On the Many-to-One Transport Capacity of a Dense Wireless Sensor Network and the Compressibility of Its Data *

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Abstract. In this paper we investigate the capability of large-scale sensor networks to measure and transport a two-dimensional field. We consider a data-gathering wireless sensor network in which densely deployed sensors take periodic samples of the sensed field, and then scalar quantize, encode and transmit them to a single receiver/central controller where snapshot images of the sensed field are reconstructed. The quality of the reconstructed field is limited by the ability of the encoder to compress the data to a rate less than the single-receiver transport capacity of the network. Subject to a constraint on the quality of the reconstructed field, we are interested in how fast data can be collected (or equivalently how closely in time these snapshots can be taken) due to the limitation just mentioned. As the sensor density increases to infinity, more sensors send data to the central controller. However, the data is more correlated, and the encoder can do more compression. The question is: Can the encoder compress sufficiently to meet the limit imposed by the transport capacity? Alternatively, how long does it take to transport one snapshot? We show that as the density increases to infinity, the total number of bits required to attain a given quality also increases to infinity under *any* compression scheme. At the same time, the single-receiver transport capacity of the network remains constant as the density increases. We therefore conclude that for the given scenario, even though the correlation between sensor data increases as the density increases, any data compression scheme is *insufficient* to transport the required amount of data for the given quality. Equivalently, the amount of time it takes to transport one snapshot goes to infinity.

1 Introduction

In this paper we investigate the ability of a dense wireless sensor network to measure and transport independent *snapshots* of a two-dimensional field to a central location, i.e. a *collector*, where reconstructions of these field snapshots are formed.

More specifically, N sensors are uniformly spaced over some finite geographical region. At regular time intervals, each sensor measures the field value at its

* This work is supported by NSF ITR grant number ANI-0112801.

location; then quantizes its value and losslessly encodes it with bits. The wireless network, which has a transceiver at each sensor, operates in slotted time steps to transport the bits generated by the sensor encoders to the central collector. Multiple hops may be required. There is a number W such that each sensor can transmit or receive at most W bits in one slot. Note that because a sensor value is known only at its own location, the quantization and encoding must be done independently at each sensor location.

When the central collector has received from each sensor the encoded quantized value corresponding to a particular sampling time, i.e. corresponding to one complete snapshot, it forms a reconstruction of that snapshot. The sampling and data transport are pipelined in the sense that further snapshots may be taken by the sensors and their transport may begin before the network has finished transporting prior snapshots to the collector.

The principal question to be addressed is how frequently can a new snapshot be taken and transported successfully to the collector. If new snapshots can be received by the collector every u slots, then we say the network has *throughput* $1/u$ snapshots per slot. Clearly, large throughput is desired. Alternatively, one may ask how many network slots are needed (i.e. how many times the network must be used) to transport a snapshot. If new snapshots can be received by the collector every u slots, then we say the network has *usage rate* u slots per snapshot, which is the inverse of the throughput. Clearly, small usage rate is desired.

One might also ask how much time must transpire between the time the snapshot is taken by the sensors and the time the collector has the data needed for its reconstruction. This *delay* will not be discussed here, except to say that due to pipelining the usage rate is at most as large as the delay, and usually substantially smaller.

We are particularly interested in how the network throughput and usage rate vary as N, the number of sensors, increases. Of course, the sensor spacing decreases with N, and the sensor density increases with N. Must the usage rate (i.e. the number of slots/snapshot) increase with N? If so, does it saturate at some finite value? Or does it increase without bound?

To answer these questions, one must answer a *compressibility* question and a *capacity* question: How many bits must be generated by each sensor's quantizer/encoder per snapshot? And how many bits can be transported on the average by the network to the collector per sensor per slot? (Here, we only count new bits generated at the sensors – not bits relayed by the sensors.) Suppose the answer to the compressibility question is b_N, i.e. b_N is the minimum number of bits per sensor per snapshot that must be generated for a network of size N, and suppose the answer to the capacity question is c_N, i.e. c_N is the maximum average number of bits that can be transported to the collector per sensor per slot. (c_N is less than W – usually much less.) Then the smallest possible usage rate is $u_N = b_N/c_N$ slots/snapshot. Equivalently, the maximum possible throughput is $t_N = c_N/b_N$ snapshots/slot.

To answer the capacity question, we adopt a transmission and interference model similar to that of Gupta and Kumar [1], and we show in Section 3 that

$$c_N = \Theta\left(\frac{1}{N}\right) \text{ bits/sensor/slot }, \tag{1}$$

where $\Theta(\frac{1}{N})$ means there exist constants a_1 and a_2 such that $\frac{a_1}{N} \leq c_N \leq \frac{a_2}{N}$ for sufficiently large N. That is, c_N, which may be considered to be the *many-to-one capacity* of the network, is bounded. This is essentially due to the fact that the number of bits per slot that the collector can receive is bounded by W. As a result, there is a bottleneck at the collector. In comparison, Gupta and Kumar [1] found the *peer-to-peer capacity* of a similar network to be $c_N = \Theta\left(\frac{1}{\sqrt{N \log N}}\right)$.

