On the Uniqueness and Stability of Equilibria of Network Games

Parinaz Naghizadeh and Mingyan Liu
Department of Electrical Engineering and Computer Science
University of Michigan, Ann Arbor
Email: {naghizad, mingyan}@umich.edu

***Abstract*— We study a class of games played on networks with general (non-linear) best-response functions. Specifically, we let each agent's payoff depend on a linearly weighted sum of her neighbors' actions through a non-linear interaction function. We identify conditions on the network structure underlying the game given which (i) the Nash equilibrium of the game is unique, and (ii) the Nash equilibria are stable under perturbations in the model's parameters. We find that both the uniqueness and stability of the Nash equilibria are related to the *lowest eigenvalue* of suitably defined matrices, which are determined by the network's adjacency matrix, as well as the slopes of the interaction functions. We show that our uniqueness result generalizes an existing uniqueness condition for games of linear best-responses to games with general best-response functions. We further identify the classes of agents that are instrumental in the spread of shocks over the network. In particular, for small shocks, we show that agents that are strictly inactive at a given equilibrium can be precluded from the equilibrium's stability analysis, irrespective of their network position or links.**

I. Introduction

The role of networks in determining outcomes of social and economic interactions is discernible in a myriad of applications; the spread of infectious diseases, marketing decisions by sellers, product adoption choices by buyers, and spread of news and information, are all affected by some underlying network of interactions.

In particular, when the network is composed of strategic agents, their decision making process can be studied as a *network game*. Examples of applications that have been studied in this framework include the local provision of public goods [1], [2], [3], the role of peer-effects in education [4], networked Cournot competitions [5], [6], pricing in the presence of network externalities [7], [8], and the spread of shocks in networks [9], particularly financial contagion in interlinked markets [10], [11].

Network games have received increasing attention in recent years; see [12], [13] for recent surveys. One of the primary goals of this line of research has been to identify how structural properties of the network of interdependencies and influences shape the equilibrium outcomes of these games. These finding are of importance in interpreting the variations in performance among different network structures, guiding the design of better networks, evaluating the efficacy of potential intervention policies, and constructing models that match empirical observations when the underlying network is not observable.

In this paper, we are similarly interested in the role of the network structure in determining the outcomes of network games. In particular, we aim to characterize how the network structure affects (i) the existence and uniqueness of Nash equilibria of network games, and (ii) the stability of the equilibria given the possibility of perturbations or estimation errors in the underlying model's parameters.

Specifically, consider a set of strategic agents interacting over a fixed network, where each agent's decision in the game is influenced by two factors. First, each agent is affected by the *aggregate effort* of agents in its local neighborhood. We choose a linearly weighted sum as our aggregation function, with the weights corresponding to the edge weights of the underlying network. These weights can be either positive or negative, reflecting strategic substitutes or complements, respectively. Second, the effort of each agent will depend on this aggregate effort through a (possibly non-linear) *interaction function*. This will lead to a family of network games with general *(non-linear) best-response* functions. Our model encompasses several utility models, including those used for the study of public good provision games in [1], [3], games with quadratic payoffs [4], [14], and the game over influence networks of [15]. We show that the problem of characterizing the Nash equilibria of these games can be formulated as a *variational inequality* problem. Variational inequality theory is a general problem formulation which encompasses a broad class of mathematical problems, including convex optimization, complementarity problems, and fixed point problems. We utilize this connection to derive several results on the uniqueness and stability of Nash equilibria of network games.

Our first contribution is to identify a sufficient condition on the structure of the underlying network, given which the game has a *unique* Nash equilibrium. This condition involves the lowest eigenvalue of a particular matrix; it depends on both the slope of the interaction function– through the first derivative of the best-response functions– as well as the structure of the network– through the intensities (i.e., magnitude of edge weights) of agents' interactions. We will discuss how this condition relates to existing results in the literature. In particular, we show that the result of [1] on games of linear best-responses can be recovered as a special case of our result. Bramoullé, Kranton, and D'amours [1] were the first to identify the importance of the lowest eigenvalue of the adjacency matrix in determining properties of network games' equilibria; our work establishes that the lowest eigenvalue of a suitably defined matrix plays the equivalent role in network games with non-linear best-responses. We will further identify a subclass of our model in which this sufficient condition is also *necessary* for uniqueness.

