Atomic Congestion Games on Graphs and its Applications in Networking

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Abstract— **In this paper, we introduce and analyze the properties of a class of game, the atomic congestion game on graphs (ACGG), which is a generalization of the classical congestion game. In particular, the ACGG captures the spatial information which is often ignored in the classical congestion game. This is useful in many networking problems, e.g., wireless networks where interference among the users heavily depend on the spatial information. In an ACGG, a player's payoff for using a resource is a function of the number of players who interact with it and use the same resource. Such spatial information can be captured by a graph. We study fundamental properties of the ACGG; under what conditions this game possesses a pure strategy Nash equilibrium (PNE), or the finite improvement property (FIP), which is sufficient for the existence of a PNE. We show that a PNE may not exist in general, but that it does exist in many important special cases including tree, loop, or regular bipartite networks. The FIP also exists for important special cases including systems with 2 resources or identical payoff functions for each resource. Finally, we present two wireless network applications of an ACGG: power control and channel contention under IEEE 802.11.**

I. Introduction

In this paper, we study an *atomic congestion game on graphs* (ACGG), which is a generalized form of the class of non-cooperative strategic games known as congestion games [1], [2]. We analyze the properties of the ACGG and further discuss its applications in networking such as spectrum sharing in a multi-channel wireless system and channel contention under IEEE 802.11.

In a classical congestion game, multiple players share multiple resources. A player's payoff[1] for using a particular resource depends on the number of players simultaneously using that resource. A formal description is provided in Section III. The congestion game framework is well suited to model resource competition where the resulting payoff is a function of the level of congestion (number of active/competing players).

An improvement step is a move where one player changes its strategy to increase its payoff. An improvement path is a sequence of asynchronous improvement steps. The congestion game enjoys many appealing properties: it has a pure strategy Nash Equilibrium (PNE), and any improvement path is finite and will lead to a PNE. The latter property is also called the *finite improvement property* (FIP): local greedy updates of selfish players collectively optimize a global objective known as the potential function, and such updates converge in a finite number of steps regardless of the updating sequence.

Due to the above reasons, congestion games are used widely in modeling networking problems, particularly routing problems (see Section II). In this paper, we introduce a generalization of this model where players can only affect with their neighbors according to a graph structure. These generalized games will be referred to as *atomic congestion games on graphs* (ACGG). In this game, an interaction graph describes the the congestion relationships between the players. A player's payoff for using a resource is a function of the total number of players who are using the same resource and are *within its interaction set* (i.e., connected to it by edges in the graph). Therefore, resources are *reusable* beyond a player's interaction set. The original congestion game is now a special case of the extended ACGG where the underlying interaction graph is complete (i.e., every player interacts with every other player).[2]

Our main motivation behind this generalization comes from applications in wireless networks, a key feature of which is *spatial reuse*: common spectrum resources may be shared by multiple players located far apart without causing congestion to each other. This feature cannot be captured by the standard congestion game, which assumes that all players have an equal impact on the congestion. Specifically, we consider a system where a user can only access one channel at a time, but can switch between channels. A user's principal interest lies in optimizing its own performance (e.g., its data rate) by selecting the best channel for itself. This and similar

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[1]One can also consider the cost of using a resource instead of payoff. If we define the cost as the inverse of the payoff, then maximizing the payoff is equivalent to minimizing the cost. For simplicity of presentation, we will only refer to the maximization of payoff in this paper.

[2]In our preliminary work [3], we used the term *network congestion games*. However, to better differentiate this class of games from routing games (see e.g., [4], [5]) which are also sometimes referred to as network congestion games, we will use the term *atomic congestion games on graphs* in this paper. Note that a routing game is essentially a classical congestion game in which a player's strategy space consists of a set of feasible routes and each route consists of multiple resources (links).

problems have recently captured increasing interest from the research community, particularly in the context of cognitive radio networks and software defined ratio technologies, where devices are expected to have far greater flexibility in sensing channel availability and moving their operating frequencies.

In addition to the above, there are other applications of the ACGG model. For instance, it can be used to model competition among local businesses, whose locations may be represented by vertices on a graph. Edges connect those within close proximity, and resources represent different business ventures. More discussions on application are given in Section VIII. It's also worth mentioning that one interpretation of the ACGG is that it models congestion games with incomplete information, i.e., all players in reality compete with everyone else, but a player is only aware of the presence of its neighbors on the interaction graph. This is actually the standard justification of graphical congestion games, see e.g., [6]. More on this is discussed in Section II.

In subsequent sections we will examine what properties an ACGG has. Our main findings are summarized as follows for *undirected* network graphs and *non-increasing* payoff functions:

1) The FIP is preserved in an ACGG with only two resources. Counter examples exist for three or more resources.
2) The FIP is preserved in an ACGG when all resources are identical to a player (but may be different to different players).
3) A PNE exists in an ACGG over a tree network, a loop, a regular bipartite network, and when there is a dominating resource.
4) We also identify counter examples showing that a PNE does not necessarily exist for an ACGG with 3 resources, over a directed graph, or with non-monotonic payoff functions.

The organization of the remainder of this paper is as follows. Related work is given in Section II. In Section III we present a brief review on the classical congestion game, and formally define the class of ACGG in Section IV. We then derive conditions under which ACGG possesses the FIP in Section V, and under which a PNE exists in VI. We provide negative results on existence of a PNE or the FIP in Section VII. We illustrate two networking applications of ACGG in Section VIII and discuss extensions to our work in Section IX. We conclude the paper in Section X.

II. Related Work

Congestion games have been extensively studied within the context of wireline network routing, see for instance the congestion game studied in [7], where each source node seeks the minimum delay path to a destination node, and the delay of a link depends on the number of flows going through that link. It has recently been used in wireless network modeling, e.g., access point selection in WiFi networks [8], [9], resource competition in multicamera wireless surveillance networks [10], uplink resource allocation in multichannel wireless access networks [11], wireless channels with multipacket reception capability [12], and the impact of interference set in studying the congestion game in wireless mesh networks [13]. In our recent work [14], we addressed the user-specific interference issue within the traditional congestion game framework, by introducing a concept called *resource expansion*, where we define virtual resources as certain spectral-spatial unit that allows us to capture pair-wise interference. This approach was shown to be quite effective for user objectives like interference minimization. Congestion games played on networks have been studied before in [6], where each user has the same linear payoff function. The authors discuss how these systems can be viewed as congestion games with limited information, where players can only observe the actions of their neighbors on a graph. Our ACGG model allows player-specific payoff functions of more general forms. In this sense our model is also a generalization of that considered in [6].

It should be mentioned that game theoretic approaches have often been used to devise effective decentralized solutions to a multi-agent system. Within the context of wireless communication networks and interference modeling, different classes of games have been studied. An example is the well-known *Gaussian interference game* [15], [16], in which a player can spread a fixed amount of power arbitrarily across a continuous bandwidth, and tries to maximize its total rate in a Gaussian interference channel over all possible power allocation strategies. The Bayesian form of the Gaussian interference game was studied in [17] in the case of incomplete information. In addition, a market based power control mechanism was investigated via supermodularity in [18], and using externality in [19]. A spectrum sharing similar to the one studied here was investigated in [20] using a mechanism design approach in seeking a globally optimal solution. In our problem the total power of a user is not divisible, and it can only use it in one channel at a time. This setup is more appropriate for scenarios where the channels have been pre-defined, and the users do not have the ability to access multiple channels simultaneously (which is the case with many existing devices).

Another approach to analyzing related networking problems is the use of evolutionary game theory [21], [22]. Evolutionary games often assume limited rationality and are applicable typically in the presence of a larger number of users. By contrast, our approach applies to any number of users, and also works under limited rationality (better response updating). Furthermore, the better response dynamics we consider are simpler and more realistic, in the context of wireless networking, than many types of evolutionary dynamics, e.g., those based on reproduction or imitation.