On the other hand, the compressibility question is not well posed until one specifies a model for the two-dimensional field being measured and the criteria with which the fidelity of the reconstructed snapshots are judged. These are described in the next two paragraphs.

The model for the field is a stationary two-dimensional, random field $X(u, v)$. That is, $X(u, v)$ is a real-valued random variable representing the field value at Euclidean coordinates (u, v), where u and v vary continuously. We make only benign assumptions about the random field. We make no assumption as to whether the random field is bandlimited or not (bandlimited refers to spatial frequency content). A principal characteristic of the random field is its autocorrelation function $R(\tau_1, \tau_2)$, which indicates the correlation between values of X separated horizontally and vertically by distances τ_1 and τ_2, respectively. For example, $R(\tau_1, \tau_2) = \exp\left\{-\sqrt{\tau_1^2 + \tau_2^2}\right\}$ is an example of an isotropic autocorrelation function that decays exponentially with Euclidean distance. We require that the autocorrelation function not be a constant, i.e. the field cannot be spatially constant, even if the constant is random. Finally, we assume that successive snapshots are independent. That is, each snapshot is modeled as a random field that is independent of the random fields modeling other snapshots.

In effect, the sensors take samples of the random field at locations denoted $(u_1, v_1), (u_2, v_2), ..., (u_N, v_N)$. It is these samples that are quantized, encoded and transported to the collector. The collector creates a reproduction $\widehat{X}(u, v)$, $(u, v) \in G$ as a reproduction of the original snapshot $X(u, v)$, $(u, v) \in G$, where G denotes the geographic region of interest over which the sensors are dispersed. This obviously involves interpolation. We quantify the fidelity of the $\widehat{X}$ reproduction with mean squared error:

$$\text{MSE} = \frac{1}{|G|} \int_G E\left(X(u, v) - \widehat{X}(u, v)\right)^2 du\, dv \ , \tag{2}$$

where E denotes expected value with respect to the random field, the integral is taken over the region G, and $|G|$ denotes its area. Note that due to interpolation and quantization errors, it is not possible to have MSE $= 0$. Therefore, the sensor network performs, in effect, lossy, rather than lossless coding of the random field. (Sampling, followed by scalar quantization and lossless binary encoding is a common method of lossy coding.) When N is large and, consequently, the sensors are closely spaced, the component of MSE due to interpolation error is negligible, and the MSE is well approximated simply by the average MSE between the N sensor samples and their reconstructions. That is,

$$\text{MSE} \cong \frac{1}{N} \sum_{i=1}^{N} E \left(X(u_i, v_i) - \widehat{X}(u_i, v_i) \right)^2 . \tag{3}$$

From now on, we will fix a positive number D, and assume throughout the paper that the goal of the sensor network is to sample, quantize, encode and transport snapshots of the field with a mean squared error of D or less, as given by (2) or (3).

We will assume also that the quantizers used by the various sensors are identical. Every such quantizer maps a sensor value $X(u_i, v_i)$ to an integer that indexes the possible quantization cells/bins. This index is then encoded in some lossless fashion. Though only the X's at the sensor locations will be quantized, we nevertheless need to assume that the random field and quantizer are such that the probability that each $X(u, v)$ in the entire region of interest G would quantize to the same integer is less than one. (Equivalently, the probability that all $X(u, v)$'s are in the same quantization cell is less than one.) This is another benign assumption, because if it does not hold, i.e. if with probability one all X's lies in the same quantization cell, then clearly the quantizer is too coarse to be of use.

We can now pose the compressibility question. With the above models for the random field and the fidelity measure, and with a fixed MSE target D, then as discussed in Section 2, one may show that $b_N \longrightarrow 0$ as $N \longrightarrow \infty$, where b_N is the minimum number of encoded bits per sensor per snapshot that must be transported to the collector to attain MSE less than or equal to D. The idea is that as N increases, the sensors become increasingly close, the correlation between the values produced by nearby sensors increases, and it is possible to exploit this correlation using schemes such as conditional coding or Slepian-Wolf distributed lossless coding[1] on the quantizer outputs to make $b_N \longrightarrow 0$. On the other hand, although $b_N \longrightarrow 0$, we also show in Section 2 that no matter how the lossless coding is done, b_N does not decrease as rapidly as $1/N$. That is,

$$N b_N \longrightarrow \infty \text{ as } N \longrightarrow \infty . \tag{4}$$

Note that $N b_N$ is the total number of bits coming from the quantizer/encoders from all sensors. Note also that the above result is quite general and not limited to a particular lossless coding scheme.

[1] Slepian-Wolf coding is a remarkable method that permits lossless coders to independently encode the data from correlated sources (such as the data produced by neighboring sensors) as efficiently as if each encoder could see the values produced by the other data sources. Also, note that Slepian-Wolf coding entails the simultaneous encoding of a block of successive outputs from the quantizer of a given sensor.