Our second contribution is to study the *stability* of the Nash equilibria under shocks to the underlying model's parameters. We identify a sufficient condition under which the perturbed game has a Nash equilibrium, which in addition remains sufficiently close to the equilibrium of the unperturbed game. This sufficient condition will again depend on the lowest eigenvalue of a suitably defined matrix; this matrix depends on the slope of the best-response functions and the intensity of interactions. However, in contrast to the matrix constructed to evaluate our uniqueness condition, the matrix determining the stability will be constructed on a network where all strictly inactive agents (those exerting zero effort at equilibrium) are removed. Intuitively, such agents would require larger shocks (through parametric changes in the model, or the propagation effects due to other agents' revised efforts) to become active. We show that with small enough shocks, this possibility can be precluded irrespective of these agents' network links or positions.

The paper is organized as follows. In the remainder of this section, we review the literature most closely related to this work. We present the network game model in Section II, followed by results on uniqueness and stability of the Nash equilibria of these games in Sections III and IV, respectively. We conclude in Section V.

A. Related literature

Network games have received increasing attention in recent years; [12], [13] are recent surveys of the area. Here, we review some of the work most closely related to this paper.

Our work contributes to the literature on uniqueness of Nash equilibria of network games. Conditions for uniqueness of Nash equilibria in games with *linear* best-responses have been studied in [1], [3], [16], [17]. Both [3] and [17] identify the strict diagonal dominance of the adjacency matrix as a sufficient condition for uniqueness of the Nash equilibrium. We show that these results can be viewed as a special case of our Proposition 4, which introduces a (modified) diagonal dominance property as a sufficient condition for uniqueness of Nash equilibria in games with general best-responses. The work most closely related to ours is that of Bramoullé, Kranton, and D'amours [1], who study a game of strategic substitutes on a symmetric network, and use a connection to potential games to identify the role of the lowest eigenvalue as the network measure of importance in determining the uniqueness of the Nash equilibrium. This result can be recovered as a corollary of our main uniqueness result. Our work further shows that the lowest eigenvalue of a suitably defined matrix provides a sufficient condition for the uniqueness of Nash equilibria in games with *non-linear* best-responses. Naghizadeh and Liu [3] provide a necessary and sufficient condition for the uniqueness of Nash equilibria in games with linear best-responses. We show that our uniqueness condition coincides with that of [3] when best-responses are linear; this comparison further illustrates that our sufficient condition is also necessary in such games.

The works in [2], [9], [15] provide sufficient conditions for uniqueness of Nash equilibria for games with general (*non-linear*) best-response functions. Acemoglu, Ozdaglar, and Tahbaz-Salehi [9] require the best-response function to be either a contraction or a non-expansive mapping for the uniqueness of the equilibrium; in contrast, we only require bounded derivatives on the best-response functions, and bring out the role of the graph structure in determining uniqueness. Allouch [2] studies a game of strategic substitutes on a symmetric, unweighted network, and identifies a condition of *network normality* for the uniqueness of the equilibrium. Despite the difference in the underlying models, the network normality condition also imposes a bound on the lowest eigenvalue of the adjacency matrix in terms of the slope of agents' Engle curves. A closely related work to this paper is that of Zhou et al. [15], who formulate the problem of finding the Nash equilibrium for games on influence networks as a nonlinear complementarity problem, and provide a sufficient condition, strong monotonicity, for uniqueness of the Nash equilibrium. Nonlinear complementarity problems are a special class of variational inequalities, and hence we also establish a similar sufficient condition in Proposition 1. Nevertheless, the strong monotonicity condition does not provide a connection to the network structure. We therefore identify another sufficient condition in Proposition 3, which allows us to bring out the role of the interaction structure.

Our work also contributes to the literature on stability of Nash equilibria of network games. Bramoullé, Kranton, and D'amours [1] and Allouch [2] both study a notion of *asymptotic stability*, which requires the stability of the best-response dynamics under shocks. Our notion of stability is different from that of [1], [2], as it instead requires the "continuity" of the equilibria given perturbations in the underlying model's parameters. Despite the different notions, both our work and [1], [2] identify a lowest eigenvalue as a network measure of interest in evaluating stability. In addition, both our work and [1] illustrate a difference between the role of active and inactive agents in determining the stability of an equilibrium. Exploring the connection between our two notions remains a direction of future work.