III. A Review of Congestion Games

In this section we provide a brief review on the definition of congestion games and their known properties.[3] We then introduce the ACGG as a generalization.

Congestion games [1], [2] are a class of strategic games given by the tuple $(\mathcal{I}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{I}}, (g_r)_{r\in\mathcal{R}})$, where $\mathcal{I} = \{1, 2, \cdots, N\}$ denotes a set of players, $\mathcal{R} = \{1, 2, \cdots, R\}$

[3]This review along with some of our notations are primarily based on references [1], [2], [23].

a set of resources, $\Sigma_i \subset 2^{\mathcal{R}}$ the strategy space of player i, and $g_r : \mathbb{N} \rightarrow \mathbb{Z}$ a payoff (or cost) function associated with resource r. The payoff (cost) $g_r(\cdot)$ of resource r is a function of the total number of players using that resource, and in general is assumed to be non-increasing (non-decreasing). A player in this game aims to maximize (minimize) its total payoff (cost) which is the sum total of payoff (cost) over all resources its strategy involves. For the rest of the paper, we will only refer to payoff maximization.

Denoting by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ a strategy profile, where $\sigma_i \in \Sigma_i$, player i's total payoff is given by

$$q^i(\boldsymbol{\sigma}) = \sum_{r \in \sigma_i} g_r(n_r(\boldsymbol{\sigma})) \ , \tag{1}$$

where $n_r(\boldsymbol{\sigma})$ is the total number of players using resource r under the strategy profile $\boldsymbol{\sigma}$, with $r \in \sigma_i$ denoting that player i selects resource r under $\boldsymbol{\sigma}$.

We can define Rosenthal's potential function $\phi : \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_N \rightarrow \mathbb{Z}$ as

$$\phi(\boldsymbol{\sigma}) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r(\boldsymbol{\sigma})} g_r(i) = \sum_{i=1}^{N} \sum_{r \in \sigma_i} g_r(m_r^i(\boldsymbol{\sigma})). \tag{2}$$

where the second equality comes from exchanging the two sums, and $m_r^i(\boldsymbol{\sigma})$ denotes the number of players who use resource r under strategy $\boldsymbol{\sigma}$ and whose corresponding indices do not exceed i (i.e., in the set $\{1, 2, \cdots, i\}$).

In [1] it is shown that the change in player i's payoff as a result of its unilateral move (i.e., all other players' strategy σ_{-i} remain fixed) is exactly the same as the change in the potential function. This implies that the potential function may be viewed as a global objective function. To see this, consider player i, who unilaterally moves from strategy σ_i (within the profile $\boldsymbol{\sigma} = (\sigma_i, \sigma_{-i})$) to strategy σ_i' (within the profile $\boldsymbol{\sigma}' = (\sigma_i', \sigma_{-i})$). The change to the potential function is

$$\begin{aligned} & \phi(\sigma_i', \sigma_{-i}) - \phi(\sigma_i, \sigma_{-i}) \\ = & \sum_{r \in \sigma_i', r \notin \sigma_i} g_r(n_r(\boldsymbol{\sigma}) + 1) - \sum_{r \in \sigma_i, r \notin \sigma_i'} g_r(n_r(\boldsymbol{\sigma})) \\ = & \sum_{r \in \sigma_i'} g_r(n_r(\boldsymbol{\sigma}')) - \sum_{r \in \sigma_i} g_r(n_r(\boldsymbol{\sigma})) \\ = & \ g^i(\sigma_{-i}, \sigma_i') - g^i(\sigma_{-i}, \sigma_i) \ . \end{aligned}$$

The second equality comes from the fact that the number of total players does not change for any resource that is used by both strategies σ_i and σ_i'. To see why the first equality is true, set $i = N$, in which case this equality is a direct consequence of Eqn (2). This is also true for any $1 \leq i \leq N$ by noting that the ordering of players is arbitrary so any player making a change may be viewed as the Nth player.

Consider now a sequence of strategy changes made by players asynchronously, in which each change improves the corresponding player's payoff (this is referred to as a sequence of improvement steps). The potential function improves in every such change sequence. Since the potential function of any strategy profile is finite, we have the following result [2]:

Proposition 1 (finite improvement property (FIP)): For every congestion game, every sequence of asynchronous improvement steps is finite and converges to a pure strategy Nash Equilibrium (PNE). Furthermore, this PNE is a local optimum of the potential function ϕ, defined as a strategy profile where changing one coordinate cannot result in a greater value of ϕ.

We note that by definition, FIP in *any* game is *sufficient* to guarantee the existence of a PNE for that game, but it is not *necessary*. It follows that (a) the non-existence of a PNE proves non-existence of the FIP, and (b) in general the existence of a PNE does not imply the existence of FIP. Therefore the existence of FIP is a much stronger result than PNE.

Also note that under the above standard definition, the payoff functions are resource-dependent but player-independent. This was relaxed in [24], where a player-specific payoff function $g_r^i(\cdot)$ was considered. It was shown that in this game a PNE continues to exist (at least in the "singleton" case where players access one resource at a time), but that the FIP no longer holds in general.

It is not difficult to see why the standard definition of a congestion game does not capture spatial reuse of wireless communication. In particular, if we consider channels as resources, then the payoff $g_r(n)$ for using channel r when there are n simultaneous players does not reflect reality: the function $g_r(\cdot)$ in general takes a player-specific argument since different players experience different levels of interaction even when using the same resource. This player specificity is also different from that studied in [24] mentioned above, where $g_r^i(\cdot)$ is a player-specific function but takes the *same* non-player specific argument n. To analyze and understand the consequence of this difference, we would need to extend and generalize the definition of the standard congestion game.

IV. Problem Formulation

In this section we formally define our generalized congestion game, the *atomic congestion game on graphs* (ACGG). Specifically, an N-player ACGG is given by $\Gamma_N = (\mathcal{I}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{I}}, \{\mathcal{K}_i\}_{i \in \mathcal{I}}, \{g_r^i\}_{r \in \mathcal{R}, i \in \mathcal{I}})$, where $\mathcal{K}_i$ is the *interaction set* of player/user i (i.e., players interacting with player i), while all other elements maintain the same meaning as in a standard CG. The payoff player i receives for using resource r is given by $g_r^i(n_r^i(\boldsymbol{\sigma}) + 1)$ where $n_r^i(\boldsymbol{\sigma}) = |\{j : r \in \sigma_j, j \in \mathcal{K}_i\}|$.

Our generalization goes in two directions: (a) player i's payoff for using resource r is a player-specific function, as evidenced by the index i in $g_r^i(\cdot)$, and (b) the argument of this function is also player-specific; it is the number of players interacting with itself, plus itself. The motivation for making the payoff functions player-specific is to capture, for example, the fact that in a wireless system players with different coding/modulation schemes may obtain different rates from using the same channel even when facing the same level of interferences.

A player's (total) payoff is the sum of payoffs from all the resources it uses. Note that if a player is allowed to simultaneously use all available resources, then its best strategy is to simply use all of them regardless of other players, provided that g_r^i is a non-negative function. If all players are allowed such a strategy, then the existence of a PNE is trivially true.

In this paper, we will limit our attention to the case where each player is allowed only one resource at a time, i.e., its strategy space $\Sigma_i = \mathcal{R}$ consists of R single resource strategies. In this case, the payoff player i receives for using a single resource r is given by $g_r^i(n_r^i + 1)$ where $n_r^i(\boldsymbol{\sigma}) = |\{j : r = \sigma_j, j \in \mathcal{K}_i\}|$.

