory

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channel probing, software-defined radio, wireless ad hoc networks, scheduling, stochastic control, dynamic programming, opportunistic spectrum access

1. INTRODUCTION

Effective transmission over wireless channels is a key component of wireless communication. To achieve this one must address a number of issues specific to the wireless environment. One such challenge is the time-varying nature of the wireless channel due to multi-path fading. Recent works such as [12, 13] have studied opportunistically transmitting when channel conditions are better to exploit channel fluctuations over time.

At the same time, many wireless systems also provide transmitters with multiple channels to use for transmission. As mentioned in [3], a channel can be thought of as a frequency in a frequency division multiple access (FDMA) network, a code in a code division multiple access (CDMA) network, or as an antenna or its polarization state in multiple-input multiple-output (MIMO) systems. In addition, software defined radio (SDR) [9] and cognitive radio systems [1] may provide users with multiple channels (e.g. tunable frequency bands and modulation techniques) by means of a programmable hardware which is controlled by software. The transmitter, for example, could be a secondary user seeking spectrum opportunities in a network whose N channels have been licensed to a set of primary users [1].

These systems share the common feature that the transmitter is generally supplied with more channels than needed for a single-transmission. Thus, it is possible for users or transmitters to utilize the time-varying nature of the channels by opportunistically selecting the best one to use for transmission [7, 14, 8]. This may be viewed as an exploitation of channel fluctuations in space (i.e., across different channels), and is akin to the idea of multiuser diversity [10].

In order to utilize such channel variation, it is desirable for the transmitter and/or receiver to periodically obtain information on channel quality. One method of accomplishing this in a distributed manner is to allow nodes to probe channels by having the transmitter send control packets. Upon receiving a control packet, the receiver sends back a response packet that may indicate the channel quality.

For example, recent works such as [7, 6] have proposed enhancing or exploiting the multi-rate capabilities of the IEEE 802.11 RTS/CTS handshake mechanism. [6] proposes a protocol called Receiver Based Auto Rate (RBAR) in which the receivers use physical-layer analysis of received RTS messages to find out the maximum possible transmission rate to achieve a specific bit error rate. The receiver controls the sender's transmission rate by inserting the maximum possible transmission rate into CTS message. In cognitive radio systems, channel probing may be accomplished by using a spectrum sensor at the physical layer [1]. At the beginning of each slot, the spectrum sensor detects whether a channel is available. This detection may be imperfect. Due to energy and hardware constraints, the sensor could be limited in the number of channels it can sense during a given slot.

In all these applications, the probing process can help nodes obtain information about channel quality and thus make better decisions concerning which channel to use for transmission. On the other hand, channel measurement and estimation consume valuable resources. In particular, the exchange of control packets or spectrum sensing in cognitive radio systems decreases the amount of time available to send actual data and consumes energy.

Thus, channel probing must be done efficiently in order to balance the trade-off between obtaining useful channel information and consuming valuable resources.

In this paper we study optimal strategies for a joint channel probing and transmission problem. Specifically, we consider a transmitter with multiple channels with known state distributions. It can sequentially probe any channel with channel-dependent costs. The goal is to decide which channels to probe, in what order, when to stop, and upon stopping which channel to use for transmission. Similar problems have been studied in [3, 4, 2, 14, 8, 7]. The commonality and differences between our study and the previous work are highlighted below within the context of our main contributions.

The main contributions of this work are as follows.

Firstly, we derive key properties of optimal strategies for the problem outlined above, and show that the optimal strategy has a threshold property, and can only take on one of a few structural forms. In contrast to [3, 4, 2], we do not restrict the channels to take a finite number of states; our work applies to both the case where the number of channel states is finite, and the case where they can take an (uncountably) infinite number of states. This generalization is very useful as many next generation physical layer technologies such as MIMO and Adaptive-Bit-Loading OFDM [5] are aiming to provide continuous range of data rates that can be adjusted according to channel quality.

Secondly, we derive explicitly the optimal channel probing strategy for a number of special cases of practical interest. In [14, 8, 7], variants of the problem outlined above were studied. In particular, [14, 8] analyzed a problem where channels can only be used immediately after probing (i.e. no recall of past channel probes), and that unprobed channels cannot be used for transmission. Under these conditions the problem reduces to an optimal stopping time problem for a given ordering of channels to be probed. In this paper we allow both recall and transmitting in unprobed channels; the resulting problem is thus quite different from the optimal stopping time problem. [7] assumes independent Rayleigh fading channels and because all channels are independent and identically distributed, does not focus on which channels should be probed and in what order. In contrast we consider channels that are not necessarily statistically identical, and we provide numerical results for a general class of channel distributions.

Finally, based on the key structural properties of the optimal strategies, we propose strategies that perform well for arbitrary number of channels and arbitrary number of channel states (finite or infinite). To the best of our knowledge, this is the first channel probing algorithm for the combined scenario of an arbitrary number of channels, arbitrary channel distributions, statistically non-identical channels, and possibly different probing costs.

The remainder of this paper is organized as follows. We formulate two channel probing problems in Section 2. Section 3 then presents preliminary results on the first problem and some important structural results on the optimal strategy. Two channel probing algorithms for the first problem are presented in Section 4, and these algorithms are shown to be optimal for a number of special cases. These results are then extended to the second problem in Section 5. Section 6 provides numerical results which show that our proposed algorithms perform very well for some practical channel models. Section 7 concludes the paper.

2. PROBLEM FORMULATION

We consider a wireless system consisting of N channels, indexed by the set $\Omega = \{1, 2, \cdots, N\}$ and a transmitter who would like to send a message using exactly one of the channels. With each channel j, we associate a reward of transmission denoted by X_j, which is a random variable (discrete or continuous) with some distribution over some bounded interval $[0, M]$ where $M < \infty$. We call this the channel reward. The $\{X_j\}$ may represent either the probability of transmission success or the data rate of using channel j. The randomness of the transmission probability or data rate comes from the time-varying and uncertain nature of the wireless medium. It is assumed that the transmitter knows *a priori* the distribution of X_j for all $j \in \Omega$.

We assume temporal independence for the channel rewards. That is, the channel state within the time frame of a single transmission (including time for probing that precedes the transmission) is independent of the state during other transmissions. This assumption allows us to focus on the transmission of a single message. We also assume independence between channels, i.e. $\{X_j\}_{j\in\Omega}$ are independent random variables. Thus, probing channel j does not provide any information about the state of any other channel in $\Omega - \{j\}$. These same assumptions were also made in [14, 3, 4, 2].

Note that in reality, the transmitter may not be probing to directly find the probability of transmission success or data rate. For example, channel probes may be used to measure the channel signal-to-noise ratio (SNR) [7, 14]. This measured SNR, however, essentially affects the probability of transmission success or data rate and translates into a measured valued of X_j. Thus X_j can be thought of as an abstraction of the information obtained through probing.

The system proceeds as follows. The transmitter first decides whether to probe a channel in Ω or to transmit using one of the channels, based only on its *a priori* information about the distribution of X_j. If it transmits over one of the channels, the process is complete. Otherwise, the sender probes some channel $j \in \Omega$ and finds out the value of X_j. Based on this new information, the sender must now decide between using channel j for transmission, probing another channel in $\Omega - \{j\}$ (will also be denoted simply as $\Omega - j$ for the rest of the paper), or using a channel in $\Omega - j$ for transmission even though it has not been probed. This decision process continues until the user decides which channel to use for transmission.

We can therefore think of the system as proceeding in discrete steps, where at each step the transmitter has a set of unprobed channels $S \subseteq \Omega$, and has determined the values of channels in $\Omega - S$ through probing. It must decide between the following actions: (1) probe a channel in S, (2) use the best previously probed channel in $\Omega - S$, for which we say the user *retires* or (3) use a channel in S for transmission, which we call *guessing*. This last action was referred to as using a *backup channel* in [2, 3, 4]. Note that actions (2) and (3) can be seen as *stopping actions* that complete the entire probing and transmission process.

In practical situations, perhaps due to regulatory spectrum policies, it may be true that only a subset of channels in Ω can be guessed or retired to. For example, the transmitter may be allowed to transmit on an ISM band without probing, but may be required to probe a TV band immediately before using it. In this work, we assume the transmitter can guess or retire to any channel in Ω, but we will discuss in Sections 3.3, 4, and 6 that our results apply to the case where only a subset of Ω can be guessed. Extending these results to the case where only a subset of channels can be retired to is part of our future work.

We will also associate a cost c_j, where $c_j > 0$, with probing channel j. This cost may vary between channels, depending on the probing time, interference caused to other users, and so on. The sequence of decisions on whether to continue to probe and which channel to probe or transmit in will be called a *strategy* or *channel selection policy*.

With the assumptions and objectives outlined above, we formulate two different problems. We first consider the *constant data time* (CDT) problem [14], where the transmitter has a fixed amount of time for data transmission, regardless of how many channels it probes, as follows.