Our work is also closely related to the study of systemic risks on networks, and in particular to the work of Acemoglu, Ozdaglar, and Tahbaz-Salehi [9], which studies how microscopic shocks translate to changes in an economy's macroscopic outcomes for games with general best-response functions. An assumption underlying the analysis of [9] is that shocks are small, so that the equilibria resulting under the shocks can be approximated using first-order (and second-order) Taylor expansions. In this work, we identify conditions on the network structure under which such first order approximations are possible; see Corollary 5. Finally, in work concurrent with ours, Parise and Ozdaglar [18] study a model of network aggregative games similar to the one in this paper. Their main focus is on the sensitivity analysis of Nash equilibria of these games, including under interventions by a social planner, while our main focus is on the importance of the lowest eigenvalue in the uniqueness and stability of the Nash equilibria.

II. Model and Preliminaries

A. Network game model

We consider a network of N agents, each represented by a node in a directed network $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$. Each agent $i \in \mathcal{N}$ chooses an *effort* level $x_i \in \mathbb{R}_{\geq 0}$. Let $\mathbf{x} := \{x_1, x_2, \dots, x_N\}$ denote the profile of all agents' efforts.

We represent the set of agent i's neighbors by $\mathcal{N}_i := \{j \in \mathcal{N} : \{i \to j\} \in \mathcal{E}\}$, and denote by $w_{ij} \in \mathbb{R}$ the weight associated with edge $i \to j$ of the network. The edge weights capture the strength and type of agents' interdependencies; positive weights reflect *strategic substitutes*, while negative weights reflect *strategic complements*. In words, a strategic substitute (complement) means that an increase in a neighbor j's effort leads to a decrease (increase) in agent i's effort. We assume $w_{ii} = 0, \forall i$, and that adjacency matrix is symmetric. Denote the adjacency matrix of the graph by $W := [w_{ij}]$.

Let $u_i(x_i, \mathbf{x}^{\mathcal{N}_i}; \mathcal{G})$ denote agent i's utility in the network game. This utility depends on the agent's own effort x_i, the vector of efforts of her neighbors $\mathbf{x}^{\mathcal{N}_i} := \{x_j : \ j \in \mathcal{N}_i\}$, and the graph structure $\mathcal{G}$. We will consider the family of games with utilities that have best-response functions of the form,

$$x_i = \max\{f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j), 0\}, \ \forall i. \tag{1}$$

Here, the maximization is taken to ensure that x_i is a feasible effort level. The function $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is the *interaction function* for agent i, determining the effect of agent i's neighbors' linearly aggregated effort $\sum_{j \in \mathcal{N}_i} w_{ij} x_j$, on her utility. We assume this is an increasing, continuously differentiable function with bounded derivatives. In particular, when $f_i(z) = \alpha_i z, \forall i$, the game is one of *linear best-responses*.

We present two examples of utility functions with best-responses of the form (1).

Example 1 (Public good provision games)**.** *Let the utility of agent i be,*

$$u_i(\mathbf{x}; \mathcal{G}) = V_i(x_i + f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)) - c_i x_i \ .$$

Here, $c_i \in \mathbb{R}_{>0}$ is the unit cost of effort for agent i, and $V_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing benefit function. Let q_i be the effort level at which $V_i'(q_i) = c_i$; this is the level of effort at which agent i's marginal benefit from effort equals her marginal cost of effort. Using the first order condition on the agent's utility, we obtain the following best-response function for agent i,

$$x_i^* = \max\{q_i - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j), 0\} \ .$$

This utility model has been studied in [1], [3], [15].

Example 2 (Quadratic payoffs)**.** *Let the utility of agent i be,*

$$u_i(\mathbf{x}; \mathcal{G}) = \theta_i x_i - \frac{1}{2} x_i^2 + x_i f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j) \ .$$

Here, $\theta_i \in \mathbb{R}_{>0}$ is a fixed parameter. Using the first order condition on the agent's utility, we obtain the following best-response function for each agent i,

$$x_i^* = \max\{\theta_i + f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j), 0\} \ .$$

This utility model has been studied in [4], [14].