It is easy to see that we can equivalently represent this problem on a directed graph, where a node represents a player and a directed edge connects node i to node j if and only if $i \in \mathcal{K}_j$. The ACGG can now be stated as a graph coloring problem[4], where each node picks a color and receives a value depending on the conflict (number of same-colored neighbors to a node); the goal is to see whether a PNE exists and whether a decentralized selfish scheme leads to a PNE. In this paper we will limit our attention to the case of undirected graphs, where there is an undirected edge between nodes i and j if and only if $i \in \mathcal{K}_j$ and $j \in \mathcal{K}_i$. This has the intuitive meaning that if node i interacts with node j, the reverse is also true. This symmetry does not always hold in reality, but is often a good approximation, and helps us obtain meaningful insight. Another reason for this assumption is that a PNE does not always exist in a directed graph (as we show in Example 4 via a counter example).

For simplicity of exposition, in subsequent sections we will often present the problem in its coloring version, and will use the terms *resource*, *color*, and *strategy* interchangeably. For the remainder of the paper, unless stated otherwise we shall assume that every ACGG we consider has the following properties: (a) players only employ one resource at a given time; (b) the payoff functions are player-specific and non-increasing; and (c) the interaction graph is undirected.

V. Existence of the Finite Improvement Property

In this section we investigate whether an ACGG always possesses the FIP as in the standard CG. Note that if a game has the FIP, it immediately follows that it has a PNE as described in Section III. Specifically, we show that in the following two cases an ACGG possesses the FIP: (a) when there are only two resources to choose from, and (b) when all resources are identical to a player, for all players.

A. The Finite Improvement Property for 2 Resources

The following theorem shows that FIP holds when each player can only select one of the two resources.

Theorem 1: An ACGG has the finite improvement property (FIP) when there are only two resources.

Proof: We prove this theorem by a potential function argument. Consider an ACGG with two resources, 0 and 1. Consider a player i. Let $n_{\max}^i = |\mathcal{K}_i|$ denote the number of neighbors that i has on the interaction graph. Define the mapping F^i such that for all $x \in \{0, 1, \ldots, n_{\max}^i\}$ we have

$$F^i(x) = g_1^i(1 + x) - g_0^i(1 + n_{\max}^i - x). \tag{3}$$

[4]We will use several colored graphs in our analysis, which may not show as effectively in a black/white version.

Since $g_1^i(1 + x)$ and $g_0^i(1 + n_{\max}^i - x)$ are non-increasing and non-decreasing in x, respectively, we have that $F^i(x)$ is non-increasing in x.

Define a *threshold* τ_i, which can be thought as the minimal number of neighbors that i must have using resource 1, so that i prefers to use resource 0. More precisely:

(a) If $F^i(x) \geq 0$, $\forall x \in \{0, \ldots, n_{\max}^i\}$, then let $\tau_i = 1 + n_{\max}^i$.

(b) If $F^i(x) < 0, \forall x \in \{0, \ldots, n_{\max}^i\}$, then let $\tau_i = -1$.

(c) Otherwise, we define τ_i to be the minimal value of $x \in \{0, \ldots, n_{\max}^i\}$ such that $F^i(x) < 0$.

To see that τ_i is well defined, note that if (a) and (b) are false we must have $F^i(0) \geq 0$ and $F^i(n_{\max}^i) < 0$ (because F^i is non-increasing). This means that τ_i, as described in (c), must exist. Also note that when condition (c) holds, we have $\forall x \in \{0, \ldots, n_{\max}^i\}$ that $x < \tau_i$ implies $F^i(x) \geq 0$ and $x \geq \tau_i$ implies $F^i(x) < 0$. Consider the function

$$V(\boldsymbol{\sigma}) = E_1(\boldsymbol{\sigma}) - \sum_{j \in \mathcal{I}} \sigma_j \tau_j, \tag{4}$$

where $E_1(\boldsymbol{\sigma}) = \frac{1}{2} \sum_{i \in \mathcal{I}: \sigma_i = 1} |\{j \in \mathcal{K}_i : \sigma_j = 1\}|$ equals to the number of edges in the interaction graph which link players of resource 1.

Consider a strategy profile $\boldsymbol{\sigma}$, from which some player i makes an improvement by changing its strategy. This improvement leads to a new strategy profile $\boldsymbol{\sigma'}$ such that $\sigma'_i \neq \sigma_i$ and $\sigma'_j = \sigma_j$ for each $j \neq i$. There are only two possible scenarios:

Case 1: $\sigma_i = 0$ and $\sigma'_i = 1$, i.e., the player switches from resource 0 to resource 1. In this case we can conclude $E_1(\boldsymbol{\sigma'}) = E_1(\boldsymbol{\sigma}) + n_1^i(\boldsymbol{\sigma})$, since there will be $n_1^i(\boldsymbol{\sigma})$ new edges in $\boldsymbol{\sigma'}$ linking players of resource 1. We also have $\sum_{j \in \mathcal{I}} \sigma'_j.\tau_j = \tau_i + \sum_{j \in \mathcal{I}} \sigma_j.\tau_j$. From these it follows that we have

$$V(\boldsymbol{\sigma'}) = V(\boldsymbol{\sigma}) + n_1^i(\boldsymbol{\sigma}) - \tau_i. \tag{5}$$

Now we claim that we must also have $n_1^i(\boldsymbol{\sigma}) < \tau_i$. To see this, note that in order for user i's channel switching (i.e., switching from resource 0 to resource 1) to be an improvement step, we must have $g_1^i(1 + n_1^i(\boldsymbol{\sigma})) > g_0^i(1 + n_0^i(\boldsymbol{\sigma}))$, where $n_0^i(\boldsymbol{\sigma}) = n_{\max}^i - n_1^i(\boldsymbol{\sigma})$. This implies $F^i(n_1^i(\boldsymbol{\sigma})) > 0$, which means either condition (a) or condition (c), listed above, must hold. If condition (a) holds, then $n_1^i(\boldsymbol{\sigma}) < \tau_i = n_{\max}^i + 1$. If condition (c) holds, then since $F^i(n_1^i(\boldsymbol{\sigma})) > 0$ and τ_i is the minimal value such that $F^i(\tau_i) < 0$, we must have $n_1^i(\boldsymbol{\sigma}) < \tau_i$. We have therefore shown that $n_1^i(\boldsymbol{\sigma}) < \tau_i$, and thus proved that $V(\boldsymbol{\sigma'}) < V(\boldsymbol{\sigma})$, for this scenario.

Case 2: $\sigma_i = 1$ and $\sigma'_i = 0$, i.e., the player switches from resource 1 to resource 0. In this case $E_1(\boldsymbol{\sigma'}) = E_1(\boldsymbol{\sigma}) - n_1^i(\boldsymbol{\sigma})$, since there will be $n_1^i(\boldsymbol{\sigma})$ less edges in $\boldsymbol{\sigma'}$ linking players of resource 1. We also have $\sum_{j \in \mathcal{I}} \sigma'_j.\tau_j = -\tau_i + \sum_{j \in \mathcal{I}} \sigma_j.\tau_j$. From these it follows that we have

$$V(\boldsymbol{\sigma'}) = V(\boldsymbol{\sigma}) - n_1^i(\boldsymbol{\sigma}) + \tau_i. \tag{6}$$

Now we claim that we must also have $n_1^i(\boldsymbol{\sigma}) \geq \tau_i$. To see this, note that in order for user i's channel switching (i.e., switching from resource 0 to resource 1) to be an improvement step, we must have $g_1^i(1 + n_1^i(\boldsymbol{\sigma})) < g_0^i(1 + n_0^i(\boldsymbol{\sigma}))$, where $n_0^i(\boldsymbol{\sigma}) =$

$n^i_{\max} - n^i_1(\boldsymbol{\sigma})$. This implies $F^i(n^i_1(\boldsymbol{\sigma})) < 0$, which means either condition (b) or condition (c), listed above, must hold. If condition (b) holds, then $n^i_1(\boldsymbol{\sigma}) > \tau_i = -1$. If condition (c) holds, then since $F^i(n^i_1(\boldsymbol{\sigma})) < 0$ and τ_i is minimal such that $F^i(\tau_i) < 0$, we must have that $n^i_1(\boldsymbol{\sigma}) \geq \tau_i$. We have therefore shown that $n^i_1(\boldsymbol{\sigma}) \geq \tau_i$, and thus proved that $V(\boldsymbol{\sigma'}) \leq V(\boldsymbol{\sigma})$, for this scenario.