Combining (4) with the many-to-one capacity result (1), we find that the smallest usage rate for which the mean squared error can be D or less is

$$U(N,D) = \frac{b_N}{c_N} = \frac{N b_N}{N c_N} \longrightarrow \infty \text{ as } N \longrightarrow \infty \ . \tag{5}$$

This indicates that to obtain a given MSE D, the number of slots per snapshot must grow without bound as N increases.

It must be said that this is somewhat disappointing, as it had been hoped that as N increases, the inter-sensor correlation would increase sufficiently rapidly to make Nb_N (and $U(N,D)$) saturate at a finite value, rather than approach infinity. Note, however, that this result does not say that sensor networks cannot do the desired job of measuring and transporting a two-dimensional field. Rather it says that the efficiency with which it does so, as expressed by the usage or throughput, degrades as the density of the sensors becomes very large.

It should be noted that the efficiency also degrades when N becomes too small. Specifically, there is some threshold value N_o such that for $N < N_o$, the interpolation error by itself exceeds D. Thus, there is no quantization-encoding-transport scheme that attains MSE D. Moreover, as N approach N_o from above, the quantizer must have increasingly fine resolution, which causes $b_N \longrightarrow \infty$. And since in this case $N > N_o$, we also have $Nb_N \longrightarrow \infty$. Thus as in (5), $U(N,D) \longrightarrow \infty$ as $N \searrow N_o$. We conclude that given a target MSE D and a random field model, there is an optimum value of N. This is the value for which Nb_N is smallest. This conclusion applies to bandlimited and non-bandlimited fields alike. For bandlimited fields the optimum value of N is not necessarily the value that leads to Nyquist sampling.

Based on the above analysis, an alternative strategy, to be pursued in future work, is to fix the number of sensors at the value of N that minimizes Nb_N, and then to permit there to be an additional set of transceivers at locations between the sensors. This is equivalent to having a network of $N' > N$ sensors, and putting all but N of them to sleep, while keeping all transceivers active.

We assert that the result in (4) is not at all obvious. Indeed, the limiting behavior of Nb_N has been a long standing question in the theory of sampling and quantization, which has only recently been resolved in [2]. The discussion we give in Section 2 is for one-dimensional random processes, but clearly extends to two-dimensional random fields as well. To see just how delicate the question is, in Section 2, we discuss how the rate-distortion theory branch of information theory shows that if *ideal* lossy coding were used instead of scalar quantization plus binary lossless coding, then Nb_N would not increase to infinity. However, the sensor network requires that coding be done independently at each sensor. This is why we use scalar quantization, rather than say vector or predictive quantization. On the other hand, it can be shown that even if one were allowed to use vector quantization, unless the dimension of the quantizer increases with N, Nb_N would still grow without bound.

Having shown that Nb_N grows to infinity, the question arises as to how fast it grows. In Section 2, we find the rate with which Nb_N increases for the special case of Gaussian random fields and a particular form of Slepian-Wolf coding. This also leads to a result on how fast $U(N, D)$ grows in this special case. Specifically, for a one-dimensional Gaussian field with exponential autocorrelation, it is shown that $U(N, D) \longrightarrow \infty$ at rate $\Theta(\sqrt{N} \log N)$.

In addition to the many-to-one capacity, we also consider the *many-to-many capacity*, which is the maximum average number of bits per sensor per slot that can be transported from each sensor to every other sensor. Section 3 shows that the many-to-many capacity is:

$$c_N = \Theta\left(\frac{1}{N}\right) \text{ bits/sensor/slot} . \tag{6}$$

This is the same as the many-to-one capacity. Thus the behavior of a network operating in many-to-many fashion, e.g. the asymptotic usage rate $U(N, D)$ is the same as the behavior of a network operating in many-to-one fashion.

We conclude this introduction with a comment on the results of a recent paper by Scaglione and Servetto [3]. The latter appears to claim that as N increases, the capability of the dense sensor network and the correlation structure of a typical random field are sufficient to permit any node to obtain the two-dimensional field quantized to within any prescribed distortion value. (It focuses on the many-to-many scenario.) If by such sufficiency the paper means to say that this can happen with bounded network usage (i.e., the number of slots per snapshot does not go to infinity), then our results show otherwise. That is, the number of slots needed between successive snapshots does indeed grows without bound. If such sufficiency does not involve any notion of time, then it is not clear to us what the claim means. The paper's intermediate results seem to indicate that a network can transport the field in $\Theta(\sqrt{N})$ slots, which is unbounded. Therefore its overall claim of sufficiency does not appear to match this result. Furthermore, the $\Theta(\sqrt{N})$ result (Equation (1) in [3]) is based on the assumption that the information theoretic rate-distortion function is attainable. However, in a sensor network, quantization must be done independently at each node, and our results show that in this case the ratio of the number of required encoded bits to the rate-distortion function approaches infinity. Therefore, the $\Theta(\sqrt{N})$ result is also in doubt.