We are primarily interested in studying the Nash equilibria resulting from the strategic interactions of the N agents with best-responses (1), over network $\mathcal{G}$. Formally, a Nash equilibrium $\mathbf{x}^*$ is the fixed point of the following set of optimization problems,

$$x_i^* = \arg\max_{x \geq 0} \ u_i(x, \mathbf{x}^{*\mathcal{N}_i}; \mathcal{G}), \ \forall i.$$

That is, each agent optimizes her effort, given others' effort levels. For the network games defined herein, the Nash equilibrium will be a fixed point of the N best-response mappings of the form (1). Equivalently, we are looking for a vector of efforts $\mathbf{x}^* \succeq \mathbf{0}$ such that,

$$(x_i - x_i^*)(x_i^* - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j^*)) \geq 0, \ \forall x_i \geq 0, \ \forall i. \tag{2}$$

B. The Variational Inequality (VI) problem

Variational inequality theory constitutes the study of solution properties and solution methods for a general problem formulation (see Definition 1 below), which encompasses a broad class of other mathematical problems, including convex optimization, complementarity problems, and fixed point problems. Formally, the variational inequality problem is defined as follows.

Definition 1. *A variational inequality VI(K, F) consists of a set $K \subseteq \mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, and is the problem of finding a vector $\mathbf{x}^* \in K$ such that*

$$(\mathbf{x} - \mathbf{x}^*)^T F(\mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in K \ . \tag{3}$$

Comparing (2) and (3), we observe that finding the Nash equilibria of the network game defined in Section II-A is equivalent to finding the solution set of the variational inequality problem VI$(\mathbb{R}^N_{\geq 0}, F(\mathbf{x}))$ with $F_i(\mathbf{x}) := x_i - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)$. We will therefore determine conditions for the existence and uniqueness of Nash equilibria of network games, as well as their stability, by exploring results from variational inequality theory.

III. Uniqueness of Nash Equilibria

We will first identify conditions under which the network games with best-responses (1) have a unique Nash equilibrium, by exploring the parallel between these games and variational inequalities. We begin by introducing the following definition.

Definition 2. *A function $F : K \to \mathbb{R}^n$, with K closed and convex, is strongly monotone on K if there exists a constant $c > 0$ such that $\forall \mathbf{x}, \mathbf{y} \in K$,*

$$(\mathbf{x} - \mathbf{y})^T (F(\mathbf{x}) - F(\mathbf{y})) \geq c||\mathbf{x} - \mathbf{y}||^2 \ .$$

Strong monotonicity plays a crucial role in the uniqueness of Nash equilibria. In particular,

Proposition 1. *If $F(\mathbf{x})$ with $F_i(\mathbf{x}) = x_i - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)$ is strongly monotone on $\mathbb{R}^N_{\geq 0}$, the network game has a unique Nash equilibrium*

The proof is straightforward, and follows from the VI formulation of the Nash equilibrium of network games. For a general VI(K, F), assume K is a closed and convex set, and that F is continuous on K. Then, if F is strongly monotone on K, the VI has a unique solution [19]. Applying this to the VI$(\mathbb{R}^N_{\geq 0}, F(\mathbf{x}))$ with $F_i(\mathbf{x}) := x_i - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)$ established the result.

The above proposition provides a sufficient condition for the uniqueness of the Nash equilibrium of network games with best-responses (1); it is equivalent to the sufficient condition identified in [15] for the uniqueness of Nash equilibria in games on influence networks (with utilities given in Example 1). Nevertheless, checking the strong monotonicity condition for a given function is not only difficult in general, but further does not provide a connection to the structural properties of the network game. Alternatively, we are interested in (sufficient) conditions on the network $\mathcal{G}$, and in particular the adjacency matrix W, under which the strong monotonicity condition holds, and consequently, the equilibrium is unique.

One sufficient condition for the strong monotonicity of a general mapping $F : K \to \mathbb{R}^n$ is identified by [19]. We begin by defining the following variables for a mapping F,

$$\zeta_i^{\min} := \inf_{\mathbf{x} \in K} \ \lambda_{\min} \left(\nabla_i F_i(\mathbf{x}) \right), \ \forall i,$$

and,

$$\zeta_{ij}^{\max} := \sup_{\mathbf{x} \in K} \ ||\nabla_j F_i(\mathbf{x})||, \ \forall i, j \neq i \ .$$

Here, $\nabla_j F_i$ denotes the Jacobian of F_i with respect to x_j. Define the matrix Υ with entires $\zeta_i^{\min}$ on the ii-th diagonal, and $-\zeta_{ij}^{\max}$ on the ij-th off-diagonal entries. That is,

$$\Upsilon := \begin{pmatrix} \zeta_1^{\min} & -\zeta_{12}^{\max} & \cdots & -\zeta_{1N}^{\max} \\ -\zeta_{21}^{\max} & \zeta_2^{\min} & \cdots & -\zeta_{2N}^{\max} \\ \vdots & \vdots & \ddots & \vdots \\ -\zeta_{N1}^{\max} & -\zeta_{N2}^{\max} & \cdots & \zeta_N^{\min} \end{pmatrix} .$$

Facchinei and Pang [19] establish the following sufficient condition on the auxiliary matrix Υ to evaluate the strong monotonicity of F.