Problem 1. *Given a set of channels, their probing costs, and statistics on the channel transmission success probabilities, the sender's objective is to choose the strategy that maximizes transmission reward less the sum of probing costs, i.e. achieving the following maximum;*

$$J^* = \max_{\pi \in \Pi} J^\pi = \max_{\pi \in \Pi} E\left[X_{\pi(\tau)} - \sum_{t=1}^{\tau-1} c_{\pi(t)}\right] , \qquad (1)$$

where π denotes the time-invariant strategy that probes channels in sequence $\pi(1), \cdots, \pi(\tau-1)$, and then transmits over channel π_τ at time τ. Π denotes the set of all possible CDT strategies and the right hand sum in (1) is set to 0 if $\tau = 1$.

Note that τ is a random stopping time that in general depends on the result of channel probes, and $P(1 \le \tau \le N+1) = 1$ since the longest strategy is to probe all N channels and then use one for transmission. For the rest of this paper, we will let π^* denote the strategy that achieves J^* in (1), and will refer to π^* as the *optimal (CDT) strategy*. It can be shown that such a strategy is guaranteed to exist since there are a finite number of channels.

Because the X_j's are bounded rewards, it can be seen that J^* is also upper bounded by M. Thus we will further assume that $0 < E[X_j] < M$ for all $j \in \Omega$. This is because if $E[X_j] = M$, then it is always optimal to use channel j without probing, and if $E[X_j] = 0$ then channel j is never probed or used; the optimal strategy becomes trivial if these assumptions are violated.

It can be shown that at any step, a sufficient information state [11] is given by the pair (u, S), where $S \subseteq \Omega$ is the set of unprobed channels and $0 \le u \le M$ is the highest value among probed channels in $\Omega - S$. We can use dynamic programming [11] to represent the decision process as follows. Let $V(u, S)$ denote the value function, i.e. maximum expected remaining reward given the system state is (u, S). This can be written mathematically as:

$$V(u,S) = \max\left\{\max_{j\in S}\left\{-c_j + E[V(\max\{u, X_j\}, S-j)]\right\},\ u,\ \max_{j\in S} E[X_j], \right\} \qquad (2)$$

where all of the above expectations are taken with respect to random variable X_j. The first term on the right hand side of (2) represents the expected reward of probing the best channel in S, the second the reward of using the best-probed channel, and the last the expected reward of guessing the best unprobed channel. Thus $V(0, \Omega)$ denotes the expected total reward of the optimal strategy.

An alternative formulation of the problem concerns the scenario where the transmitter has a fixed amount of time T available for both probing and transmitting, and each probing action takes Δ amount of time, i.e., the probing cost is $c_j = \Delta$, where $N\Delta < T$ is assumed so that the transmitter has the option of probing every channel. The corresponding problem will be called the *constant access time* (CAT) problem [14]. In this scenario, transmitting at a rate u after probing k channels gives final reward $u(T - k\Delta)$. More generally, this problem is described as follows.

Problem 2. *We seek the strategy that achieves the following maximum:*

$$\bar{J}^* = \max_{\lambda \in \Lambda} E\left[X_{\lambda(\tau)} \cdot (T - \Delta(\tau - 1))\right] , \qquad (3)$$

where $\lambda(\tau)$ is the channel that strategy λ uses for transmission after $\tau - 1$ probes, and Λ is the set of all possible CAT policies. We will denote by λ^* the strategy that maximizes the expectation given by (3).

Similar to the CDT problem, it can be shown that (u, S) is a valid information state for this problem given the set Ω. However, note that whereas previously in the CDT problem $V(u, S)$ does not depend on Ω, in the CAT problem the value function does depend on Ω. Specifically, the state depends on the amount of time left for transmission, denoted by $\bar{T}$, given by $\bar{T} = T - |\Omega - S| \cdot \Delta$. To emphasize this difference and remove the dependency of the value function on Ω, we will adopt the triple $(\bar{T}, u, S)$ as the information state, while noting that $\bar{T}$ is obtainable from S if Ω is also given. Then the maximum expected remaining reward $H(\bar{T}, u, S)$, analogous to (2), is given by:

$$H(\bar{T}, u, S) = \max\left\{\bar{T}u, \max_{j\in S}\left\{\bar{T}E[X_j]\right\},\ \max_{j\in S}\left\{E\left[H\left(\bar{T} - \Delta, \max\left\{X_j, u\right\}, S - j\right)\right]\right\}\right\} \qquad (4)$$

where the three terms represent, respectively, the reward of retiring, using channel j without transmission, and probing channel j and then proceeding according to the optimal strategy.

Note that while the dynamic programs are readily available in both cases, computing the value function $V(\cdot,\cdot)$ and $H(\cdot,\cdot,\cdot)$ for every state is very difficult and practically impossible because the state space is potentially infinite and uncountable, since u can be any real number in $[0, M]$ if the X_j's are continuous random variables. Specifically, to compute $V(u, S)$, we may need to know $V(\tilde{u}, S - j)$ for all $j \in S$ and all $u \leq \tilde{u} \leq M$. Consequently, instead of trying to compute these values and the associated strategies, we will use the above two formulations to derive fundamental properties of optimal strategies and use them to find simpler ways of determining optimal strategies in Section 4.

For the CDT problem, any strategy can be defined by the set of actions it takes with respect to its entire set of information states, $\cup_S \cup_u (u, S)$. We let retire(u) denote the action that the sender retires and uses the best previously probed channel in $\Omega - S$, which has value u; probe(j) denotes the action that channel j is probed, for some $j \in S$; and guess(j) denotes the action that channel j, for $j \in S$, is guessed (i.e., used even though it has not been probed). For state (u, S), a strategy must decide between retire(u), probe(j), and guess(j), for all $j \in S$. We let $\pi(u, S)$ denote the action taken by strategy π when state is (u, S). For example, $\pi(u, S)$ =probe(j) means the sender probes channel j when the state is (u, S). Similarly, we will denote a CAT strategy by λ and use similar notations, e.g., $\lambda(\bar{T}, u, S)$ =probe(j) means the strategy probes channel j in state $(\bar{T}, u, S)$.

The detailed analysis in this paper primarily deals with the CDT problem due to space limitation as well as its relative simplicity in presentation. As we will discuss in Section 5, many of our results on CDT strategies are also applicable to CAT strategies.

3. PROPERTIES OF THE OPTIMAL STRATEGY

In this section, we establish key properties of the optimal CDT strategy. Unless otherwise stated, all proofs are given in the Appendix.

3.1 Threshold Property of the Optimal Strategy

We first note that for all $S \subseteq \Omega$ and any $\tilde{u} \geq u$,

$$V(u, S) \leq V(\tilde{u}, S) \ . \tag{5}$$

This inequality follows from (1) and (2). In particular, consider any channel selection strategy starting from state (u, S), and apply the same strategy starting from state $(\tilde{u}, S)$. Clearly the expected reward of the strategy cannot be less in the latter starting scenario, since the set of unprobed channels is the same for both cases, while the best probed channel for the latter case is better than that of the former scenario. Thus, $V(\cdot, S)$ is a nondecreasing function. Similarly, it can be established that $V(u, \cdot)$ is a nondecreasing function, i.e. for all $u \in [0, 1]$ and any $\tilde{S} \supseteq S$:

$$V(u, S) \leq V(u, \tilde{S}). \tag{6}$$

We have the following lemmas:

Lemma 1. *Consider any state (u, S). If $V(u, S) = u$, then $V(\tilde{u}, S) = \tilde{u}$ for all $\tilde{u} \geq u$.*

Lemma 2. *If $V(u, S) = E[X_j]$ for some $j \in S$, then $V(\bar{u}, S) = E[X_j]$ for all $\bar{u} \leq u$.*

Lemma 2 follows directly from (2) and (5), since these equations imply $E[X_j] \leq V(\bar{u}, S) \leq V(u, S) = E[X_j]$. Therefore, its proof is not included in the Appendix.

The above two lemmas imply that for fixed S, the optimal strategy has a *threshold* structure with respect to u. In particular, for any set $S \subseteq \Omega$, we can define the following quantities:

$$a_S = \inf \{u : V(u, S) = u\} \tag{7}$$

$$b_S = \sup \{u : V(u, S) = E[X_j], \text{ some } j \in S\} \ , \tag{8}$$

where the right hand side of (7) is nonempty since it is always true that $V(M, S) = M$. We will set $b_S = 0$ if the set on the right hand side of (8) is empty. Note that both a_S and b_S are completely determined given the set S. It follows from Lemmas 1 and 2 that $0 \leq b_S \leq a_S \leq M$. Thus we have the following corollary:

Corollary 1. *For any state (u, S), there exists an optimal strategy π^* and constants $0 \leq b_S \leq a_S \leq M$ satisfying:*

$$\pi^*(u, S) = \begin{cases} \text{retire}(u) & \text{if } u \geq a_S \\ \text{probe}(j_u), \ j_u \in S & \text{if } b_S < u < a_S \\ \text{guess}(j), \ j \in S & \text{if } u < b_S \end{cases} .$$

It should be noted that at $u = b_S$, $\pi^*(u, S) = \text{guess}(j)$ if $b_S > 0$; otherwise, $\pi^*(u, S) = \text{probe}(j)$. Also note that the optimal channel to probe, j_u, in general depends on the value of u. This corollary indicates that there exists an optimal strategy with the described threshold structure. It remains to determine these thresholds, which can be very difficult especially for large S. It also remains to determine which channel should be probed if we are in the "probe" region above.