The remainder of the paper is organized as follows. The next section presents the results on the number of bits b_N resulting from quantizing and encoding the sensor samples. Section 3 derives the many-to-one and many-to-many transport capacity of the sensor network when N is large. Section 4 summarizes and concludes.

2 The Compressibility of Sensor Data

We need to assess the minimum number of bits that an encoder could produce when encoding a quantized sensor value, when sensors are densely placed, and consequently, their values are highly correlated. We will summarize and use the recent results of [2].

As stated in the introduction, we view the sensors as taking uniformly spaced samples of a stationary two-dimensional random field over a finite geographical region. The collection of all samples taken at one time instance form a snapshot. Successive snapshots are assumed to be independent.

Though the field is two-dimensional, the basic ideas are more readily apparent and simpler to describe in one dimension. Therefore, we will focus on the case that N sensors are uniformly spaced on a straight line of length $L < \infty$. In this case, let $X(s)$, $0 \le s \le L$ denote the field value at location s. $X(s)$ is assumed to be a continuous parameter stationary random process. Let $(X_1, \dots, X_N)$ denote the N sensor values taken at a spacing of $d = L/N$. Let $(I_1, \dots, I_N)$ denote the integers resulting from quantizing $(X_1, \dots, X_N)$ with some fixed quantizer.

2.1 $b_N \longrightarrow 0$

From basic information theory we know that no lossless compression technique could compress the output of the quantizer with fewer than

$$H(I_1, \dots, I_N) \text{ bits.} \tag{7}$$

Equivalently, it requires on average at least

$$\frac{1}{N} H(I_1, \dots, I_N) \text{ bits per sample} \tag{8}$$

to losslessly encode each quantized sensor value.

The lower bound in (7) can in fact be attained using Slepian-Wolf distributed lossless coding. This requires every sensor to simultaneously encode a block of, say, M successive outputs from *its* quantizer. Observe that the block of outputs is a temporal block rather than a spatial one. Temporal blocks are needed in order for the encoder, at each sensor, to operate at rate close to some conditional entropy value (these conditional entropies will be stated shortly). Spatial blocks, however, are not used since every sensor knows only its own values and so the quantization and encoding must be done independently at each sensor.

The lower bound in (7) is attained in the following way. Let all sensors quantize their values independently. Let sensor 1 losslessly encode its block of M successive quantizer outputs into approximately $MH(I_1)$ bits using conventional block lossless coding[2], where $H(I_1)$ denotes the entropy of one of its quantizer outputs, and where the independence of successive outputs has been used. Let sensor 2 encode its values using Slepian-Wolf style coding with respect to sensor 1. Then, it losslessly encodes its block of M successive quantizer outputs into approximately $MH(I_2|I_1)$ bits, where $H(I_2|I_1)$ denotes the conditional entropy

[2] This and subsequent similar approximations can be made arbitrarily tight by choosing M large. Moreover, this and subsequent block encodings are *nearly* rather than *perfectly* lossless, meaning that there is a nonzero probability that the decoder output does not match the encoder input. However, such decoding error probabilities can be made arbitrarily small by choosing M large, thereby having negligible effect on the overall MSE.

of an output of sensor 2 given an output of sensor 1 in the same snapshot. (The decoder will already have decoded the I_1's, before decoding the I_2's.) Similarly, sensor 3 uses Slepian-Wolf coding with respect to sensors 1 and 2, thus mapping its M quantizer outputs into approximately $MH(I_3|I_2, I_1)$ bits. And so on. It follows that for the kth sensor, the number of bits per snapshot generated by its quantizer/encoder is approximately $b_N(k) = H(I_k|I_1, \ldots, I_{k-1})$. It is well known that $b_N(k)$ decreases monotonically with k. Thus, for large N, most of the $b_N(k)$'s are approximately the same. That is, there is a value b_N such that $b_N(k) \cong b_N$ for most k. It is this value to which Section 1 refers when prescribing the number of bits per sensor per slot produced by each sensor's quantizer/encoder.

It also follows that the total number of bits B_N produced by all the sensors is given by:

$$\begin{aligned} B_N &= \sum_{k=1}^{N} b_N(k) \\ &= H(I_1) + H(I_2|I_1) + \ldots + H(I_N|I_{N-1}, I_{N-2}, \ldots, I_1) \\ &= H(I_1, \ldots, I_N) \ , \end{aligned} \tag{9}$$

where the last equality is an elementary property of entropy. This shows that the Slepian-Wolf approach does indeed attain the lower bound in (7).

We now show $b_N \longrightarrow 0$ as $N \longrightarrow \infty$. Using elementary information theory relations,

$$\begin{aligned} b_N &= \sum_{k=1}^{N} b_N(k) \\ &= \frac{1}{N} \sum_{k=1}^{N} H(I_k|I_{k-1}, I_{k-2}, \ldots, I_1) \\ &\leq \frac{1}{N} \sum_{k=1}^{N} H(I_k|I_{k-1}) \\ &= \frac{H(I_1)}{N} + \frac{(N-1)}{N} H(I_2|I_1) \\ &\longrightarrow H(I_2|I_1) \text{ as } N \longrightarrow \infty \ . \end{aligned} \tag{10}$$

As N increases the sensors become closer and closer. Consequently their correlation increases. Specifically, as $N \longrightarrow \infty$, the distance between sensors 1 and 2 goes to zero. Thus their sample values become essentially identical resulting in $H(I_2|I_1) \longrightarrow 0$, which in turn implies that $b_N \longrightarrow 0$.