Proposition 2 ([19, Proposition 12.13])**.** *If Υ is a P-matrix (i.e., the determinants of all its principal minors are strictly positive), then F is strongly monotone.*

We can now apply Proposition 2 to the VI formulation of network games. We begin by constructing the corresponding auxiliary matrix Υ for the functions $F_i(\mathbf{x}) = x_i - f_i(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)$. First, note that $\nabla_i F_i(\mathbf{x}) = 1, \forall i$, so that the diagonal entries of Υ are all 1. For the off-diagonal entires, we have $\nabla_j F_i(\mathbf{x}) = w_{ij} f_i'(\sum_{j \in \mathcal{N}_i} w_{ij} x_j)$. Let $m_i := \max_z f_i'(z)$. In addition, let $G = [|w_{ij}|]$ be the matrix containing the absolute values (strengths) of the edge weights. We refer to G as the *intensity* matrix. Consequently, the auxiliary matrix for our network games will be $\Upsilon = \mathbf{I} + \Gamma$, where $\mathbf{I}$ is the $N \times N$ identity matrix, and Γ is given by:

$$\Gamma := \begin{pmatrix} 0 & -g_{12} m_1 & \cdots & -g_{1N} m_1 \\ -g_{21} m_2 & 0 & \cdots & -g_{2N} m_2 \\ \vdots & \vdots & \ddots & \vdots \\ -g_{N1} m_N & -g_{N2} m_N & \cdots & 0 \end{pmatrix} . \quad (4)$$

The above matrix has negative entries on all its off-diagonal entries; such matrix is referred to as a *Z-matrix*. Note that both Υ and Γ are Z-matrices. Establishing conditions under which Z-matrices are P-matrices has received considerable attention in the literature; we will invoke these results in the subsequent proof.

We now present our main uniqueness result.

Proposition 3. *Consider a network game with best-response functions* (1)*, and the corresponding matrix Γ in* (4)*. If $|\lambda_{min}(\Gamma)| < 1$, the game has a unique Nash equilibrium.*

Proof. By Propositions 1 and 2, if $\Upsilon = \mathbf{I} + \Gamma$ is a P-matrix, then the Nash equilibrium is unique. We therefore identify conditions under which Υ is a P-matrix.

We know that a symmetric matrix is a P-matrix if and only if it is positive definite [20]. Note that the matrix Υ that has the particular decomposition $\mathbf{I} + \Gamma$. This matrix is therefore positive definite if and only if $|\lambda_{\min}(\Gamma)| < 1$.

□

We derive corollaries of the above for two special cases: symmetric and linear interaction functions. We begin with games with symmetric interaction functions.

Corollary 1. *Let $f_i(z) = f(z), \forall i$, with $m := \max_z |f'(z)|$. Then, if $|\lambda_{\min}(G)| < \frac{1}{m}$, the network game has a unique Nash equilibrium.*

The above corollary can be intuitively interpreted as follows. The smallest eigenvalue of the intensity matrix G can be interpreted as the extent to which changes in agents' actions reverberate in the network. Corollary 1 states that these reverberations must be small enough, and in particular, bounded by the inverse of the rate of change in agents' interaction functions. Specifically, when $f'(\cdot)$ is large, it means that each agent's best-response function (1) is relatively sensitive to changes in her neighbors actions. Hence, for a unique Nash equilibrium to exist, the interaction intensities G must limit the rebound of changes; equivalently, the lowest eigenvalue of G must be sufficiently small.

We next consider games with linear interaction functions, which lead to games with linear best-response functions.