To help overcome the difficulty in determining a_S and b_S for a general S, we first focus on quantities $a_{\{j\}}$ and $b_{\{j\}}$ (subsequently simplified as a_j and b_j) for a single element $j \in \Omega$, which can be determined relatively easily from (7) and (8), respectively, as shown below. These are indices concerning channel j that are *independent of other channels*. We will see that they are very useful for deciding the optimal strategies, thus significantly reducing the complexity of the problem.

It is thus worth taking a closer look at a_j and b_j. Note that at state (u, j), probing channel j results in an expected reward $-c_j + E[\max\{u, X_j\}]$, since there are no channels to probe after j. Action guess(j) gives the expected reward $E[X_j]$ while retiring gives reward u. Because of the assumptions that $0 < E[X_j] < M$ and $c_j > 0$, for sufficiently small u the probing reward becomes less than the guessing reward. By comparing the rewards of the three options, it can be seen that guessing is optimal if: $E[(u - X_j)I_{\{X_j<u\}}] \leq c_j$ and $u \leq E[X_j]$, where $I_{\{\cdot\}}$ is the indicator function. We will also adopt the notation that, for any random variable Z, $E[Z^+] = E[Z \cdot I_{\{Z>0\}}]$.

Similarly, when u is sufficiently large the probing and guessing reward become less than the reward for retiring,

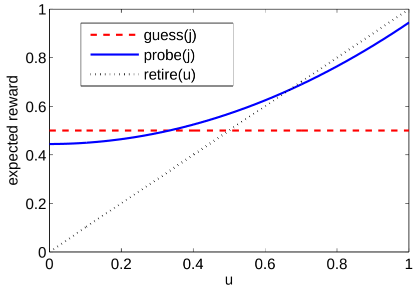

Figure 1: As described in Section 3.1: when j is the only unprobed channel and X_j is uniformly distributed in $[0, 1]$, the expected reward from actions guess(j), probe (j) and retire(u) as functions of u. Note that $a_j = 2/3$ (the crossing point of solid and dotted lines) and $b_j = 1/3$ (the crossing point of solid and dashed lines).

u. Thus for any $j \in S$ we have the following:

$$a_j = \min \left\{ u : u \geq E[X_j], c_j \geq E\left[(X_j - u)^+\right] \right\} \tag{9}$$

$$b_j = \max \left\{ u : u \leq E[X_j], c_j \geq E\left[(u - X_j)^+\right] \right\} \tag{10}$$

Note that $a_j \geq E[X_j] \geq b_j$. In addition, $a_j = b_j$ if and only if $E[X_j] = a_j = b_j$. It also follows that for $b_j < u < a_j$ probing is strictly an optimal strategy. It can be seen from the above that c_j essentially controls the width of this probing region; for larger c_j, both a_j and b_j will be closer to $E[X_j]$.

The above discussion is depicted in Figure 1 where we have plotted the expected reward of the three actions guess(j) (dashed line), probe(j) (solid line), and retire(u) (dotted line) as functions of u when X_j is uniformly distributed in $[0, 1]$ and $c_j = 1/18$. In this case, $a_j = 2/3$ and $b_j = 1/3$. Note that increasing (decreasing) c_j would shift the solid curve down (up), thus decreasing (increasing) the width of the middle region where probe(j) is the optimal action.

This example demonstrates a method for computing a_j and b_j for any channel j. Note that to determine these two constants simply requires taking the intercepts between the following three functions of u: $f_1(u) = E[X_j]$, $f_2(u) = u$, and $f_3(u) = -c_j + uP(X_j \leq u) + E[X_j I_{\{X_j > u\}}]$. Thus regardless of whether X_j is continuous or discrete, computing a_j and b_j is not very complex.

In the rest of this section we derive properties of the optimal strategy expressed in terms of these individual indices a_j and b_j.

3.2 Structure of The Optimal Strategy

In this section we provide a summary of results on the structure of the optimal strategy. These results are essential in reducing the space of policies within which the optimal strategy lies.

Theorem 1. *For any set S, define R and j^* as follows:*

$$R = \left\{ j \in S : a_j = \max_{k \in S} a_k \right\} .$$

$$j^* = \operatorname*{argmax}_{j \in R} \left\{ I_{\{a_j = b_j\}} E[X_j] + I_{\{a_j > b_j\}} \left(E[X_j \,|X_j \geq a_j] - \frac{c_j}{P(X_j \geq a_j)} \right) \right\}$$

Then we have the following, where d_S is some constant such that $d_S \leq b_{j^*}$:

1. *For all $u \geq a_{j^*}$, $\pi^*(u, S) =$retire(u). If $u < a_{j^*}$ then $\pi^*(u, S) \neq$retire(u).*

2. *For all $d_S < u < a_{j^*}$, $\pi^*(u, S) =$probe(j^*).*

3. *For all $0 \leq u < d_S$, exactly one of the following three possibilities holds for π^*:*

   (a) $\pi^*(u, S) = \text{probe}(j^*)$
   (b) $\pi^*(u, S) = \text{guess}(j^*)$
   (c) $\pi^*(u, S) = \text{probe}(k),\ k \in S - j^*$

   where channel k does not vary with u.

The proof of Theorem 1 is broken down into separate sections in the Appendix as follows. Part 1 of Theorem 1 is proven in Section 9.2. This result provides both a necessary and sufficient condition for the optimality of retiring and using a previously probed channel. A very appealing feature of this result is that it allows us to decide when to retire based only on *individual* channel indices that are calculated independently of other channels, thus reducing computational complexity.

Part 2 of Theorem 1 is proven in Section 9.3. This result implies that by first ordering the individual channels by functions of the indices a_j, we can determine the optimal channel to probe for u in the interval (d_S, a_{j^*}).

Finally, Part 3 of Theorem 1 gives three possibilities on the structure of the optimal strategy. In particular, Part 3(b) narrows down the set of possible channels we can guess. In words, the best channel among those achieving the highest value of a_j in S (which we have called j^*) is the only possible channel we can guess. This result is proven in Section 9.5. A key result in that section is that if there are multiple channels in R, then we can easily check whether $a_j = b_j$ is true in order to determine whether probing or guessing is the optimal action. Section 9.5 also provides some necessary and sufficient conditions for guessing to be optimal. In addition, parts 3a and 3c indicate that the optimal channel to probe does not vary with u in the region $(0, d_S)$. This result is proven in Section 9.4.

Theorem 1 significantly reduces the number of possibilities on the structure of the optimal strategy, but it remains to determine when cases $3a$, $3b$, $3c$ of Theorem 1 hold along with the value of d_S. In general, this structure will depend on the specific values of S and the indices a_j, b_j. One can use the results of Sections 9.2, 9.3, 9.5 to determine some necessary or sufficient conditions for any particular case of the above theorem to hold. In the next section, we will propose a suboptimal algorithm, based on these three possible forms, which we show to be optimal under a number of special cases of interest.

Figure 2 summarizes the main results from Theorem 1. We have shown that for all $u \geq a_{j^*}$, i.e. right region of the line, retire(u) is optimal. For $\max_{j \in S} b_j < u < a_{j^*}$, i.e. the middle region of the line, probe(j^*) is optimal. Note that it is possible this region may be empty if the probing costs become too high. Finally, the optimal action in the left region will depend on S, and thus remains to be determined. Note that guess(j^*) is the only possible guessing action for this region, as proven in Lemma 9 and Corollary 2.

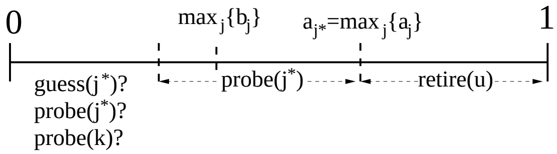

Figure 2: Summary of main results from Section 3: figure depicts optimal strategy $\pi^*(u, S)$ as a function of u. For the middle and right regions of the line, the optimal strategy is well-defined for any S. For the left region, the optimal action may depend on S.

3.3 Decomposition of Problem 1

The following result on the structure of the optimal strategy allows us to decompose Problem 1 into N subproblems. To begin, define $\Phi(j), j = 1, 2, \cdots, N$, to be the set of strategies which do not guess any channel except possibly channel j. Within each set $\Phi(j)$ we define the best strategy, i.e. achieves the value function in (2), by π_j^*:

$$\pi_j^*(u, S) = \operatorname*{argmax}_{\pi \in \Phi(j)} V^\pi(u, S) \ , \tag{11}$$

where $V^\pi(u, S)$ is the expected remaining reward under policy π given the system state (u, S). It can be shown in a similar manner to Theorem 5.2 in [3] (where a 3-channel system is considered) that the optimal strategy satisfies $\pi^* \in \cup_{j\in\Omega}\{\Phi(j)\}$. In other words, we have:

Lemma 3. *For any (u, S), there exists an optimal strategy π^* such that:*

$$\pi^*(u, S) = \operatorname*{argmax}_{\pi\in\Pi} V^\pi(u, S) = \operatorname*{argmax}_{j\in S} V^{\pi_j^*}(u, S) \tag{12}$$

That is, the optimal strategy will only guess one channel (if it guesses at all). Thus, the optimal strategy π^* among all strategies is the best π_j^* among all $j \in \Omega$. This result again reduces the number of possible optimal strategies. As the proof of this lemma is similar to that of Theorem 5.2 in [3], it is omitted for brevity. However, we would like to point out that it can be shown that Lemma 3 can be extended to the case where the transmitter is only allowed to guess a subset $\bar{S} \subseteq S$ of the channels. In this case, one can replace S under the argmax in Lemma 3 with $\bar{S}$.