With the above two cases, we have shown that for every improvement step, either the value of the function V decreases, or the value of V remains constant but the number of players of resource 1 decreases. We next show that an improvement loop (where a sequence of improvement steps leads to the same state being visited more than once) is impossible. This is done by contraction.

Suppose that a sequence of strategy profiles $\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, ..., \boldsymbol{\sigma}^T$ form an improvement loop, such that $T > 0$, $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}^0$, and for each $t \in \{0, 1, ..., T-1\}$ we have that $\boldsymbol{\sigma}^{t+1}$ is obtained by taking profile $\boldsymbol{\sigma}^t$ and having some player perform an improvement step. Now, since V never increases during an improvement step, we must have $V(\boldsymbol{\sigma}^0) \geq V(\boldsymbol{\sigma}^1) \geq ... \geq V(\boldsymbol{\sigma}^T)$. At the same time, since $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}^0$ we have $V(\boldsymbol{\sigma}^T) = V(\boldsymbol{\sigma}^0)$. Therefore we must have $V(\boldsymbol{\sigma}^0) = V(\boldsymbol{\sigma}^1) = ... = V(\boldsymbol{\sigma}^T)$. But we have just proved that whenever V does not decrease during an improvement step, the number of players of resource 1 must decrease. So this implies profile $\boldsymbol{\sigma}^T$ has less players of resource 1 than $\boldsymbol{\sigma}^0$, which contradicts our assumption that $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}^0$. This contradiction implies that, in fact, an improvement loop does not exist.

The fact that improvement loops cannot exist means that every improvement path must be finite (because the set of different possible profiles is finite, and no profile can be visited more than once within an improvement path), so every finite improvement path must eventually terminate at a profile from which no further improvement steps can be performed. Such a terminal profile must be a pure Nash equilibrium, hence our system has the finite improvement property. ■

Theorem 1 establishes that when there are only two resources, the FIP holds, and consequently a PNE exists. Note that the above proof uses a potential function argument, but technically the function V is not a "proper" potential function because it does not (strictly) decrease with every improvement step. However the function $U(\boldsymbol{\sigma}) = V(\boldsymbol{\sigma}) + \epsilon \cdot H(\boldsymbol{\sigma})$ is a proper potential function, which strictly decreases with every improvement step. Here $H(\boldsymbol{\sigma})$ is the number of players of resource 1 in profile $\boldsymbol{\sigma}$ and $\epsilon > 0$ is a real chosen to be suitably small (smaller than the amount that V decreases by, whenever V decreases due to an improvement step).

B. The Finite Improvement Property for Identical Resources for Each player

The next theorem shows the second case in which the FIP holds, when all resources are identical to each player, but different players can have different payoff functions. In the context of a multi-channel wireless system, this can represent the case where all channels have the same bandwidth and statistically similar channel quality to each player (e.g, either with frequency flat fading or with proper channel interleaving such as the IEEE 802.16d/e standard [25]), but from player to player, their perceived channel conditions may vary.

Theorem 2: For an ACGG, if for all $r \in \mathcal{R}$, $i \in \mathcal{I}$, and $n \in \{1, \ldots, N\}$, we have $g^i_r(n) = g^i(n)$, then the game has the finite improvement property (FIP).

Proof: We prove this theorem by using a potential function argument. Recall that player i's total payoff under the strategy profile $\boldsymbol{\sigma}$ is given by $g^i(\boldsymbol{\sigma}) = g(n^i(\boldsymbol{\sigma}) + 1)$, with $n^i(\boldsymbol{\sigma}) = |\{j : \sigma_j = \sigma_i, j \in \mathcal{K}_i\}|$, where $\sigma_i \in \mathcal{R}$, and we have suppressed the subscript r since all resources are identical.

Now consider the following function defined on the strategy profile space:

$$\phi(\boldsymbol{\sigma}) = \sum_{i,j \in \mathcal{K}} \mathbf{1}(i \in \mathcal{K}_j)\mathbf{1}(\sigma_i = \sigma_j) = \frac{1}{2}\sum_{i \in \mathcal{K}} n^i(\boldsymbol{\sigma}) \ , \qquad (7)$$

where the indicator function $\mathbf{1}(A) = 1$ if A is true and 0 otherwise. For a particular strategy profile $\boldsymbol{\sigma}$, this function ϕ is the sum of all pairs of players that are connected (neighbors of each other) and have chosen the same resource under this strategy profile. Viewed on a graph, this function is the total number of edges connecting nodes with the same color.

We see that every time player i improves its payoff by switching from strategy σ_i to $\sigma_i^{'}$ and thus reducing $n^i(\boldsymbol{\sigma}^{-i}, \sigma_i)$ to $n^i(\boldsymbol{\sigma}^{-i}, \sigma_i^{'})$ (as g^i is a non-increasing function), the value of $\phi(\cdot)$ strictly decreases accordingly.[5] Since our potential function (which is equal to the number of edges linking a pair of players of the same resource) takes values $\{0, 1, ..., \frac{n(n-1)}{2}\}$ and decreases with each asynchronous improvement step, our game converges to a PNE in quadratic time, when it evolves via asynchronous improvement steps. Hence the game has the FIP. ■

VI. Existence of a Pure Strategy Nash Equilibrium

In this section, we examine what graph properties will guarantee the existence of a PNE in the absence of the FIP. Specifically, we show that a PNE always exists for ACGGs defined on the following types of graphs: (a) a tree, (b) a loop, and (c) a regular, bipartite graph with non-player specific payoff functions.

A. Existence of PNE on a Tree Graph

We show that a PNE exists when the underlying graph is given by a tree. We denote by G_N the network (graph) of the N-player ACGG Γ_N. As before, the payoff functions $g^i_r(n^i_r)$ are non-increasing, and $n^i_r(\boldsymbol{\sigma})$ denotes the number of neighbors of player i (excluding i) using strategy r.

Lemma 1: If every N-player ACGG Γ_N has at least one PNE, then every $(N+1)$-player ACGG Γ_{N+1} formed by connecting a new player to an existing player in a N-player network G_N has at least one PNE.

Remark 1: Note that in this lemma, the network G_N itself does not have to be a tree. The lemma states that as long as a

[5] It's easy to see that a non-increasing function $G(\sum_{i,j \in \mathcal{K}} \mathbf{1}(i \in \mathcal{K}_j)\mathbf{1}(\sigma_i = \sigma_j))$ is an ordinal potential function of this game, as its value improves each time a player's individual payoff is improved (which decreases the value of its argument).

PNE exists for one class of networks, then by adding one more node through a single link, a PNE exists in the new network.

Proof: By assumption Γ_N has a PNE denoted by $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$. Suppose Γ_N is in such a PNE. Now connect new player $N+1$ to an arbitrary player j in G_N. This is illustrated in Figure 1.

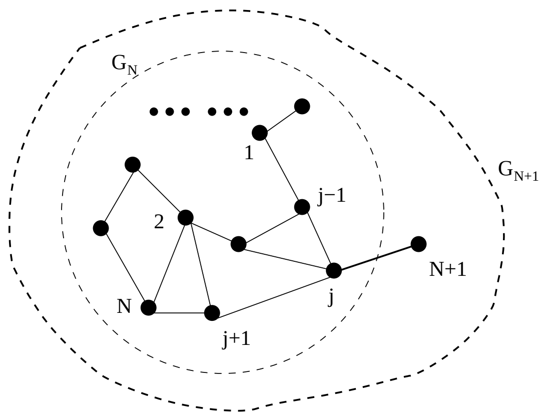

Fig. 1. Adding one more player to the network G_N with a single link.