Corollary 2. *Let $f_i(z) = z, \forall i$. Then, if $|\lambda_{\min}(G)| < 1$, the network game has a unique Nash equilibrium.*

This result is equivalent to that of Bramoullé, Kranton, and D'amours [1], which analyzed the Nash equilibria of network games with linear best-responses and symmetric adjacency matrices using the theory of potential functions, and was the first to identify the role of the lowest eigenvalue of the adjacency matrix as a sufficient condition for the uniqueness of the Nash equilibrium of these games. The work of Naghizadeh and Liu [3] has further used the theory of linear complementarity problems to show that the Nash equilibrium of network games with linear best-responses (with either symmetric and asymmetric adjacency matrices) is unique *if and only if* its adjacency matrix W is a P-matrix. In particular, a symmetric matrix G is a P-matrix if and only if it is positive definite [20], and it is positive definite if and only if $|\lambda_{\min}(G)| < 1$.

Putting these together, we conclude that the condition of Proposition 3 is not only sufficient, but also *necessary*, for the uniqueness of the Nash equilibrium in the class of games with linear interaction functions and symmetric adjacency matrices.

Finally, we present a diagonal dominance property on the matrix Υ, which provides an alternative sufficient condition that will allow us to apply Proposition 2 to network games.

Proposition 4. *If $\sum_{j\in\mathcal{N}_i} g_{ij} < \frac{1}{m_i}, \forall i$, then the network game has a unique Nash equilibrium.*

The proof is straightforward and follows from the fact that a strictly diagonally dominant matrix is a P-matrix [20].

Intuitively, Proposition 4 requires that $m_i \sum_{j\in\mathcal{N}_i} g_{ij}$, which is an upperbound on the aggregate rate at which each agent's neighbors can affect her best-response function in (1), be smaller than the rate of change in the agent's own effort. Note also that by setting $m_i = 1, \forall i$, we can recover the results of [16], [17], which show that strict diagonal dominance of the intensity matrix as a sufficient condition for the uniqueness of the Nash equilibrium in games of linear best-responses.

We end this section with a comparison of the two sufficient conditions of Propositions 3 and 4. By the Gershgorin circle theorem, all eigenvalues of Γ lie within discs with center 0 and radius $R_i := m_i \sum_{j\in\mathcal{N}_i} g_{ij}$. Therefore, the condition of Proposition 3, which requires $|\lambda_{\min}(\Gamma)| < 1$, is also satisfied under the diagonal dominance property of Proposition 4. The reverse however does not necessarily hold. In other words, Proposition 3 provides a more general (weaker) sufficient condition. Proposition 4 may however be of interest due to its intuitive interpretation and simpler evaluation.

IV. STABILITY OF NASH EQUILIBRIA

We now turn to the question of *stability* of Nash equilibria in network games. We are interested in identifying conditions under which small changes in the underlying model's parameters lead to solutions that are not substantially divergent from the starting equilibrium.

Formally, we will generalize our best-response model in (1) to the family of parametrized functions $f_i(z, \epsilon) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where each agent i is affected by a *perturbation parameter* or *shock* ϵ. The perturbation parameter can be used to model a variety of changes in agents' payoffs, including price shocks and variations in edge weights.

Let $\mathbf{x}(\boldsymbol{\epsilon})$ denote the Nash equilibrium of the game under the vector of shocks $\boldsymbol{\epsilon}$, and let $\mathbf{x}^*$ be the Nash equilibrium of the unperturbed game. We ask whether for small $\boldsymbol{\epsilon}$, the perturbed game has an equilibrium $\mathbf{x}(\boldsymbol{\epsilon})$, and if so, whether $\mathbf{x}(\boldsymbol{\epsilon})$ is close to $\mathbf{x}^*$. We formalize these statements in the following definition. Denote the ball of radius β centered at $\mathbf{y}$ by $\mathcal{B}(\mathbf{y}, \beta) := \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{y}|| < \beta\}$. Then,

Definition 3. *A Nash equilibrium $\mathbf{x}^*$ is stable if there exists $\eta > 0$ and $c > 0$, such that $\forall \boldsymbol{\epsilon} \in \mathcal{B}(\mathbf{0}, \eta)$, the Nash equilibrium $\mathbf{x}(\boldsymbol{\epsilon})$ exists and satisfies,*

$$||\mathbf{x}(\boldsymbol{\epsilon}) - \mathbf{x}^*|| \le c||F(\mathbf{x}(\boldsymbol{\epsilon}), \boldsymbol{\epsilon}) - F(\mathbf{x}(\boldsymbol{\epsilon}), \mathbf{0})|| \ ,$$

where, $F_i(\mathbf{x}, \epsilon_i) = x_i - f_i(\sum_{j\in\mathcal{N}_i} w_{ij}x_j, \epsilon_i)$.