2.2 $Nb_N \longrightarrow \infty$

It has recently been shown [2] that $H(I_1, \ldots, I_N) \longrightarrow \infty$ as $N \longrightarrow \infty$. The following briefly sketches the basic idea. Let T_o denote a quantization threshold that $X(s)$ crosses with probability one in the interval $[0, L]$. Let S_o denote the location of the first crossing of this threshold. The assumptions in Section 1 about the random field and quantizer insure the existence of T_o. Furthermore, they imply that S_o is a continuous random variable, thus having infinite entropy. When N is large and consequently the sample spacing d is small, from the quantizer outputs $(I_1, \ldots, I_N)$, one can immediately and easily determine in which time interval of length d the first threshold crossing occurs. Thus one obtains an estimate $\widehat{S}_o$ of S_o that is accurate to within d. Since $d \longrightarrow 0$ as $N \longrightarrow \infty$ and since S_o has infinite entropy, it follows from elementary information theory that the entropy $H(\widehat{S}_o)$ tends to infinity. Finally, since $\widehat{S}_o$ is a function of $(I_1, \ldots, I_N)$,

$$H(I_1, \ldots, I_N) \geq H(\widehat{S}_o) \longrightarrow \infty \ . \tag{11}$$

Since from (7) $B_N = Nb_N$ can be no smaller than $H(I_1, \ldots, I_N)$, we see that $Nb_N \longrightarrow \infty$.

This argument can be generalized to the case of a two-dimensional field. We note also that if the snapshots of the field were dependent, it can be shown that using an encoding scheme that encodes based on previous snapshots will do no better.

2.3 The Growth of Rate For a Gaussian Random Field

As mentioned, although the encoding of the sensor value X_i must be done without knowledge of the other sensor values with which it is correlated, one could nevertheless losslessly encode it with approximately $b_N = \frac{1}{N} H(I_1, \ldots, I_N)$ bits, provided Slepian-Wolf distributed coding is used [4]. A suboptimal but easier to analyze case is where Slepian-Wolf coding is used to encode each sensor value with approximately $b_N = H(I_2|I_1)$ bits. For this situation, it has been shown in [2] that when $X(s)$ is a stationary Gaussian random process and the scalar quantizer is uniform with step size Δ and an infinite number of levels, then

$$\lim_{\rho \to 1} \frac{H(I_2|I_1)}{-\sqrt{1-\rho^2}\, \log \sqrt{1-\rho^2}\, M_{\sigma,\Delta}} = 1 \tag{12}$$

where ρ is the correlation coefficient of X_1, X_2 and $M_{\sigma,\Delta}$ is a constant that depends on the variance σ^2 of $X(s)$ and the quantization step size Δ.

Let us consider now, as examples, two autocorrelation functions for the random process $X(s)$. To keep notation simple, let the process $X(s)$ have unit variance.

1. $R_X(s) = e^{-|s|}$: The correlation coefficient in this case is $\rho = e^{-d}$, recalling that $d = L/N$ is the spacing between adjacent sensors. It follows from the usual expansion of the exponential that $\sqrt{1-\rho^2} \longrightarrow \sqrt{2d}$ as $d \longrightarrow 0$. Therefore, (12) can be rewritten as follows:

$$\lim_{d \to 0} \frac{H(I_2|I_1)}{-\sqrt{2d}\, \log \sqrt{2d}\, M_{1,\Delta}} = 1 \ . \tag{13}$$

Consequently for large N,

$$B_N \approx -N\sqrt{2d}\,M_{1,\Delta}\ \log\sqrt{2d} = \sqrt{2L}\,M_{1,\Delta}\ \sqrt{N}\log\sqrt{\frac{N}{2L}} \longrightarrow \infty \text{ as } N \longrightarrow \infty\,. \tag{14}$$

In this case, B_N increases as $\sqrt{N}\log N$.

2. $R_X(s) = e^{-s^2}$: The correlation coefficient in this case is $\rho = e^{-d^2}$. It follows from the usual expansion of the exponential that $\sqrt{1-\rho^2} \longrightarrow \sqrt{2}d$ as $d \longrightarrow 0$. Therefore, (12) can be rewritten as follows:

$$\lim_{d\to 0} \frac{H(I_2|I_1)}{-\sqrt{2}d\ \log\sqrt{2}d\,M_{1,\Delta}} = 1\,. \tag{15}$$

Consequently for large N,

$$B_N \approx -N\sqrt{2}d\,M_{1,\Delta}\ \log(\sqrt{2}d) = \sqrt{2}L\,M_{1,\Delta}\ \left(\log\frac{N}{\sqrt{2}L}\right) \longrightarrow \infty \text{ as } N \longrightarrow \infty\,. \tag{16}$$

In this case B_N increases as $\log N$.