Let player $N+1$ select its best response strategy:

$$\sigma_{N+1} = r_o = \text{argmax}_{r \in \mathcal{R}} g_r^{N+1}\left(n_r^{N+1}(\boldsymbol{\sigma}) + 1\right),$$

where $n_r^{N+1}(\boldsymbol{\sigma})$ is defined on the extended network G_{N+1}, and it counts the $(N+1)$-th node's neighbors using resource r under $\boldsymbol{\sigma}$. We now consider three cases depending on j's strategy change in response to the network expansion from G_N to G_{N+1}.

Case 1: $\sigma_j \neq r_o$. In this case, player $N+1$ selected a resource different from j's, so j has no incentive to change its strategy in response to the addition of player $N+1$. In turn player $N+1$ will remain in r_o as this is its best response, and no other players are affected by this single-link network extension. Thus the strategy profile $(\sigma_1, \cdots, \sigma_N, r_o)$ is a PNE for the game Γ_{N+1}.

Case 2: $\sigma_j = \sigma_{N+1} = r_o$, and player j's best response to the network expansion remains $\sigma_j = r_o$. That is, even with the additional interfering neighbor $N+1$, the best choice for j remains r_o. In this case again we reach a PNE for the game Γ_{N+1} with the same argument as in Case 1.

Case 3: $\sigma_j = \sigma_{N+1} = r_o$, and player j's best response to this network expansion is to move away from strategy r_o. In this case more players may in turn change strategies. Suppose we hold player $(N+1)$'s strategy fixed at r_o. Consider now a new N-player ACGG $\bar{\Gamma}_N$, defined on the original network G_N, but with the following modified payoff functions for $r \in \mathcal{R}$ and $i \in \mathcal{I}$:

$$\bar{g}_r^i(n_r^i + 1) = \begin{cases} g_r^i(n_r^i + 2) & \text{if } i = j, r = r_o \\ g_r^i(n_r^i + 1) & \text{otherwise} \end{cases}.$$

In words, the game $\bar{\Gamma}_N$ is almost the same as the original game Γ_N, the only difference being that the addition of player $(N+1)$ and its strategy r_o is built into player j's modified payoff function. By the assumption of Lemma 1, this game with N players has a PNE and we denote that by $\bar{\boldsymbol{\sigma}}$. Suppose $\bar{\boldsymbol{\sigma}}$ is reached in the network G_N with player $(N+1)$ fixed at $\sigma_{N+1} = r_o$. If we have $\bar{\sigma}_j = r_o$, then obviously player $(N+1)$ has no incentive to change its strategy because as far as it is concerned its environment has not changed. In turn no player in G_N will change its strategy because they are already in a PNE with player $(N+1)$ held at r_o. If $\bar{\sigma}_j \neq r_o$, then player $(N+1)$ has no incentive to change its strategy because j moved away from r_o which does not decrease player $(N+1)$'s payoff on this resource, and at the same time its payoff for using any other resource is no better. Again r_o is player $(N+1)$'s best response. In either case, strategy profile $(\bar{\boldsymbol{\sigma}}, r_o)$ is a new NE for the game Γ_{N+1}. ■

Theorem 3: Any ACGG defined over a *tree* has at least one PNE.

Proof: The proof is easily obtained by noting that any tree can be constructed by starting from a single node and adding one node (connected through a single link) at a time. Formally, we prove this by induction. Start with a single player indexed by 1. This game has a PNE, in which the player selects $\sigma_1 = \text{argmax}_{r \in \mathcal{R}} g_r^1(1)$ for any payoff functions. Assume that any N-player game Γ_N over a tree G_N with any set of non-increasing payoff functions has at least one PNE. Any tree G_{N+1} may be constructed by adding one more leaf node to some other tree G_N by connecting it to only one of the players in G_N. Lemma 1 guarantees that such a formation will result in a game with at least one PNE. ■

B. Existence of PNE on a Loop

Theorem 4: Any ACGG defined over a *loop* network has at least one PNE.

Proof (Sketch): The complete proof, which is lengthy, can be found in [26]. Here we only provide a sketch.

We begin this proof by assuming that every player on the loop always has a unique best response. In the event of a tie where $g_r^i(n) = g_{r'}^i(n')$ for some n and n', we can impose a unique best response by assuming that each player has a preference order among colors when the payoffs are the same.[6] Note that this assumption does not affect the validity of the proof, because relaxing it only widens the set of PNE a given game on the loop has.

Under our assumption, we show that every player i can be associated with a triple $(a(i), b(i), c(i)) \in \mathcal{R}^3$ of possible best responses to different scenarios. The triple has the following properties.

1) If i has no neighbors playing $a(i)$, then i's best response is $a(i)$, where $a(i) = \text{argmax}_{i \in \mathcal{R}}(g_r^i(1))$.
2) If i has one neighbor playing $a(i)$, with the other neighbor not playing $b(i)$, then i's best response is to play $b(i)$.
3) If i has one neighbor playing $a(i)$ and one neighbor playing $b(i)$, then i's best response is $c(i)$.

The main idea of the proof is to show the existence of PNE given the existence of players with various kinds of triples. We start by showing that if there exists a player i^* such that $a(i^*) = b(i^*)$, then a PNE exists. This is done by holding i^* fixed at $a(i^*)$ and letting the other players alter their strategies freely. Since the other players are essentially

[6]For example, a player with a color preference of "red>blue>green" will pick red if the payoffs of choosing red or blue are the same.

playing on a line graph (which is a type of tree graph) we use Theorem 3 to construct a strategy configuration within which each player in $\mathcal{I} - \{i\}$ employs their best response. We then show that allowing i^* to employ its best response under this configuration constitutes a PNE.

Next we show that if no such player i^* exists (so that $a(i) \neq b(i), \forall i$), a PNE must also exist. This is done by constructing an algorithm which produces strategy configurations that satisfy many of the players around the loop. The algorithm begins by assigning player 1 a strategy $\sigma_1 \in \{a(1), b(1)\}$. After this, the algorithm continues to allocate strategies σ_i to $i \in \{2, 3, ..., N\}$ in such a way that $\sigma_i = a(i)$ unless $a(i) = \sigma_{i-1}$ in which case $\sigma_i = b(i)$. We use this algorithm repeatedly to demonstrate the existence of PNE under several cases. The entire set of cases we consider exhausts all possible games where $a(i) \neq b(i), \forall i$. ■

C. *Existence of PNE on a Regular Bipartite Graph*

A graph is regular when all its vertices have the same number of connections. A graph is bipartite when its vertices can be numbered 1 and 2 (only two numbers) so that no edge connects a pair of vertices with the same number. Many well known graphs are regular and bipartite including hypercubes and rectangular lattices.

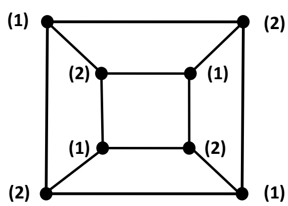

Fig. 2. The cube graph is regular and bipartite. (The numbers in the brackets near each player represents the resource selected by that player.)

Theorem 5: Any ACGG defined over a regular and bipartite network, with non-player specific payoff functions, has at least one PNE.

Proof: As payoff functions are not player-specific, we will suppress the superscript i in the function $g_r^i(\cdot)$. Suppose each vertex has degree d (so d denotes the number of connections each vertex has, e.g., $d = 3$ in Fig. 2). Without loss of generality, we order the resources such that the payoff functions satisfy $g_1(1) \geq g_2(1) \geq ... \geq g_R(1)$. If $g_1(d+1) \geq g_2(1)$, then resource 1 dominates and we can trivially construct an NE by allowing each player to use resource 1.