Intuitively, a Nash equilibrium is said to be stable if for sufficiently small perturbations, the game will still have a Nash equilibrium that is also sufficiently close to the starting equilibrium.

To identify conditions for equilibrium stability, we further differentiate between the following sets of agents in the starting equilibrium $\mathbf{x}^*$,

$$\begin{aligned} A &:= \{i \in \mathcal{N} |\ x_i^* > 0,\ x_i - f_i(\mathbf{x}^{\mathcal{N}_i}) = 0\}, \\ I &:= \{i \in \mathcal{N} |\ x_i^* = 0,\ x_i - f_i(\mathbf{x}^{\mathcal{N}_i}) > 0\}, \\ B &:= \{i \in \mathcal{N} |\ x_i^* = 0,\ x_i - f_i(\mathbf{x}^{\mathcal{N}_i}) = 0\}. \end{aligned}$$

In words, A denotes the set of *active* agents, while I and B are the inactive agents at the Nash equilibrium, with I denoting *strictly inactive* agents, and B denoting *borderline inactive* agents.

We define the following block-partitioned matrix for the sets of active and borderline inactive agents,

$$\Gamma_{A,B} := \begin{pmatrix} \Omega_{AA} & \Omega_{AB} \\ \Omega_{BA} & \Omega_{BB} \end{pmatrix} . \tag{5}$$

Here, given two sets S_1 and S_2, $\Omega_{S_1S_2}$ denotes the $|S_1|\times|S_2|$ matrix with entires $-g_{ij}m_i$, for $i \in S_1, j \in S_2$, where $m_i = \max_z \frac{\partial f_i(z,0)}{\partial z}$ and $g_{ij} = |w_{ij}|$. It is worth noting that $\Gamma_{A,B}$ is (a rearrangement of) the matrix Γ defined earlier in (4) when all rows and columns corresponding to the strictly inactive agents I are dropped.

We now present our sufficient condition for the stability of the Nash equilibrium.

Proposition 5. *Consider an equilibrium $\mathbf{x}^*$ with active agents A and borderline inactive agents B, and $\Gamma_{A,B}$ defined in* (5)*. If $|\lambda_{min}(\Gamma_{A,B})| < 1$, the Nash equilibrium is stable.*

Proof. We use results on the stability of the solution $\mathbf{x}^*$ of the parametrized variational inequality $\text{VI}(K, F(\mathbf{x}, \boldsymbol{\epsilon}))$, when applied to the VI formulation of Nash equilibria of network games. Specifically, consider the block partitioned matrix,

$$\nabla_{A,B} F_{A,B} := \begin{pmatrix} \nabla_A F_A & \nabla_B F_A \\ \\ \nabla_B F_B & \nabla_A F_B \end{pmatrix} \ . \tag{6}$$

where $\nabla_{S_1} F_{S_2}$ denotes the gradient of functions in the set S_2 with respect to variables in the set S_1. [21][Theorem 3.1] shows that if the above matrix is positive definite, then the Nash equilibrium $\mathbf{x}^*$ is stable in the sense of Definition 3.

Let $\Sigma_{A,B}$ denote (6) for the network game. We will identify conditions under which this matrix is a P-matrix, which will mean that it is also positive definite [20]. We begin by noting that this is a Z-matrix, and that is lower-bounded by $\Sigma_{A,B} \succeq \mathbf{I}_{|A|+|B|} + \Gamma_{A,B}$, where the lower bound is also a Z-matrix. As a result, if $\mathbf{I}_{|A|+|B|} + \Gamma_{A,B}$ is a P-matrix, so is $\Sigma_{A,B}$. A symmetric matrix is a P-matrix if and only if it is positive definite [20]. Noting that $\mathbf{I}_{|A|+|B|} + \Gamma_{A,B}$ is positive definite if and only if $|\lambda_{\min}(\Gamma_{A,B})| < 1$ completes the proof.