In light of the previous discussion that the total number of bits must increase to infinity as N increases, it should not be surprising that (14) and (16) increase without bound as $N \longrightarrow \infty$. Note that in these examples the number of bits per sensor $b_N = B_N/N$ goes to 0.

On the other hand, suppose that instead of independently scalar quantizing each sensor value, a hypothetical omniscient encoder could jointly quantize a block of, say, K_N adjacent sensor values from the same snapshot. Then if K_N is permitted to grow with N, information theoretic rate-distortion theory can be used to show that B_N, the number of bits per snapshot required to attain a target MSE D, will remain bounded rather than grow to infinity. However, if K_N is not permitted to grow with N, then an argument like that used above for scalar quantization shows that B_N must again go to infinity. This indicates the criticality of the independent quantization/encoding requirement. Moreover, it indicates that even if the latter were not required, it would still be very difficult to have B_N remain bounded.

3 Transport Capacity

In this section we analyze the transport capacity of a network where communication is of a many-to-one fashion (or more specifically all-to-one in this case). This follows from the motivating application illustrated in Section 1 whereby all sensors send sampled data to a single collector/receiver. We will extend this analysis to discuss the capacity when the communication is of a many-to-many fashion as well. We present two types of results in this section. The first type of result is in the form of an upper bound, i.e., a level that the transport capacity cannot possibly exceed given our assumptions. The second type is in the form of a constructive lower bound, i.e., the transport capacity that is achievable via a particular construction of routing and scheduling mechanisms. These two results serve different purposes in this paper. The upper bound is used jointly with Sections 2.1 and 2.2 to show that the number of slots required per snapshot grows without bound. The lower bound is used jointly with Section 2.3 to characterize the usage rate in the special case of a Gaussian random field with known autocorrelation functions.

Capacity of wireless networks has attracted much attention in recent studies with the assumption that source traffic is uncorrelated. The seminal work by Gupta and Kumar [1] first developed the transport capacity of a wireless network where sources and destinations are randomly chosen. The main results of [1] state that the total transport capacity of a network of N nodes is $\Theta(\frac{\sqrt{N}}{\sqrt{\log N}})$. Equivalently, the per source transport capacity is $\Theta(\frac{1}{\sqrt{N \log N}})$. Both are throughput capacities in amount of data transported end-to-end per unit of time. The main difference of the scenario studied in this section is that there is a single receiver.

Throughout this section the transport capacity is defined in two ways, the *total transport capacity*, which is the total rate at which the network transports data to the single receiver, and the *per-node transport capacity*, which is the rate at which each sensor transports to the single receiver. When each sensor has equal amount of data to send these two definitions become equivalent. We will use terms *collector*, *sink*, and *receiver* interchangeably, and use terms *sensor*, *node*, and *source* interchangeably.

We assume that the network used for our calculations is deployed following a uniform distribution over a field of area A. For simplicity we also assume that this field has a circular shape and that the collector is located at the center of the field. We assume that the collector cannot simultaneously receive from multiple sensors. The sensors are stationary once deployed and cannot transmit and receive simultaneously. As mentioned before, time is slotted, and all nodes share a channel with capacity of W bits per slot. We assume nodes use omnidirectional antennas, and use a fixed transmission power and achieve a fixed transmission range, denoted by r. We use transmission and interference models similar to those used in [1]. Let X_i and X_j be two sources with distance $d_{i,j}$ between them. Then the transmission from X_i to X_j will be successful if and only if

$$d_{i,j} \leq r \ \text{ and } d_{k,j} > r + \delta, \quad \delta \geq 0 \qquad (17)$$

for any other source X_k that is simultaneously transmitting. Here δ denotes the interference range. We assume that the transmission range r is sufficiently large to guarantee connectivity with high probability.

3.1 Capacity Upper Bound

We first consider an obvious upper bound on the total transport capacity in the case of a single receiver. From the collector's point of view, the maximum rate of transport is achieved when it is receiving 100% of the time. Since W is the transmission capacity of the shared channel, it follows that the collector cannot receive at rate faster than W. We thus have the following result:

Theorem 1. *The total transport capacity in a wireless network featuring many-to-one communications is upper bounded by W.*

Equivalently, if each sensor sends an equal amount then the per-node transport capacity is upper bounded by $\frac{W}{N}$.

Note that this result is independent of the assumption of the shape of the field, the location of the collector and the interference model. It also is not an asymptotic result so it can be applied to networks with finite N. It is simply a (direct) consequence of the assumption that the collector cannot receive simultaneously from multiple sensors. In [5] we show that this upper bound is in general not achievable with high probability as the number of sensors increases to infinity.