Now consider the case where $g_1(d+1) < g_2(1)$. Since our graph is bipartite, we may assign the vertices numbers 1 and 2 in such a way that no edge connects a pair of vertices with the same number. We can think of this numbering as a resource allocation $\boldsymbol{\sigma}$. Under this allocation each employer of 2 will receive payoff $g_2(1)$ (because they have no neighbors employing 2) whereas they *would* get $g_1(d+1) \leq g_2(1)$ if they played 1, which is no better. So each employer of 2 is playing its best response under $\boldsymbol{\sigma}$. In a similar way, the fact that $g_1(1) \geq g_2(1) \geq g_2(d+1)$ implies that each employer of 1 is playing its best response. ■

D. *Existence of PNE for Complete Graphs and a Dominant Resource*

We end this section by stating that an ACGG defined over a fully connected graph always has a PNE: ACGG over a complete graph simply reduces to the standard CG with player-specific payoff functions. The result has been given in [24].

Theorem 6: Any ACGG defined over a complete graph has at least one PNE.

We also note that regardless of the type of graphs, whenever there is a dominant resource r, i.e., its payoff function is such that $g_r^i(K_d+1) \geq g_{r'}^i(1)$, where $K_d = \max\{|\mathcal{K}_i|, i = 1, 2, \cdots, N\}$, for all $r' \in \mathcal{R}$ and all $i \in \mathcal{I}$, then a PNE obviously exists where all players share the same dominant resource.

VII. Counter Examples

In this section we present counter examples on the existence of a PNE and the FIP. Our main result in this section is Theorem 7 which shows that a PNE does not always exist in an ACGG.

A. *ACGG with 3 Resources*

Theorem 7: For an ACGG with 3 resources: a PNE does not always exist. Moreover, the FIP may not hold even when a PNE exists.

In Example 1 we give two instances of ACGG which justify Theorem 7.

Example 1: Consider the network topology given in Figure 3. Assume that the following set of inequalities holds for decreasing payoff functions.

$$\begin{aligned}
&g_r^1(1) < g_b^1(2); \quad g_b^2(2) < g_r^2(2); \quad g_r^3(2) < g_p^3(3);\\
&g_p^4(2) < g_b^4(2); \quad g_p^5(2) < g_b^5(2); \quad g_b^1(3) < g_r^1(2);\\
&g_r^2(2) < g_b^2(1); \quad g_p^3(1) < g_r^3(1); \quad g_b^4(1) < g_p^4(1);\\
&g_b^5(1) < g_p^5(1).
\end{aligned}$$

According to this set of inequalities, a best response loop exists which is given in Table I. Moreover, one can check[7] that a PNE does not exist for the payoff vectors $\boldsymbol{g}_j^i = (g_j^i(1), \ldots, g_j^i(n_{\max}^i + 1))$ given below, where $n_{\max}^i$ is the number of neighbors of i. They satisfy the above set of inequalities:

$$\begin{aligned}
&\boldsymbol{g}_b^1 = (12, 11, 8, 6), \quad \boldsymbol{g}_r^1 = (10, 9, 7, 5), \quad \boldsymbol{g}_p^1 = (4, 3, 2, 1)\\
&\boldsymbol{g}_b^2 = (8, 6, 4), \quad \boldsymbol{g}_r^2 = (9, 7, 5), \quad \boldsymbol{g}_p^2 = (3, 2, 1),\\
&\boldsymbol{g}_b^3 = (6, 5, 3, 2), \quad \boldsymbol{g}_r^3 = (12, 8, 4, 1), \quad \boldsymbol{g}_p^3 = (11, 10, 9, 7),\\
&\boldsymbol{g}_b^4 = \boldsymbol{g}_b^5 = (8, 7, 3), \quad \boldsymbol{g}_r^4 = \boldsymbol{g}_r^5 = (5, 4, 2),\\
&\boldsymbol{g}_p^4 = \boldsymbol{g}_p^5 = (9, 6, 1).
\end{aligned}$$

We also provide a counter example where a PNE exists but there is a best response loop. Simply let $g_r^5(1) = 10$

[7] The checking unfortunately can only be done numerically and exhaustively to the best of our knowledge.

time step	1	2	3	4	5
0	r	b	r	p	p
1	r → b				
2		b → r			
3			r → p		
4				p → b	
5					p → b
6	b → r				
7		r → b			
8			p → r		
9				b → p	
10					b → p

TABLE I
A BEST-RESPONSE LOOP IN THE 3-RESOURCE PNE COUNTER EXAMPLE.

while keeping all other payoffs the same. Then one can easily check that (b, r, p, b, r) is the unique PNE. Since the set of inequalities above does not include $g_r^5(1)$, they are still satisfied, thus a best response loop exists.

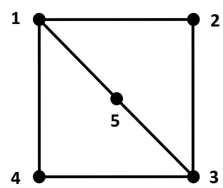

Fig. 3. A PNE counter example of 3 resources.

B. ACGG with 3 Resources and Non-Player Specific Payoffs

Proposition 2: In an ACGG with non-player specific payoffs, the FIP does not always hold even when a PNE exists.

Remark 2: Recall that the existence of FIP is sufficient but not necessary for the existence of a PNE. Both Theorem 7 and Proposition 2 are negative results. Theorem 7 implies the FIP does not generally hold in an ACGG with 3 resources, because otherwise a PNE would exist. Proposition 2 is a slightly weaker negative result; it says that in the special case with non-player specific payoff functions the FIP does not hold, i.e., there may be improvement loops. This however does *not* suggest a PNE does not exist either in this case, for the latter can exist without the former. Indeed it remains an intriguing open question whether a PNE always exists in an ACGG with non-increasing, non-player specific payoff functions.

In Example 2 we give an instance of ACGG which justifies Proposition 2.

Example 2: Suppose we have three colors to assign, denoted by r, p, and b. Consider a network topology shown in Figure 4, where we will primarily focus on nodes A, B, C and D. In addition to node C, node A is also connected to A_r, A_p and A_b nodes of colors red, green and blue, respectively. B_r, B_p, B_b, C_r, C_p, C_b, and D_r, D_p, D_b and similarly defined and illustrated in Figure 4. Note that these sets may not be disjoint, e.g., a single node may contribute to both A_r and B_r, and so on.

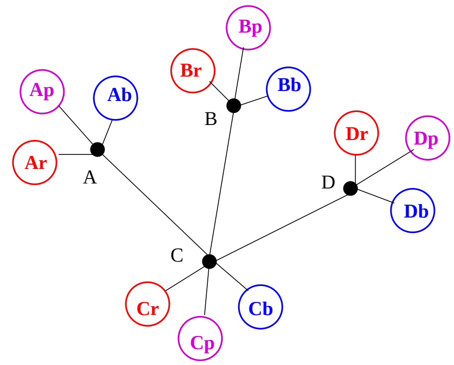

Fig. 4. An FIP counter example of 3 resources and non-player specific payoffs.

Consider now the sequence of improvement updates shown in Table II involving only nodes A, B, C, and D, i.e., within this sequence none of the other nodes change color (note that this is possible in an asynchronous improvement path), where the notation $s_1 \to s_2$ denotes a color change from s_1 to s_2. At time 0, the initial color assignment is given.

time step	A	B	C	D
0	b	p	p	b
1	b → r			
2		p → r		
3				b → r
4			p → r	
5	r → p			
6				r → b
7		r → b		
8			r → b	
9	p → b			
10			b → p	
11		b → p		

TABLE II
A BEST-RESPONSE LOOP IN THE 3-RESOURCE FIP COUNTER EXAMPLE.

We see that this sequence of color changes form a loop. If we can show that such loop is feasible, then we have found a counter example. For this to be an improvement loop such that each color change results in an improved payoff, it suffices for the following sets of conditions to hold. Since we assume all players have the same payoff function, we have suppressed the superscript i in $g_r^i(\cdot)$, and the notation "$>_k$" denotes that the improvement occurs at time k.

$$\begin{aligned}
& g_r(A_r+1) >_1 g_b(A_b+1) > g_b(A_b+2) \\
& >_9 g_p(A_p+1) >_5 g_r(A_r+2) \ ; \\
& g_r(B_r+1) >_2 g_p(B_p+2) >_{11} g_b(B_b+1) \\
& >_7 g_r(B_r+2) \ ; \\
& g_b(C_b+3) >_8 g_r(C_r+1) > g_r(C_r+4) \\
& >_4 g_p(C_p+1) >_{10} g_b(C_b+4) \ ; \\
& g_r(D_r+1) >_3 g_b(D_b+1) >_6 g_r(D_r+2)
\end{aligned}$$

It is straightforward to verify the sufficiency of these conditions by following a node's sequence of changes.