□

The above theorem can be intuitively interpreted as follows. We begin by noting that the condition $|\lambda_{min}(\Gamma_{A,B})| < 1$ of Proposition 5 imposes a restriction on the mutual effects of variations in the best-response functions of the active and borderline inactive agents to guarantee stability. Consider a small change in one of the active agents' efforts due to a parametric shock to her utility function. When not sufficiently bounded, this can turn a borderline inactive agent active, which could in turn initiate fluctuations in other active and inactive agents' efforts. Such changes can reverberate through the network, leading to a new Nash equilibrium that is no longer close to the starting equilibrium point. In contrast, when $|\lambda_{min}(\Gamma_{A,B})| < 1$, such fluctuations will be bounded, leading to stable equilibria. It is also interesting to highlight that strictly inactive agents do not play a critical role in the stability of the equilibrium; after all, such agents would require larger shocks (through parametric changes, or propagation effects due to other agents' revised efforts) to become active. Proposition 5 states that with small enough shocks, this possibility can be effectively precluded irrespective of the network structure.

We next obtain a corollary of Proposition 5 for the stability of unique equilibria.

Corollary 3. *If $|\lambda_{\min}(\Gamma)| < 1$, the Nash equilibrium is unique and stable.*

Proof. Uniqueness follows from Proposition 3. For stability, we begin by noting that $\Gamma_{A,B}$ in (5) is (a rearrangement of) the matrix Γ in (4), when all rows and columns corresponding to the strictly inactive agents are dropped. Using the Cauchy interlacing theorem, we know that $\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma_{A,B})$. As the trace of these matrices is zero, we conclude that $\lambda_{\min}$ is negative for both $\Gamma_{A,B}$ and Γ. We conclude that $|\lambda_{\min}(\Gamma_{A,B})| \leq |\lambda_{\min}(\Gamma)|$. This, together with Proposition 5, imply that the equilibrium is stable. □

We next look at the special case of games of linear best-responses. Let $G_{A,B}$ denote the intensity matrix restricted to the set of active and borderline inactive agents.

Corollary 4. *Let $f_i(z) = z, \forall i$. Then, if $|\lambda_{\min}(G_{A,B})| < 1$, the Nash equilibrium is stable.*

That is, for games with linear beat-responses, it is sufficient to check the lowest eigenvalue of the intensity matrix (or the adjacency matrix for games of pure substitutes) on a network restricted to active and borderline inactive agents, to determine equilibrium stability.

Finally, we consider the case where the Nash equilibrium solution is *non-degenerate*, that is, when all agents are either active or strictly inactive, and $B = \emptyset$. For this case, we can characterize the sensitivity of the equilibrium efforts to shocks.

Corollary 5. *Let $\mathbf{x}^*$ be a non-degenerate Nash equilibrium. If for Ω_{AA} defined in* (5), *$|\lambda_{\min}(\Omega_{AA})| < 1$, the Nash equilibrium is stable. In addition,*

$$x_i(\boldsymbol{\epsilon}) = x_i^* - \frac{\partial f_i(\mathbf{x}^{*\mathcal{N}_i}, 0)}{\partial x} \sum_{j \in \mathcal{N}_i} w_{ij} \frac{\partial f_j(\mathbf{x}^{*\mathcal{N}_i}, 0)}{\partial \epsilon} \ \epsilon_i, \ \ i \in A \ ,$$

$$x_i(\boldsymbol{\epsilon}) = 0, \ \ i \in I \ .$$

where $\mathbf{x}^{*\mathcal{N}_i} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^*$.

The proof is straightforward and follows from [21][Corollary 3.4]. This characterization can be used in evaluating how microscopic shocks translate into changes in macroscopic measures of aggregate output.

V. Conclusion

We studied the Nash equilibria of network games where agents' payoffs depend on a linearly weighted sum of their local neighbors' efforts through a non-linear interaction function. For this class of games, which have non-linear best-response functions, we showed that both the Nash equilibrium uniqueness, as well as the sensitivity of the equilibria to shocks, are determined by the lowest eigenvalues of suitably defined matrices. These matrices depend on the slope of agents' interaction functions, as well as the intensity of their interactions. Our sufficient conditions can then be interpreted as a measure of how shocks reverberate through the network, as a function of both the structure of the network of interactions, as well as the sensitivity of each agent to the aggregate changes in her neighbors' efforts.

We further provided a characterization of the equilibria resulting under small shocks for non-degenerate Nash equilibria. As a direction of future work, we are interested in using this characterization to quantify the effect of local shocks on (macroscopic) aggregate outcomes of the network game. This will in turn enable the design of optimal intervention policies, which can be used to modify the aggregate outcome through small changes in the network structure or agents' payoffs.

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