Finally, it remains to determine the structure of π_j^*. We have the following useful result.

Lemma 4. *For any S and $j \in S$, define R as in Theorem 1, replacing, a_k with $\bar{a}_k$ (defined in Equation (18)) for every channel except for j. If $j \in R$, then let $j^* = j$. Otherwise, define j^* as in Theorem 1. Then if $j^* \neq j$, the optimal strategy π_j^* is:*

$$\pi_j^*(u, S) = \begin{cases} \text{retire}(u) & \text{if } u \geq a_{j^*} \\ \text{probe}(j^*) & \text{if } u < a_{j^*} \end{cases} .$$

It can be shown that Theorem 1 also holds for each strategy π_j^*. Thus, Lemma 3 can be seen as arising from Case 1 in Theorem 1. This result uniquely describes the optimal strategy π_j^* for any set of channels S, if $j \neq j^*$. When $j = j^*$, then the optimal strategy has a more complicated structure. In the next section, we propose a suboptimal algorithm which approximates the optimal strategy when $j = j^*$.

4. JOINT PROBING AND TRANSMISSION STRATEGIES

As stated earlier, the state space for our problem is the set of all (u, S), which is infinite. This makes it very difficult to recursively apply dynamic programming to evaluate all $V(u, S)$ and solve for the optimal strategy. In this section we propose two algorithms for channel probing for arbitrary number of channels with arbitrary distributions. They are motivated by the properties derived in the previous section. We show they are optimal for a number of special cases of practical interest.

4.1 Channel Probing Algorithms

In order to motivate our first algorithm, consider when there are two unprobed channels $S = \{j \cup k\}$. As described in the previous section, the ordering of the constants a_j, b_j, a_k, and b_k will help determine the optimal strategy. Note that due to Corollary 1, it is not hard to calculate the expected reward of probing j or k for state (u, S). For example, if $u < b_k$, then probe(j) at state (u, S) incurs the following expected reward:

$$\begin{aligned}
&- c_j + E[V(\max(u, X_j), k)] \\
&= -c_j + P(X_j < b_k)E[X_k] + E[X_j I_{\{X_j \geq a_k\}}] \\
&- c_k \cdot P(b_k \leq X_j < a_k) + E[\max(X_k, X_j) I_{\{a_k > X_j \geq b_k\}}] \ .
\end{aligned}$$

The above calculation can similarly be applied to the other two separate cases of $u > a_k$ and $a_k > u > b_k$, and they can also similarly be applied to determine the expected reward of action probe(k). Note that this procedure computes the expected probing reward in a finite number of steps, whereas not using the threshold properties given by Corollary 1 would first require the computation of $V(u, j)$ and $V(u, k)$ for all $u \in [0, 1]$, thus requiring an infinite number of computations.

Motivated by the above, the proposed algorithm is as follows. It essentially finds two channels indexed by j^* and k, and use these to define the strategy γ. We use the following notation in describing the algorithm:

$$f_{j,k}(u) = -c_j + E[V(\max(X_j, u), k)] \ , \tag{13}$$

which is the expected reward of probe(j) at state $(u, j \cup k)$.

Algorithm 1. *(A two-step lookahead policy γ for a given set of unprobed channels $S \subseteq \Omega$)*
Step 1*: Compute j^* as defined in Theorem 1.*
Step 2*: Replace S with $S - j^*$ and repeat Step 1, with k denoting the result of this step.*
Then strategy γ is defined as follows for state (u, S):

1. *If $u \geq a_{j^*}$, then $\gamma(u, S) =$retire(u).*
2. *If $a_{j^*} > u > \max\{b_{j^*}, b_k\}$, then $\gamma(u, S) =$probe(j^*).*
3. *If $u \leq \max\{b_{j^*}, b_k\}$, we have the following cases:*
   (a) *If $b_{j^*} \geq a_k$, then $\gamma(u, S) =$guess(j^*).*
   (b) *If either $f_{j^*,k}(0) \geq \max\{E[X_{j^*}], f_{k,j}(0)\}$ or $b_k \geq b_{j^*}$ is true, then $\gamma(u, S) =$probe(j^*).*
   (c) *Otherwise, there exists a unique b_0 such that $b_{j^*} > b_0 > b_k$ and $f_{j^*,k}(b_0) = \max\{E[X_{j^*}], f_{k,j^*}(0)\}$. For $b_0 \leq u \leq b_{j^*}$, we have $\gamma(u, S) =$probe(j^*). For $u < b_0$, we have $\gamma(u, S) =$guess(j^*) if $E[X_{j^*}] \geq f_{k,j^*}(0)$. Otherwise, $\gamma(u, S) =$probe(k).*

It is worth describing this strategy in the context of results derived in the previous section. For u satisfying Case 1 of the algorithm description, γ is optimal from Theorem 1 and Lemma 6. For some of the u values described in Case 2, γ is optimal from Theorem 1 and Lemma 7. For Case 3(a), γ is optimal from Theorem 1, Lemma 9 and Corollary 2. Thus γ is optimal for most values of u. For Cases 3(b) and 3(c) of Algorithm 1, the procedure essentially computes the expected probing cost if we are forced to retire in two steps.

An important reason for proposing a two-step policy arises from Theorem 1 which states that the optimal strategy takes one of three possible forms. For fixed S, it was shown that as u varies there can be at most two possible channels to probe, one of which must be j^*. This gives rise to the strategy above that only considers two channels, j^* and a second channel k.

We also propose a second two-step lookahead algorithm, which we call β, that is motivated by Algorithm γ and Lemmas 3 and 4. Due to space limitations and its similarity to γ, we present only a brief description.

Algorithm 2. *(Two-Step Lookahead Policy β) For each channel $j \in S$ and the corresponding set of strategies $\Phi(j)$ defined in Section 3.2, find the best two-step policy (analogous to Algorithm 1) as follows. If $j^* \neq j$, we can set β_j to be strategy π_j^* of Lemma 4. Otherwise, determine the best two-step strategy in $\Phi(j)$ using the two channels j^* and k, similar to Algorithm 1 but replacing a_k with $\bar{a}_k$ and setting $b_k = 0$.*

After determining β_j for all $j \in S$, using Lemma 3 take the best strategy among all $\{\beta_j\}_{j \in S}$ to obtain algorithm β.

In the case when the transmitter can only guess a subset $\bar{S} \subseteq S$ of channels then we can modify Algorithm 2 by replacing S with $\bar{S}$ in the above description.

Note that determining algorithm β requires running a similar algorithm to γ for each channel in S, thus requiring more computation. However, this strategy generally performs better than γ as we will show in Section 6. We next consider a few special cases and show that γ is optimal in these cases. It can also be shown that these results hold for β as well.

4.2 Special Cases

We first consider a two channel system. Since Algorithm 1 is essentially a two-step lookahead policy, we have the following (the proof is omitted for brevity):

Theorem 2. *For any given set of unprobed channels S, where $|S| = 2$, γ is an optimal strategy.*

We next consider the case when all channels are statistically identical, with an arbitrary number of them, each having different probing costs.

Theorem 3. *Suppose $|S| \geq 2$, and all channels in S are identically distributed, with possibly different probing costs. Then the optimal strategy π^* is described as follows, with j^* being a channel in S satisfying $c_{j^*} = \min_{j \in S}\{c_j\}$.*

Case 1*: If $a_{j^*} > b_{j^*}$, we have:*

$$\pi^*(u, S) = \begin{cases} \text{retire}(u) & \text{if } u \geq a_{j^*} \\ \text{probe}(j^*), & \text{otherwise} \end{cases}$$

Case 2*: If $a_{j^*} = b_{j^*}$, the optimal strategy is: retire(u) if $u \geq a_{j^*}$; otherwise, $\pi^*(u, S) =$guess(j^*).*

This theorem implies that if we have a set of statistically identical channels Ω, then the initial step of the optimal strategy is uniquely determined by a_{j^*} and b_{j^*}, where j^* is the channel with smallest probing cost. If $a_{j^*} = b_{j^*}$, then $\pi^*(u, \Omega) =$guess(j^*) and it is not worth probing any channels. If $a_{j^*} > b_{j^*}$, then we should first probe j^*. Let k denote the channel with the smallest probing cost in $S - \{j^*\}$. If the probed value of X_{j^*} is higher than a_k, then it is optimal to retire and use j^* for transmission. Otherwise, if $a_k > b_k$ then probe(k) is optimal; if $a_k = b_k$ then guess(k) is the optimal action. This process continues until we retire, guess, or $|S| = 1$. When $|S| = 1$, then we compare the best probed channel, which has value u, to a_j and b_j, where $S = \{j\}$. If $u \geq a_j$, then we retire; if $a_j > u \geq b_j$, then we probe the last remaining channel; finally, if $u < b_j$ then we should just use the last remaining channel without probing it.