To complete this counter example, it remains to show that the above set of inequalities are feasible given appropriate choices of A_x, B_x, C_x and D_x, $x \in \{r, p, b\}$. There are many such choices; one example is $A_x = 5, B_x = 3, C_x = 7, D_x = 1$, for all $x \in \{r, p, b\}$. With such a choice, and substituting them into the earlier set of inequalities and through proper reordering, we obtain the following single chain of inequalities:

$$\begin{aligned} & g_r(2) > g_b(2) > g_r(3) > g_r(4) > g_p(5) > g_b(4) \\ > \quad & g_r(5) > g_r(6) > g_b(6) > g_b(7) > g_p(6) > g_r(7) \\ > \quad & g_b(10) > g_r(8) > g_r(11) > g_p(8) > g_b(11) \end{aligned} \quad (8)$$

It should be obvious that this chain of inequalities can be easily satisfied by the right choices of non-increasing payoff functions. It is easy to see how if we have more than 3 colors, this loop will still be an improving loop as long as the above inequalities hold. This means that for 3 colors or more the FIP does not hold in general. Note that the updates in this example are not always best response updates; they can be better responses which still result in payoff improvements.

Remark 3: The topology of Figure 4 can easily be made to represent a tree topology (i.e., any neighbor of A has A as the only neighbor, and so on). Then by Theorem 3, there exists at least one PNE of this game. However, we have just shown that the FIP does not hold.

C. ACGG with non-Monotonic Payoffs or Directed Interaction Graph

We further identify two cases where a PNE does not necessarily exist (and thus the FIP does not hold) with non-player specific payoff functions: when the payoffs are non-monotonic, and when the graph is directed; these are given in Examples 3 and 4, respectively.

Example 3: Consider a 3-player, 2-resource network given in Figure 5 with non-monotonic, non-player specific payoffs given as $g_1(1) = 2, g_1(2) = 5, g_1(3) = 3, g_2(1) = 4, g_2(2) = 6$, and $g_2(3) = 1$. One can easily check that there is no PNE from the game matrix corresponding to these payoff functions are given in Table III.

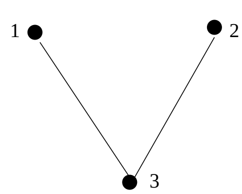

Fig. 5. Counter example of non-monotonic payoff functions.

Example 4: Consider the 3-player, 2-resource network given in Figure 6. Assume that the payoffs are $g_1(1) = 3, g_1(2) = 2, g_2(1) = 4, g_2(2) = 1$, which are non-player specific and non-increasing. It is easy to check that PNE does not exist by the game matrix given in Table IV.

U 3/ U 1,2	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
1	5,5,3 (3)	5,4,5 (3)	4,5,5 (3)	4,4,2 (1)
2	2,2,4 (2)	2,6,6 (1)	6,2,6 (2)	6,6,1 (3)

TABLE III
COUNTER EXAMPLE FOR PNE FOR NON-MONOTONIC PAYOFFS (FOR EACH MATRIX ENTRY a, b, c (d), FIRST THREE NUMBERS REPRESENT THE PAYOFFS TO PLAYERS 1, 2, 3 RESPECTIVELY WHILE (d) REPRESENTS THE INDEX OF THE PLAYER WHO CAN IMPROVE HIS PAYOFF BY DEVIATING FROM HIS STRATEGY).

U 3/ U 1,2	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
1	2,2,2 (3)	3,4,2 (3)	4,2,3 (2)	1,4,2 (1)
2	2,3,4 (1)	3,1,4 (2)	4,3,1 (3)	1,1,1 (3)

TABLE IV
PNE COUNTER EXAMPLE FOR DIRECTED GRAPH (FOR EACH MATRIX ENTRY a, b, c (d), FIRST THREE NUMBERS REPRESENT THE PAYOFFS TO PLAYERS 1, 2, 3 RESPECTIVELY WHILE (d) REPRESENTS THE INDEX OF THE PLAYER WHO CAN IMPROVE HIS PAYOFF BY DEVIATING FROM HIS STRATEGY).

VIII. APPLICATIONS OF ACGG

In this section we illustrate some applications of the ACGG model, and give example scenarios which can be modeled by the special interaction graphs studied in Section VI. We then discuss in more detail two applications in wireless networks: power control in multi-channel CDMA wireless network and IEEE 802.11 channel contention. We end this section with some numerical results on the convergence in an ACGG.

As mentioned in the introduction, in addition to networking applications, the ACGG can model the following scenario of business competition, where each vertex in a graph represents a different shop/business premise; two business premises are linked if they are close to one another; and resources represent business ventures (or average foot traffic per unit area). In a similar way the ACGG can be used to model how industrial organizations decide which natural resources (e.g., lumber, coal, gold) to harvest within their vicinity. More broadly, the ACGG can be used to model a congestion game with incomplete information. In a congestion game with incomplete information, all players in reality compete with everyone else, but a player is only aware of the presence of its neighbors on the graph.

Within the context of wireless networks, topologies like tree and loop may correspond to wireless devices deployed in a subway system, mine or along highways, while a bipartite network may correspond to a scenario where nodes are located in two separated areas with transmitter and receiver of the same user on different areas with directional antennas. An example of a regular bipartite graph topology is a wireless sensor network spaced out regularly to form a 2 dimensional grid.

A. Power Control in multi-channel CDMA wireless network

In the multi-channel CDMA wireless power control problem, the utility/rate player/user i gets for using channel k is often taken to be (see e.g., [27]).

$$\log\left(1 + \gamma \frac{h_{ii}^k P_i^k}{N_o + \sum_{j \neq i} h_{ji}^k P_j^k}\right),$$

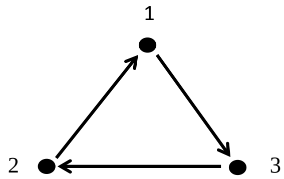

Fig. 6. Counter example for directed graphs.

where h_{ji}^k is the channel gain between the transmitter of user j and the receiver of user i, P_j^k is the transmit power of user j on channel k, N_o is the noise power, and $\gamma > 0$ is the spreading gain. Suppose we adopt the assumptions that (a) each user can only choose a single channel to transmit on at full power, and (b) any user j with $h_{ji}^k P_j^k < \epsilon$, for some $\epsilon > 0$ does not cause interference to user i, and (c) $h_{ji}^k P_j^k$ is (approximately) the same for all users $j \neq i$ who does not cause interference to i. The latter two assumptions validates the graph based model used here. Then the resulting power control problem can be modeled as an ACGG. Note that if we assume there are only two channels, then by Theorem 1 this game possesses the FIP and a PNE can be reached in a distributed way.

B. IEEE 802.11 Channel Contention

Another application of an ACGG is to analyze the random access scheme under the collision model such as IEEE 802.11. In this case only a single user can access a channel at each time slot, and the reward a user obtains from selecting a channel is the probability of accessing the channel multiplied by the rate that channel offers to the user. Suppose we assume, as is commonly done, that users sufficiently far apart from each other do not interfere with each other, and thus can access the same channel in the same slot. Let $p_j^i(n)$ be the probability that user i accesses channel j when the number of i's neighbors competing for channel j is n, and R_j^i be the rate of channel j seen by user i. Then the resulting channel contention can be modeled as an ACGG with user-specific payoff functions given by $g_j^i(n) = R_j^i p_j^i(n)$.