We now extend the above result to the many-to-many case. More specifically we consider the all-to-all broadcast scenario where data generated at each sensor is to be delivered to all other sensors in the network. Note in this case there is not a single collector, but rather that every sensor is a collector. Again we note that receiving at a rate of W for a given sensor can only be achieved when the sensor is continuously receiving. This is clearly infeasible since each sensor also needs to transmit its own data. Thus in this case the total transport capacity is also upper bounded by W bits per slot. Here the transport capacity refers to the number of distinct bits delivered per slot, thus a bit that reaches multiple destinations (since each bit has a destination of all other sensors in the network) is not counted multiple times.

3.2 Achievable Capacity

In this subsection we show constructively that a transport capacity on the order of W (but less than W) can be achieved. Here we will explicitly assume that all nodes need to transmit the same number of bits, or need to achieve a same rate. This assumption coincides with the suboptimal encoding scheme in subsection 2.3 where each sensor value is encoded using approximately $b_N = H(I_i|I_{i-1}) = H(I_2|I_1)$ bits. Consequently we will determine the achievable per-node capacity or per-node throughput, denoted by λ, and then multiply this result by N to obtain the total transport capacity instead of considering the total transport capacity directly. The result here is obtained with high probability in the asymptotic regime as N goes to infinity. We assume that the area A contains at least a circular area of radius $2r + \delta$. This is not an unreasonable assumption since the range r required to maintain connectivity decreases as $N \longrightarrow \infty$. We begin with the following lemma.

Denote by A_R the area of a circle of radius R, i.e., $A_R = \pi R^2$. Let random variable V_R denote the number of nodes within an area of size A_R. We then have the following lemma.

Lemma 1. *In a randomly deployed network with N nodes,*

$$\mathit{Prob}\left(\frac{NA_R}{A} - \sqrt{\alpha_N N} \leq V_R \leq \frac{NA_R}{A} + \sqrt{\alpha_N N}\right) \longrightarrow 1 \quad \mathit{as} \quad N \longrightarrow \infty, \tag{18}$$

where the sequence $\{\alpha_N\}$ is such that $\lim_{N\to\infty} \frac{\alpha_N}{N} = \epsilon$, ϵ positive but arbitrarily small.

This result can be easily shown using Chebychev's inequality and noting that the mean of V_R is $\frac{NA_R}{A}$ and the variance σ^2 is $\frac{NA_R}{A}(1-\frac{A_R}{A})$:

$$\text{Prob}\left(\frac{NA_R}{A} - \sqrt{\alpha_N N} \leq V_R \leq \frac{NA_R}{A} + \sqrt{\alpha_N N}\right) \geq 1 - \frac{\sigma^2}{\alpha_N N} = 1 - \frac{\frac{A_R}{A}(1-\frac{A_R}{A})}{\alpha_N}. \tag{19}$$

The second term on the right hand side of (19) goes to zero since $\alpha_N \longrightarrow \infty$ as $N \longrightarrow \infty$.

This lemma shows that the number of nodes in a fixed area is bounded within $\sqrt{\alpha_N N}$ of the mean where α_N goes to infinity as $N \longrightarrow \infty$ but $\lim_{N\to\infty} \frac{\alpha_N}{N}$ is arbitrarily small.

Using this lemma, the following theorem constructs capacity that can be achieved with high probability as $N \longrightarrow \infty$ in the many-to-one case. Note that our result is as a function of the transmission range r and we have assumed that r is sufficiently large to guarantee connectivity. We construct this bound assuming that the routing and relaying scheme is such that each of the nodes one hop away from the sink carries an equal share of the total traffic. This is feasible given that the collector is at the center, the nodes are uniformly distributed and each sensor generates the same amount of bits.

Theorem 2. *A uniformly deployed network using multi-hop transmission for many-to-one communication can achieve per-node throughput $\lambda \geq \frac{W}{N} \frac{\pi r^2 - \sqrt{\epsilon}}{4\pi r^2 + 4\pi r\delta + \pi\delta^2 + \sqrt{\epsilon}}$ with high probability as $N \to \infty$, where ϵ is as given in Lemma 1.*

To see this, consider a source that is at least $2r+\delta$ away from the closest border of the network. The area of interference is thus a circle of radius $r' = 2r+\delta$ centered at this source. Using Lemma 1, with high probability the number of interfering neighbors including the source, k_1, is

$$\frac{NA_{r'}}{A} - \sqrt{\alpha_N N} \leq k_1 \leq \frac{NA_{r'}}{A} + \sqrt{\alpha_N N}. \tag{20}$$