Note that the optimal strategy described above is the same as strategy γ of Algorithm 1 applied to statistically identical channels. This is true because within Case 3 in the description of Algorithm 1, 3(b) will occur whenever $a_{j^*} > b_{j^*}$ for statistically identical channels, and Case 3(a) occurs whenever $a_{j^*} = b_{j^*}$. Collectively, Cases 1, 2, 3(a) and 3(b) all describe the optimal strategy of Theorem 3. Note that this theorem applies to *all* cases of statistically identical channels, regardless of their distribution or probing costs. Changing the channel distribution and probing costs will affect the values of a_j or b_j, but they do not alter the general structure of the optimal strategy as given by the theorem.

Finally, we consider the case where the number of channels is very large and not statistically identical.

Infinite Number of Channels (INC) Problem: Consider Problem 1 with the following modification: we have N different types of channels, but an infinite number of each channel type.

Note that Theorem 3 solves this problem if $N = 1$. When referring to the state space for this problem, we will let S denote the set of available channel *types*. Theorem 3 says that if we have many statistically identical channels of one type, then whether $a_j > b_j$ or $a_j = b_j$ determines if we will probe or guess channel j. Analogously, we have the following result:

Theorem 4. *For any set of channels S, define j^* according to* Step 1 *of Algorithm 1. Then for the INC Problem, there exists an optimal strategy π^* satisfying:*

1. *If $u \geq a_{j^*}$, then $\pi^*(u, S) =$retire(u).*
2. *If $u < a_{j^*}$ and $a_{j^*} > b_{j^*}$, then $\pi^*(u, S) =$probe(j^*).*
3. *If $u < a_{j^*}$ and $a_{j^*} = b_{j^*}$, then $\pi^*(u, S) =$guess(j^*).*

Due to space limitations, proof of the above theorem is not included; however, it should be noted that it essentially follows from Theorem 3. This theorem implies that when the number of channels is infinite, and there are an arbitrary number of channel types, then we will only probe or guess one channel, i.e. the other channels become irrelevant. In addition, note that Algorithm 1 is also the optimal strategy for the INC Problem, because Case 3(c) of the description of Algorithm 1 does not occur. For all the other cases, Algorithm 1 reduces to the optimal strategy described in Theorem 4.

To summarize, in this subsection we have shown that Algorithm 1 reduces to the optimal strategies for the above special cases based on Theorems 2, 3, and 4.

5. CONSTANT ACCESS TIME POLICIES

In this section we present results on the optimal channel access time (CAT) policy.

Similarly to Corollary 1, we can show that for any state $(\bar{T}, u, S)$, there exists an optimal strategy λ^* and constants $0 \leq b_{S,\bar{T}} \leq a_{S,\bar{T}} \leq 1$ such that:

$$\lambda^*(\bar{T}, u, S) = \begin{cases} \text{retire}(u) & \text{if } u \geq a_{S,\bar{T}} \\ \text{probe}(j_u),\ j_u \in S & \text{if } b_{S,\bar{T}} < u < a_{S,\bar{T}} \\ \text{guess}(j^{'}),\ j^{'} \in S & \text{if } u < b_{S,\bar{T}} \end{cases}$$

Thus for each set of channel j, we can define indices $a_{j,\bar{T}}$ and $b_{j,\bar{T}}$ similar to (9) and (10) but note that these indices are time-variant, which makes the analysis significantly more complex. The following results show a similarity between λ^* and π^*.

First, we note that any individual channel index $a_{j,\bar{T}}$ can be calculated for each $\bar{T}$ as follows:

$$a_{j,\bar{T}} = \min\left\{u : u \geq E[X_j], u \geq E\left[\max(X_j, u)\right] \cdot \frac{\bar{T}-\Delta}{\bar{T}}\right\}$$

where $a_{j,\bar{T}}$ is the smallest u such that $\lambda(\bar{T}, u, j)$ =retire(u). The indices $b_{j,\bar{T}}$ can be calculated similarly. We can use these individual channel indices to obtain the following.

Lemma 5. *For any set of channels S and $\bar{T} > 0$, we have $a_{S,\bar{T}} = \max_{j\in S}\ a_{j,\bar{T}}$.*

Thus, the index $a_{S,\bar{T}}$, and therefore the set of states where retirement is optimal, can be determined using only individual channel indices from time $\bar{T}$. As in the case of CDT, it is important to note these indices do not depend on other indices $\{a_{j,\bar{T}-k\Delta}\}$, which simplifies computation.

In general, due to the time-varying nature of these indices, it becomes very difficult to determine the structure of the optimal strategy. However, the similarity in the index properties between the CDT and CAT policies can be used to propose the following two-step lookahead algorithm, similar to Algorithms γ and β. For any particular $\bar{T}$ and set of channels S, we first determine the two channels with the highest indices $a_{j,\bar{T}}$. Then compute the optimal strategy if we are forced to retire within two steps, similarly to Algorithm 1. We evaluate the performance of this strategy in the next section.

6. NUMERICAL RESULTS

In this section we examine the performance of the proposed algorithms under a practical class of channel models.

For both CDT and CAT policies, we will consider a two-state channel model where, for each channel $P(X_j = r_j) = p_j = 1 - P(X_j = 0)$ for some $r_j > 0$. This models the case, for example, where channels are either off or on, and have data rate r_j in the on state [1]. Under this setting, the set of information states is $\cup_j \cup_S (r_j, S)$.

For our numerical results, we chose parameters r_j, p_j, and c_j for each channel as follows. First, r_j and p_j were modeled as independent random variables, uniformly distributed in the interval [2] $(0, 1)$. After the particular realization of these parameters was chosen, the cost for channel j was uniformly chosen in the interval [3] $(0, p_j(1-p_j)r_j + 0.01)$.

For each realization of r_j, p_j, c_j, the expected rewards of the following strategies were computed for the CDT problem: the optimal strategy (determined via dynamic programming), algorithms γ and β from Section 4.1, and the optimal algorithm if guessing is not allowed (no-guess), as described in Section 9.3 and Theorem 5. A total of 10^3 random realizations are generated and then averaged for each value N. Figure 3 (TOP) depicts the performance of these strategies as the number of channels N varies. The average rewards of these strategies are normalized by dividing the average reward of the optimal strategy. We note that both Algorithm γ and β perform very close to the optimal, with β performing slightly better. This is because Algorithm γ and β are optimal when Case 3(a) from Theorem 1 holds. In general, this case holds for most values of p_j, r_j, and c_j. When Case 3(b) or 3(c) of Theorem 1 holds, Algorithms γ and β only differ with the optimal algorithm in the parameter d_S. Thus in general they are very close numerically to the optimal strategy.

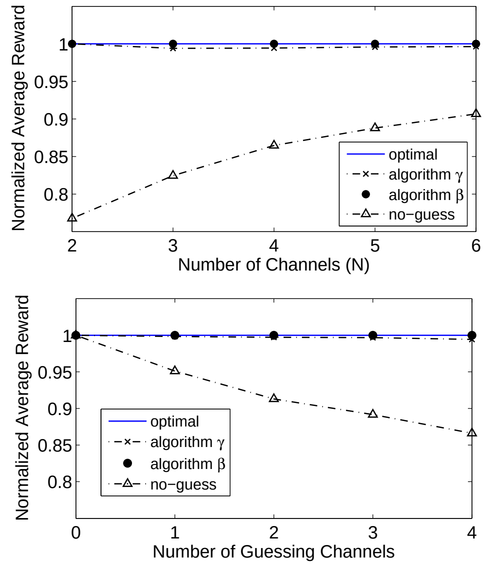

Figure 3: (TOP): Average performance of optimal CDT strategy, algorithms γ and β of Section 4.1, and the optimal strategy without guessing. Rewards are normalized by the average reward of the optimal strategy. (BOTTOM): Average performance of these strategies for a four-channel system where the number of channels that can be guessed varies between 0 and 4.

[1] [4] has considered optimal CDT strategies for two-state channels, each with identical data rate. When the parameters r_j differ between different channels, it can be shown the strategies of [4] are not necessarily optimal.

[2] Note that the upperbound on r_j of 1 is chosen for simplicity; it could be any positive value M, which simply causes the reward and the cost to scale simultaneously.

[3] This choice of upperbound on c_j is to ensure that some channels will be probed, as it can be shown that if $c_j > p_j(1-p_j)r_j$ then channel j should never be probed and only guessed. The additional 0.01 to $p_j(1-p_j)r_j$ is to ensure that some channel will be guessed, but the value 0.01 is an arbitrary choice.

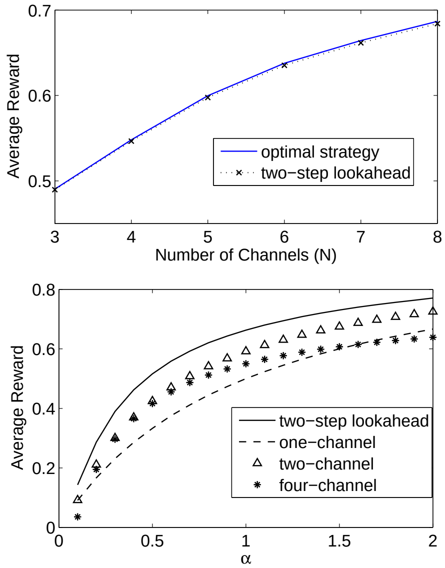

Figure 4: (TOP): Performance of optimal CAT strategy and a two-step lookahead CAT policy as the number of channels (N) varies. (BOTTOM) Performance of the two-step lookahead, one-channel, two-channel, and four channel algorithms as described in Section 6, when all channels are statistically identical with $P(X_j \leq x) = x^\alpha$ and different probing costs $c_j = 0.01j$.