C. Numerical Results

We now present some numerical results on the expected number of asynchronous improvement steps needed to converge to a PNE (T_c) over random graphs, in a context similar to that of channel contention described above. Consider three resources (or channels) indexed $1, 2, 3$, and the following two cases. The first case is the random access scheme with identical channels and non-user specific payoffs. We assume that the payoff a user i gets from using a channel k when there are $n^i(k)$ neighbors using channel k is $g(n^i(k)+1) = 1/(n^i(k)+1)$; this can model a fair share of the channel among all users contending for the same channel under random access. The second case is the random access scheme with non-identical channels and non-user specific payoffs, where payoffs of user i are given by $g_k(n^i(k)+1) = k/(n^i(k)+1)$, $k \in \{1, 2, 3\}$; this would model the scenario that each competing user may get a different data rate when it has successfully obtained the right to use the channel due to different modulation/coding schemes, and so on. We have shown that in the first case FIP exists, therefore users will converge to a PNE in finite time by improvement steps. In the second case, we have a counter-example showing that better response loops may exist, but we have not shown an instance for which a PNE does not exist. Thus, we only consider best response updates, and for the second case we also check if the players ever enter a best-response loop ($T_c = \infty$ if this ever happens).

For each simulation, we consider 20 users randomly placed on a 10 by 10 square area. For each random placement of users, we generate the interference graph based on a threshold γ. If the distance between two users is greater than γ, we assume that they do not interfere with each other. We calculate T_c for each γ by averaging over 100 random placements of users and 100 runs for each random placement of users, where each run starts with a random strategy profile and the improving user in each step is randomly selected among all users who can improve their payoff by changing their strategies. We plot the average value of T_c for each threshold $\gamma \in \{1, 2, \ldots, 10\}$ for both cases in Figure 7, where the unit of T_c is the number of improvement steps. The results shows that the convergence is fairly fast. In the case of identical channels, convergence tends to be fast when the network is either sparsely connected (low threshold), or when the network is approximately fully connected (high threshold).

These observations are intuitively satisfying: when the network is sparsely connected (or even disconnected), smaller number of interconnected users leads to fewer number of updates; when the network is near fully connected, the impact of any update is immediately known to the users, which again leads to fewer number of updates needed. In between these two extremes, when the network is connected but not densely connected, an update can potentially impact all users in the network, but this impact may take much longer to propagate through the network. The non-identical channel case is more complex; however, it does follow the same trend except at very low thresholds. In addition, it generally takes longer to converge in this case than in the case of identical channels when all other parameters are the same.

We also note for all 10000 simulations we did not observe a single case where the users enter a best response loop. This is consistent with the difficulty we had in searching for counter examples (see Remark 2): in general, the instances in which a PNE or best-response FIP does not exists for ACGGs with non-user specific payoffs are very rare.

IX. Discussion

In this section, we discuss the relevance and limitations of the ACGG model and the results obtained in the context of wireless applications, and point to directions of further studies.

Two results obtained in this paper are of particular interest in the context of wireless networking, namely Theorem 1 and Theorem 2. Theorem 1 showed that when users are limited to only two channels, the finite improvement property holds over

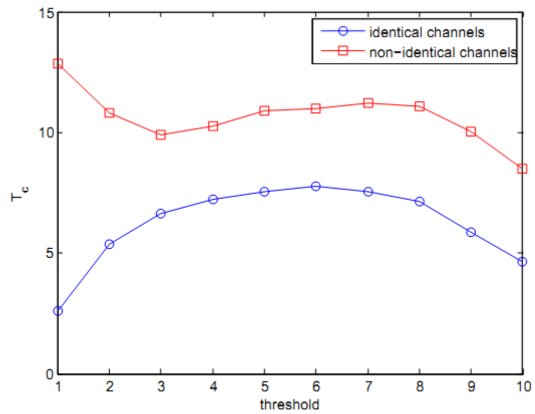

Fig. 7. T_c value depending on the threshold γ.

arbitrary graphs with user-specific payoff functions. Theorem 2 showed that when channels are of equal width and propagation characteristics for each user (as is the case when a contiguous block of bandwidth is evenly sliced into smaller channels), the finite improvement property holds. This is true even if the channels are of different quality to different users, e.g., due to the use of different modulation schemes. This latter scenario is a very realistic one, as this is the case with multiple channels in WiFi, bluetooth, and so on. The finite improvement property suggests that in such systems greedy user updates (i.e., using best-reply dynamics) will lead, in a finite number of steps, to a PNE, which is also the local minimizer of the explicit potential function (Eqn (7) in this case). This means that we not only have an easy way of obtaining a PNE, but also have a sense of the (local) efficiency of this NE.

In the weaker case where the existence of PNE is established, as shown in Theorems 3-5, players can similarly reach a PNE with probability one by using the uncoupled dynamics with 2-recall given in Theorem 3 of [28].

Our primary focus in this paper has been the existence of PNE and not its performance. As noted above, when the FIP holds, we can attain a PNE which is also a local optimal solution to the potential function, but there may be multiple PNE that result in different objective function values. In general the performance of a reached PNE is not guaranteed. It's also worth noting that although convergence to a PNE implies stationary behavior, even when the players cycle there can be cases where the average reward of all the players during a cycle will be higher than when they play at a PNE. Thus if the cost of switching is negligible, this type of cycling behavior may indeed be better than a PNE in terms of the objective function. An example of such a situation is given in [29], where the authors show that a natural learning algorithm cycles but this results in social welfare higher than that achieved at the unique (mixed) NE.

One limitation of the ACGG model is that it treats all interference relationships equally, i.e., the underlying network graph is unweighted (see e.g. the power control problem illustrated in Section VIII-A). In reality the channel quality perceived by a user depends not only on who else is using the same channel and can potentially interfere, but also its distances to these interfering users. One way to address this is to define the congestion game over a weighted network graph, and define the user payoff as a function of the weights on links connecting interfering users who use the same channel. Analysis along this line will be very interesting yet challenging.

Throughout our discussion, we have limited our attention to the case where each user can access one resource/channel at a time. In reality it's also possible for a user to access multiple channels at a time. As mentioned earlier, if all users can access all channels simultaneously and the available transmission power is decoupled across the channels, then the resulting congestion game is not particularly interesting, as an obvious PNE is where all users use all resources. A more interesting case is when users are limited to the number of channels they can access simultaneously. An additional feature may be that different users have different sets of channels they are allowed to access, i.e., user i's strategy space $\sigma_i \subset 2^{\mathcal{R}_i}$, where $\mathcal{R}_i \subset \mathcal{R}$ is user i's set of allowed channels. Finally, a user may need to spread the communication resource such as transmission power among multiple channels, thus transmitting over one or multiple channels implies different payoff functions for each channel. All these features will make the resulting game much more complicated and are subjects of future study.

X. Conclusion

In this paper, we considered an extension to the classical congestion games by allowing resources to be reused among non-interacting or non-interfering users. This extension is applicable in the context of wireless network, including spectrum sharing in multi-channel wireless systems, where spatial reuse is frequently exploited to increase spectrum utilization due to decay of wireless signals distance.

The resulting game, the atomic congestion game on graphs (ACGG), is a generalization of the original congestion game. We showed that the finite improvement property (FIP) holds when there are only two resources or the resources are identical to each user (but may be different between users). The FIP guarantees the existence of a pure strategy NE. We provided a negative result on the existence of FIP in the general case. We also showed that a pure strategy NE exists without the FIP if the network can be modeled by a tree, a loop, a regular bipartite graph, or with a dominant resource.

This work represents the first step in understanding how spatial relationship affects the properties of congestion games and its potential applications in networking. There are several ways of extending this work. One possibility is to consider more general directed weighted graphs, where the users have asymmetric interference relationship (both in direction and weight). The other direction is to understand how users will interact in these games with limited information, for example, through learning or imitation algorithms. Some preliminary work along these directions can be found in [30]–[32].

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