Consider the entire network represented as a connected graph G(V,E), with edges connecting nodes that are within each other's interference range. Then the highest degree of this graph is $k_1 - 1$, since k_1 is the number of nodes within any interference area. Using the known result from graph theory, see for example [6, 7], that the chromaticity of such a graph is upper bounded by the highest degree plus one, i.e., $k_1 - 1 + 1 = k_1$ in this case, there exists a schedule of length at most $l \leq k_1$ slots that would allow all nodes to transmit at least once during this schedule. The nodes one hop away from the sink carry the traffic of the entire network. Denote the number of these one hop nodes by k_2, there thus exists a schedule of length l such that $\frac{N}{k_2}\lambda = \frac{W}{l}$. Note that k_2 is bounded with high probability by Lemma 1: $\frac{NA_r}{A} - \sqrt{\alpha_N N} \leq k_2 \leq \frac{NA_r}{A} + \sqrt{\alpha_N N}$. Therefore we have

$$\begin{aligned}
\frac{N}{\frac{NA_r}{A} - \sqrt{\alpha_N N}}\lambda \geq \frac{N}{k_2}\lambda = \frac{W}{l} \geq \frac{W}{k_1} \geq \frac{W}{\frac{NA_{r'}}{A} + \sqrt{\alpha_N N}} \\
\frac{1}{\frac{A_r}{A} - \sqrt{\alpha_N / N}}\lambda \geq \frac{W}{\frac{NA_{r'}}{A} + \sqrt{\alpha_N N}} \\
\text{as } N \longrightarrow \infty, \quad \lambda \geq \frac{W}{N} \cdot \frac{\frac{A_r}{A} - \sqrt{\epsilon}}{\frac{A_{r'}}{A} + \sqrt{\epsilon}} \\
= \frac{W}{N} \cdot \frac{\pi r^2 - \sqrt{\epsilon}}{4\pi r^2 + 4\pi r\delta + \pi\delta^2 + \sqrt{\epsilon}} \\
\text{(since } \sqrt{\epsilon} \text{ arbitrarily close to 0)} \approx \frac{W}{4N\left(1 + \delta\left(\frac{1}{r} + \frac{\delta}{4r^2}\right)\right)}.
\end{aligned} \tag{21}$$

Since there are N nodes transmitting with $\lambda \geq \frac{W}{N}\frac{\pi r^2 - \sqrt{\epsilon}}{4\pi r^2 + 4\pi r\delta + \pi\delta^2 + \sqrt{\epsilon}}$, and considering the result of Section 3.1 the achievable total transport capacity of the network is $\Theta(1)$.

We now briefly discuss the many-to-many case. Consider a node at any location in the network. When it first transmits its data, the data reaches every node within a distance r from the this node. Nodes on the edge of this area then retransmit the data to other nodes which were not reached in the first transmission. Because the size of the field is finite, it takes a finite number of transmissions k to cover the whole field. Once the whole field is covered, all intended destinations must have received the data. Consider a network where each node transmits its data this way, one starting as soon as the previous one has just finished. Under such a construction it would take at most Nk transmissions to transmit data from every node to every other nodes in the network. Therefore

$$\lambda \geq \frac{W}{kN}. \tag{22}$$

Since there are N nodes in the network, each with $\lambda \geq \frac{W}{kN}$, the total transport capacity of the network will again be $\Theta(1)$.

Note that the parameter k does not depend on N since an increase in N only means an increase in density when the size of the field is fixed. An increase in density means that every transmission reaches more nodes, but does not affect the number of transmissions needed to cover the field. An increase in the field size or a decrease in r will increase k, but as long as $r > 0$ and the field size is finite, k will be finite.

To summarize we have shown in this section that overall the total transport capacity of the network is $\Theta(1)$ in both the many-to-one and the many-to-many cases. Equivalently the per-node capacity is $\Theta(\frac{1}{N})$. The key is that the total capacity does not grow as the size of the network increases. This is a major difference from what was derived in [1] for the peer-to-peer case. At the same time, the per-node throughput decays as fast as $\frac{1}{N}$ as N increases.

4 Conclusion

In this paper we characterized the amount of data required to sample, quantize, and encode a field densely deployed with wireless sensors, and the amount of data that can be transported by the wireless sensor network, motivated by an imaging application where there is a single receiver/collector. We showed that as the number of sensors increases to infinity, the total amount of data generated for every snapshot also goes to infinity. At the same time, while the number of bits generated per sensor per snapshot may go zero, it can only do so at a rate strictly less than $\frac{1}{N}$. On the other hand, as the size grows, the total transport capacity of the network remains constant on the order of 1, and the transport capacity per node is on the order of $\frac{1}{N}$. Therefore the amount of data required for a fixed MSE cannot be transported within finite network usage. We would like to emphasize that this result holds for both a bandlimited and non-bandlimited random field, regardless of the encoding scheme used.

We showed that in the special case of a one-dimensional Gaussian random field with two example autocorrelation functions, there exists a coding scheme with which the number of bits per sensor per snapshot is on the order of $\frac{\log N}{\sqrt{N}}$ and $\frac{\log N}{N}$. We also constructively showed that the achievable per node capacity is on the order of $\frac{1}{N}$. Therefore in this special case the network usage is $\Theta(\sqrt{N}\log N)$ and $\Theta(\log N)$, respectively.

We also discussed that since the number of slots per snapshot increases with the number of sensors, there should exist an optimal number of sensors that minimizes the number of slots per snapshot. We do not know what this optimum is, but if we did, it would place a limit on how densely sensors should be deployed, beyond which one should *suppress* sensors, e.g. put sensors to sleep, to prevent over-sampling.

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