As mentioned in Section 2, it may be the case that some regulatory spectrum policies do not allow all channels to be guessed. Figure 3 (BOTTOM) analyzes the performance of these strategies when $N = 4$ and only a subset $\bar{S} \subseteq S$ of these channels can be guessed. For this case, we modify Algorithm γ as follows. If $j \notin \bar{S}$, we set $a_j = \bar{a}_j$ as given by (18), and set $b_j = 0$. For $j \in \bar{S}$, the indices remain unchanged. These changes are made to remove guess(j) as a possible action. For Algorithm β, we replace S in its definition with $\bar{S}$. We see that the relative performance between the optimal strategy and algorithms γ and β does not significantly change as $|\bar{S}|$ varies. On the other hand, by definition the no-guess strategy is optimal when $|\bar{S}| = 0$, but as expected its average reward decreases as $|\bar{S}|$ increases.

Similarly, Figure 4 (TOP) analyzes the performance of the optimal strategy and a two-step lookahead algorithm (similar to γ) for the CAT problem. As can be seen, the two-step lookahead algorithm, calculated similarly to Algorithm 1 but for each $\bar{T}$ as mentioned in the previous section, performs very well in comparison to the optimal strategy.

Figure 4 (BOTTOM) analyzes performance when the channels are statistically identical with CDF $P(X_j \leq x) = x^\alpha$ for all j and with different probing costs $c_j = 0.01j$. From Theorem 3, the two-step lookahead algorithm is optimal. Its performance is compared in the figure to the following algorithms. The *one-channel* algorithm does not probe and simply transmits using the "best" channel (lowest cost). Thus comparing the two-step lookahead algorithm to this strategy gives an indication of the gain from using probing. The *two-channel* (*four-channel*) algorithm depicted in the figure probes the best two (four) channels and then uses the best channel (among those probed) for transmission. Thus, comparing these strategies to the two-step lookahead strategy indicates the gain from using a more efficient probing algorithm over simple heuristics.

In all cases, these results confirm that the two-step lookahead policy performs very similarly and close to the optimal strategy, but with much less computational overhead. From the dynamic programming formulation given in (2), even when the channel rewards are discrete random variables, computing the optimal strategy at state (u, S) still requires us to take combinations of all subsets of S. By comparison, the two-step lookahead policy determines the optimal strategy by only considering the best two channels in S.

7. CONCLUSION

In this paper, we analyzed the problem of channel probing and transmission scheduling in wireless multichannel systems. We derived some key properties of optimal channel probing strategies, and showed that the optimal policy has a threshold structure and can only take one of a few forms. Using these properties, we proposed two channel probing algorithms which we showed are optimal for some cases of practical interest, including statistically identical channels, a few nonidentical channels, and a large number of nonidentical channels. These algorithms were also shown to perform very well compared to the optimal strategy under a practical class of channel models.

Future work includes extending results to cover regulatory spectrum policy constraints, such as the case where the transmitter can only retire to a subset of channels. It would also be interesting to simulate the proposed algorithms with realistic physical layer assumptions.

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9. APPENDIX

9.1 Proof of Lemma 1

It suffices to prove for all S and $0 \le u \le \tilde{u} \le M$:

$$V(\tilde{u}, S) - V(u, S) \le \tilde{u} - u \ . \tag{14}$$

If (14) is true, then $V(u, S) = u$ implies $V(\tilde{u}, S) \le \tilde{u}$. Combining this with (2), which says $V(\tilde{u}, S) \ge \tilde{u}$, proves the lemma. We prove (14) by induction on the cardinality of S.

Induction Basis: Consider any $S \subseteq \Omega$ such that $|S| = 1$. Let $j = S$. From equation (2) defining the value function, $V(\tilde{u}, \{j\})$ (simplified as $V(\tilde{u}, j)$ below) has three possible values. We show (14) holds for all three cases.

Case 1: $V(\tilde{u}, j) = \tilde{u}$. From (2), $V(u, j) \ge u$. Therefore, equation (14) easily follows.

Case 2: $V(\tilde{u}, j) = -c_j + E[\max(X_j, \tilde{u})]$. From (2), we have $V(u, j) \ge -c_j + E[\max(X_j, u)]$. Therefore,

$$\begin{aligned} V(\tilde{u}, j) - V(u, j) &\le E\left[\max(X_j, \tilde{u})\right] - E\left[\max(X_j, u)\right] \\ &= E[\tilde{u} - \max(X_j, u) | X_j < \tilde{u}] P(X_j < \tilde{u}) \le \tilde{u} - u \ , \end{aligned}$$

which proves that (14) holds.

Case 3: $V(\tilde{u}, j) = E[X_j]$. From (2), $V(u, j) \ge E[X_j]$. Therefore, equation (14) easily follows.

Induction Hypothesis: Consider any $S \subseteq \Omega$ such that $|S| \ge 2$ and suppose (14) holds for all $\tilde{S} \subseteq \Omega$ such that $|\tilde{S}| < |S|$. Again we prove (14) holds for all possible values of $V(\tilde{u}, S)$. If $V(\tilde{u}, S) = \tilde{u}$, then (2) implies $V(u, S) \ge u$ which implies (14) holds. Similarly, if $V(\tilde{u}, S) = E[X_j]$ for some $j \in S$, then $V(u, j) \ge E[X_j]$ implies (14). Finally, suppose $V(\tilde{u}, S) = -c_j + E[\max(X_j, \tilde{u})]$ for some $j \in S$. Then because $V(u, S) \ge -c_j + E[V(\max(X_j, u), S - j)]$ for this j, we have:

$$\begin{aligned} & V(\tilde{u}, S) - V(u, S) \\ & \le E\left[V(\max(X_j, \tilde{u}), S - j) - V(\max(X_j, u), S - j)\right] \\ & = E\left[\left(V(\tilde{u}, S - j) - V(\max(X_j, u), S - j)\right) I_{\{X_j < \tilde{u}\}}\right] \\ & \le V(\tilde{u}, S - j) - V(u, S - j) \le \tilde{u} - u \ , \end{aligned}$$

where the last two inequalities follow from (5) and the induction hypothesis, respectively. □

9.2 Proof of Part 1 in Theorem 1

We will prove the following lemma:

Lemma 6. *For any (u, S), $\pi^*(u, S) =$ retire(u) if and only if $u \ge \max_{j \in S} a_j$. Equivalently,*

$$a_S = \max_{j \in S} a_j \ . \tag{15}$$

Proof. We use proof by contradiction on two cases.

Case 1: Suppose $a_S < \max_{j \in S} a_j$. Equivalently, $a_S < a_k$ for at least one $k \in S$. Fix u such that $a_S < u < a_k$. By definition of a_S, we have $V(u, S) = u$. On the other hand, the definition of a_k and $u < a_k$ implies:

$$V(u, k) = \max\left\{E[X_k], -c_k + E\left[\max(X_k, u)\right]\right\} > u \ ,$$

Finally, (6) gives $V(u, S) \ge V(u, k) > u$, which contradicts the assumption that $u > a_S$. Thus, $a_S < a_k$ is not possible.

Case 2: Suppose $a_S > \max_{j \in S} a_j$. Fix any u such that $\max_{j \in S} a_j < u < a_S$. By definition of a_S, the optimal strategy at state (u, S) is to either probe or guess a channel in S, but retiring is not optimal. Suppose the optimal strategy is to probe $k \in S$. This implies:

$$\begin{aligned} -c_k + E\left[V(\max(u, X_k), S - k)\right] &\ge V(u, S) \\ &\ge V(u, S - k) \ , \end{aligned} \tag{16}$$

where the last inequality follows from (6). Since $u > a_k$, then by definition of a_k we have: $-c_k + E\left[\max(u, X_k)\right] < u$ Combining this with (16), we have:

$$\begin{aligned} E\left[V(\max(u, X_k), S - k) - V(u, S - k)\right] &> c_k \\ &> E\left[\max(u, X_k) - u\right] \end{aligned}$$

Conditioning the above expectations on events $\{X_k > u\}$ and $\{X_k \le u\}$ gives us:

$$\begin{aligned} E\left[V(X_k, S - k) - V(u, S - k) | \, X_k > u\right] P(X_k > u) \\ > E\left[X_k - u | \, X_k > u\right] P(X_k > u) \ , \end{aligned} \tag{17}$$

which contradicts (14).

If the optimal strategy is to guess a channel $k \in S$, then $V(u, S) = E[X_k]$. However, since $V(u, k) \le V(u, S)$ then $V(u, k) = E[X_k]$ as well. This implies $a_k \ge u$, which contradicts the assumption that $u > a_k$.

Combining Cases 1 and 2 proves Lemma 6. □

9.3 Proof of Part 2 in Theorem 1

We prove the following for any set of channels S.

Lemma 7. *For any S, define R, j^* as in Theorem 1. For all $\max_{j \in S} b_j < u < \max_{j \in S} a_j$, $\pi^*(u, S) =$probe(j^*).*

To prove this, we define and solve another problem.

No Guessing (NG) Problem: Consider Problem 1 with the following modification: at each step, the user must choose between the following *two* actions: (1) probe a channel that has not yet been probed, or (2) retire and use the best previously probed channel. Therefore, the user is not allowed to use a channel that has not yet been probed.

The NG Problem can be seen as a generalization of Theorem 4.1 considered in [3], which restricted X_j to be discrete random variables. To describe the theorem, we use the following notation for any channel $j \in \Omega$:

$$\bar{a}_j = \min\left\{u : c_j \geq E\left[(X_j - u)^+\right]\right\} \ , \qquad (18)$$

where $\bar{a}_j = 0$ if the above set is empty. Note that from equations (9) and (10), we see that $\bar{a}_j = a_j$ if and only if $a_j > b_j$. If $a_j = b_j$, then $\bar{a}_j < a_j$. We use these indices in the following theorem, which can be seen as a generalization of Theorem 4.1 in [3].

Theorem 5. *For state (u, S), the optimal strategy $\hat{\pi}$ for the NG Problem is described as follows:*

1. *If $u \geq \max_{j\in S}\{\bar{a}_j\}$, then $\hat{\pi}(u, S)$ =retire(u).*
2. *Otherwise, define R and j^* as in Theorem 1 by replacing a_j with $\bar{a}_j$ for each $j \in S$. Then $\hat{\pi}(u, S)$ =probe(j^*) .*

Due to space constraints, we provide a sketch proof. The proof that $\hat{\pi}(u, S)$ =retire(u) for all $u \geq a^*_j = \max_{j\in S}\{\bar{a}_j\}$ uses Lemma 6. For $u < \max_{j\in S}\{\bar{a}_j\}$, we use induction on the cardinality of S. It can be shown by computation that at $u = a_{j^*}$, probe(j^*) is optimal. Then, it can be shown that for all $u \leq a^*_j$, the difference between (1) the expected reward of probing j^* first and (2) probing any other channel k first, does not depend on u. Using this result, we have $\pi^*(a_{j^*}, S)$ =probe(j^*) implies $\pi^*(u, S)$ =probe(j^*) for all $u < a_{j^*}$.

Even though the NG Problem is different from Problem 1, its optimal strategy is also optimal for Problem 1 if guessing is not optimal for all future time steps. From Lemma 6 and Corollary 2, probing occurs if $\max_{j\in S} b_j < u < \max_{j\in S} a_j$. Thus, we have proven Lemma 7.

9.4 Proof of 3a, 3c in Theorem 1

To prove 3c, we prove the following lemma:

Lemma 8. *If $\pi^*(u, S)$ =probe(k) for some $u > 0$ and $k \neq j^*$, then $\pi^*(\tilde{u}, S)$ =probe(k) for all $0 \leq \tilde{u} \leq u$.*

Proof. It suffices to prove $\pi^*(u, S)$ =probe(k) implies:

$$E[V(\max(X_k, u), S - k)] = E[V(X_k, S - k)]. \qquad (19)$$

Proving that (19) suffices because if this equation were true, then it implies: $V(u, S) = -c_k + E[V(\max(X_k, u), S - k)] = -c_k + E[V(X_k, S - k)]$. However, the last term of this equation is the expected reward of probing channel k at state $(0, S)$. Since (5) implies that $V(0, S) \leq V(u, S)$, then probe(k) must be the optimal strategy at $(0, S)$. Similarly, using (5) again we have $\pi^*(\tilde{u}, S)$ =probe(k) all $0 \leq \tilde{u} \leq u$.

Now we prove (19) holds. In order for the optimal strategy at (u, S) to be probe(k), from Lemma 6 it must be true that $u < a_{j^*}$. We prove (19) by backward induction on the cardinality of S, starting with $|S| = 2$ as the lemma requires non-singleton S.

Induction Basis: Suppose $|S| = 2$. From Lemma 7, we know that $\pi^*(u, S)$ =probe(j^*) for all $\max\{b_{j^*}, b_k\} < u < a_{j^*}$. Suppose $\pi^*(u, S)$ =probe(k) for some $u \leq \max\{b_{j^*}, b_k\}$. We prove (19) for two cases.

Case 1: $b_{j^*} \leq b_k$. Taking the difference in expected reward between probe(j^*) and probe(k) for $b_k \leq u < a_{j^*}$, it is straightforward to show (similar to the proof of Theorem 5) that this difference is invariant to u. Since we know $\pi^*(u, S)$ =probe(j^*) for all $b_k < u < a_{j^*}$, then it must also be true that $\pi^*(b_k, S)$ =probe(j^*). It can be shown that $E\left[V\left(\max\{X_{j^*}, b_k\}, k\right)\right] = E\left[V\left(\max\{X_{j^*}, \tilde{u}\}, k\right)\right]$, for any $0 \leq \tilde{u} \leq b_k$, because $V(\tilde{u}, S)$ is constant when $\tilde{u} \leq b_k$. Thus, the expected reward of probe(j^*) is contant for all $\tilde{u} \leq b_k$. Combining this with (5) implies $\pi^*(\tilde{u}, S)$ =probe(j^*) for all $\tilde{u} \leq u$, which contradicts $\pi^*(u, S)$ =probe(k) for some $u > 0$.

Case 2: $b_{j^*} > b_k$. In this case, $\pi^*(u, S)$ =probe(k) for some $u \leq \max\{b_{j^*}, b_k\} = b_{j^*}$. On the other hand, by definition of b_{j^*}, we know that $\pi^*(u, j^*)$ =guess(j^*) for all $u \leq b_{j^*}$. Thus, $V(u, j^*) = E[X_{j^*}] = V(0, j^*)$, proving (19).

Induction Hypothesis: Consider any S such that $|S| = n$ for some $n > 2$. Suppose (19) is true for all $\bar{S} \subset S$ such that $2 \leq |\bar{S}| < n$. We prove (19) holds for S by contradiction.

Suppose $E[V(\max(X_k, u), S-k)] > E[V(X_k, S-k)]$. This implies by conditioning on $\{X_k \leq u\}$:

$$0 \ < E\left[\{V(u, S - k) - V(X_k, S - k)\}\, I_{\{X_k \leq u\}}\right] \qquad (20)$$

Note that if $V(u, S-k) = V(0, S-k)$, then by (5) we have $V(\tilde{u}, S - k) = V(0, S - k)$ for all $0 \leq \tilde{u} \leq u$. This would imply the righthandside of (20) is equal to 0, which contradicts the equation. Thus (20) implies that $V(u, S-_k) \neq V(0, S - k)$. From the induction hypothesis, this inequality means that $\pi^*(u, S-k)$ ≠probe(i) for any $i \neq j^*$. Similarly, $\pi^*(u, S - k)$ ≠guess(j^*) since this would imply $\pi^*(0, S - j^*)$ =guess(j^*), which would mean $V(u, S-k) = V(0, S - k)$. Finally, using $u < a_{j^*}$ implies that $\pi^*(u, S - k)$ =probe(j^*).

From repeating the argument, the optimal strategy is the following. First probe(k). If $X_k \geq a_{j^*}$, then retire. Otherwise, probe(j^*), and retire if $\max\{X_k, X_{j^*}\} \geq a_{j_2}$; otherwise probe$(j_2)$, where j_2 is the channel determined similarly to j^* in Theorem 1 but replacing S with $S - j^* - k$. This process continues until the transmitter retires. Thus, we see the optimal strategy either probes or retires, but never guesses a channel. However, from the NG problem and Theorem 5, we know that the strategy which first probes j^* obtains a higher expected reward than any strategy that probes k first and never guesses. Thus, it cannot be optimal to first probe k, which contradicts the assumption that $\pi^*(u, S)$ =probe(k).

Therefore, it must be true that $E[V(\max(X_k, u), S-k)] = E[V(X_k, S - k)]$, which completes the proof. □

Proving 3a can occur is straightforward. These results imply the optimal channel to probe is constant for $0 \leq u < d_S$.

9.5 Proof of 3b in Theorem 1

We first derive conditions for guessing to be optimal.

Lemma 9. *Given a set of unprobed channels S, define R as in Theorem 1. Then we have the following:*

1. *If there exists $j^* \in R$ such that $a_{j^*} > b_{j^*}$ and $b_{j^*} \leq \max_{j\in S-j^*} E[X_j]$, then $b_S = 0$.*
2. *If there exists $j^* \in R$ such that $b_{j^*} \geq \max_{j\in S-j^*} a_j$, then $b_S = b_{j^*}$.*

Proof. We prove the result for the two cases.

Case 1: For notation, let $E[X_k] = \max_{j\in S-j^*} E[X_j]$ for $j^* \in R$ satisfying $a_{j^*} > b_{j^*}$ as described by the lemma. From the lemma, we know that $b_{j^*} \leq E[X_k]$. If we can show that for every u, probing some channel in S or retiring is better than guessing any channel in S, then this will prove there exists an optimal strategy with $b_S = 0$. Note that the expected reward of guessing the best channel is $\max_{j\in S} E[X_j] = \max\{E[X_k], E[X_{j^*}]\}$. Thus it suffices to show that for all $u \leq a_{j^*}$, there exists a probing strategy with higher expected reward than $E[X_k]$ and $E[X_{j^*}]$.

As described in Section 3.1, $E[X_k] \le a_k$. Thus we have $b_{j^*} \le E[X_k] \le a_{j^*}$ since j^* is in R. From the definition of b_{j^*}, a_{j^*} and by the assumption that $a_{j^*} > b_{j^*}$, then $\pi^*(u, j^*)$ =probe(j^*) whenever $b_{j^*} \le u \le a_{j^*}$. Therefore, $\pi^*(E[X_k], j^*)$ =probe(j^*) and we have: $V(E[X_k], j^*) = -c_{j^*} + E[\max(X_{j^*}, E[X_k])] \ge E[X_k]$, However, note that the lefthandside of this equation is the expected reward of the following strategy: probe j^* first, and use this channel for transmission if its value is higher than $E[X_k]$; if its value is lower than $E[X_k]$, then guess(k), i.e. use channel k for transmission. Thus the expected reward of this two-step strategy is always at least the reward of simply using channel k for transmission. This result holds for all u. Also, by definition of a_{j^*} and b_{j^*}, $\pi^*(u, j^*)$ =guess(j^*) for all $u < b_{j^*}$. Thus $V(u, j^*) = E[X_{j^*}]$ for all such u. However, from (5), we also know that $V(E[X_k], j^*) \ge V(u, j^*) = E[X_{j^*}]$. Thus we have shown that for all u, there exists a strategy of probing j first which does at least as good as the strategy of guess(k) or guess(j^*), which are the only two possible guessing actions. Thus, there exists an optimal strategy which never guesses for all u, i.e. $b_S = 0$.

Case 2: From the lemma, $a_{j^*} \ge b_{j^*} \ge \max_{j \in S - j^*} a_j$. In addition, from Lemma 6 and from the threshold properties described in Section 3.1, we have $b_S \le a_S = a_{j^*}$. Thus, $V(u, b_S) = u$ for all $u \ge a_{j^*}$. Now we have two cases for the relationship between a_{j^*} and b_{j^*}. First, suppose $a_{j^*} = b_{j^*}$. From equations in Section 3.1, we see that this equality implies that $E[X_{j^*}] = b_{j^*}$. Finally, using (5), we see that $V(u, S) \le V(b_{j^*}, S) = b_{j^*} = E[X_{j^*}]$ for all $u < b_{j^*}$. Combining this with (2) we have $b_S = b_{j^*}$.

Now suppose $a_{j^*} > b_{j^*}$, which implies $a_{j^*} > \max_{j \in S - j^*} a_j$. From Lemma 7, $\pi^*(u, S)$ =probe(j^*) for all $a_{j^*} > u \ge b_{j^*}$. Thus, $V(b_{j^*}, S) = -c_{j^*} + E[\max(X_{j^*}, b_{j^*})]$, where we do not probe anything after j^* because of Lemma 6. From Section 3.1, we know $a_{j^*} > b_{j^*}$ implies $-c_{j^*} + E[\max(X_{j^*}, b_{j^*})] = E[X_{j^*}]$. Thus, $V(b_{j^*}, S) = E[X_{j^*}]$, which again implies that $V(u, S) = E[X_{j^*}]$ for all $u < b_{j^*}$. Thus we have shown there exists an optimal strategy with $b_S = b_{j^*}$. □

Thus conditions 1) and 2) of the lemma provide separate necessary and sufficient conditions for guessing to be optimal. Note that this lemma also has further implications. When $|R| \ge 2$, and $a_j = b_j$ for at least one $j \in R$, then condition 2) of Lemma 9 is always satisfied. Thus $b_S = b_j$ in this case. Otherwise, $a_j > b_j$ for all $j \in R$ and condition 1) of Lemma 9 is always satisfied.

When $|R| = 1$ and letting $j^* = R$, suppose $\pi^*(u, S) =$ guess(k) for some $k \ne j^*$, $u > 0$. This implies $E[X_k] > E[X_{j^*}] \ge b_{j^*}$, which leads to condition 1) of Lemma 9. This lemma implies $b_S = 0$, which contradicts $\pi^*(u, S) =$ guess(k). Thus, if $|R| = 1$ then $\pi^*(u, S) \ne$guess(k) for $k \notin R$. This leads to the following corollary:

Corollary 2. *Given a set S, define R as in Theorem 1. Then if $|R| \ge 2$ and $a_j = b_j$ for at least one $j \in R$, then $b_S = b_j$. Otherwise, $b_S = 0$. If $|R| = 1$, let $\{j^*\} = R$. Then $\pi^*(u, S) \ne$guess(j) for all u and $j \in S - j^*$.*

Finally, Lemma 8 shows $b_S > 0$ implies $\pi^*(u, S) \ne$probe(k) for all $k \ne j^*$ and u, where j^* is defined in Theorem 1.

9.6 Proof of Theorem 3

When probing costs are equal for all channels, this theorem follows from Corollary 2 since all channels in S are statistically identical, $R = S$, and $|R| = |S| \ge 2$

When probing costs differ between channels, we can use induction on the cardinality of S to prove the result.

Induction Basis: Suppose $|S| = 2$. From Theorem 2, the strategy given in Theorem 3 is optimal.

Induction Hypothesis: Consider any $S \subseteq \Omega$, where $|S| \ge 3$, and suppose the theorem holds for all $\tilde{S} \subseteq \Omega$ such that $|\tilde{S}| < |S|$. For notation, let $S = \{j_1, j_2, \cdots, j_n\}$ where $c_{j_1} \ge c_{j_2} \ge \cdots \ge c_{j_n}$. Note that from the discussion in Section 3, when $c_j \le c_k$ but X_j and X_k have the same distribution, then $a_j \ge a_k$ while $b_j \le b_k$. Thus, $a_{j_n} \ge a_{j_{n-1}} \ge \cdots \ge a_{j_1}$ and $b_{j_n} \le b_{j_{n-1}} \le \cdots \le b_{j_1}$.

From Lemma 6, we know $a_S = a_{j_n}$. Thus, it only remains to determine $\pi^*(u, S)$ for $u < a_{j_n}$.

Case 1: Suppose $a_{j_n} > b_{j_n}$. From Corollary 2, only channel j_n can be guessed. However, because $b_{j_n} \le b_{j_{n-1}}$, then Lemma 9 implies $b_S = 0$. Thus, we only need to decide which channel to probe when $u < a_{j_n}$.

We derive π^* for two separate subcases. First, suppose $a_{j_l} > b_{j_l}$ for all $1 \le l \le n$. In particular, this implies $a_{j_n} > b_{j_1}$. Let $V^*(u, S)$ denote the expected reward of the following strategy: first probe j_n and then proceed according to π^* as determined by the induction hypothesis. Meanwhile, let $H(u, S)$ denote the expected reward of first probing some channel j_k, where $k < n$, and proceeding according to π^*. For any $b_{j_1} \le u < a_{j_n}$, it can be shown similar to the proof of Theorem 5 that $V^*(u, S) - H(u, S)$ is invariant to u. However, from Lemma 7 we know $\pi^*(u, S)$ =probe(j_n) for all $b_{j_1} \le u < a_{j_n}$. Thus, $V^*(u, S) > H(u, S)$ for these u. Combining everything implies $\pi^*(b_{j_1}, S)$ =probe(j_n) and $V(b_{j_1}, S) = V^*(b_{j_1}, S)$. Finally, it can be easily shown that $V^*(u, S) = V^*(b_{j_1}, S)$ for any $u < b_{j_1}$. From (5), this implies $V(u, S) = V^*(u, S)$ for all $u < b_{j_1}$; therefore, $\pi^*(u, S)$ =probe(j_n) for all $u < b_{j_1}$.

Now suppose $a_{j_l} = b_{j_l}$ for some $1 \le l < n$ (we let l denote the largest index satisfying $a_{j_l} = b_{j_l}$). Consider probing any channel j_k where $k < n$ and $k \ne l$. Then from the induction hypothesis, after probing j_k we will either retire or continue to probe channels in decreasing order of the indices $\{a_{j_n}, a_{j_{n-1}} \cdots, a_{j_l}\}$. If the state is reached where channel j_l has the highest index value, then from the induction hypothesis we will retire if $\max\{X_{j_k}, X_{j_n}, \cdots, X_{j_{l+1}}\} \ge a_{j_l}$; otherwise, the optimal action is guess(j_l) which collects a reward of $E[X_{j_l}]$. Since j_l is never probed, the total expected reward of this strategy is exactly the same as the reward of a strategy in the NG Problem where initially $u = E[X_j]$, channels are probed in the order: $\{j_k, j_n, \cdots, j_{l+1}, j_l\}$ and retirement occurs according to Theorem 5. Similarly, first probing j_n has the same expected reward as a strategy in the NG Problem that probes channels in the order $\{j_n, j_{n-1}, \cdots, j_l\}$ and retires according to Theorem 5, where again $u = E[X_j]$. Thus, we can use Theorem 5 to show that the latter strategy must have higher expected reward. Similar steps can be used to show that probe(j_l) cannot be optimal for any u.

Case 2: Suppose $a_{j_n} = b_{j_n}$. Then Corollary 2 implies $\pi^*(u, S)$ =guess(j_n) for all $u \le a_{j_n} = b_{j_n}$. □