Noname manuscript No.
(will be inserted by the editor)

Fixed-Time Stable Proximal Dynamical System for Solving Mixed Variational Inequality Problems

Kunal Garg · **Mayank Baranwal** · **Rohit Gupta** · **Ramanarayan Vasudevan** · **Dimitra Panagou**

arXiv:1908.03517v2 [math.OC] 12 Aug 2019

Received: date / Accepted: date

Abstract In this paper, the *fixed-time* stability of a novel proximal dynamical system is investigated for solving mixed variational inequality problems. Under the assumptions of strong monotonicity and Lipschitz continuity, it is shown that the solution of the proposed proximal dynamical system exists in the classical sense, is uniquely determined and converges to the unique solution of the associated mixed variational inequality problem in a fixed time. As a special case, the proposed proximal dynamical system reduces to a novel fixed-time stable projected dynamical system. Furthermore, the fixed-time stability of the modified projected dynamical system continues to hold, even if the assumptions of strong monotonicity are relaxed to that of strong pseudomonotonicity. Connections to convex optimization problems are discussed, and commonly studied dynamical systems in the continuous-time optimization literature are shown as special cases of the proposed proximal dynamical system considered in this paper. Finally, several numerical examples are presented that corroborate the fixed-time convergent behavior of the proposed proximal dynamical system.

Keywords Fixed-Time stability · Proximal dynamical system · Mixed variational inequality problem

Kunal Garg
Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: kgarg@umich.edu

Mayank Baranwal
Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: mayankb@umich.edu

Rohit Gupta
Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: rohitgpt@umich.edu

Ramanarayan Vasudevan
Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: ramv@umich.edu

Dimitra Panagou
Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109, USA
E-mail: dpanagou@umich.edu

Mathematics Subject Classification (2000) 34D20 · 37N40 · 47J20

1 Introduction

In this paper, the *mixed variational inequality problem* (MVIP) of the form:

$$\text{Find } x^* \in \mathbb{R}^n \text{ such that } \langle F(x^*), x - x^* \rangle + g(x) - g(x^*) \geq 0 \text{ for all } x \in \mathbb{R}^n, \tag{1}$$

is considered, where $F : \operatorname{dom} g \to \mathbb{R}^n$ is an operator and $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous convex function with $\operatorname{dom} g := \{x \in \mathbb{R}^n : g(x) < \infty\}$. The MVIP is succinctly represented by $\mathrm{MVI}(F, \operatorname{dom} g)$. Note that solving the MVIP is equivalent to the problem of solving a generalized equation of the form:

$$\text{Find } x^* \in \mathbb{R}^n \text{ such that } 0 \in F(x^*) + \partial g(x^*),$$

where the sub-differential mapping $\partial g : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone (see [36]).

In particular, the focus of this paper is on designing dynamical systems such that their solutions converge to the solution of the MVIP in a *fixed time*, starting from any given initial condition.[1] MVIPs have numerous applications in optimization (see, e.g., [14, 18]), game theory (see, e.g., [8, 37]), control theory (see, e.g., [5, 29]), and other related areas (see, e.g., [26]). Many numerical methods ranging from discrete-time gradient based approaches to continuous-time gradient flows have been proposed for solving MVIPs. Proximal point algorithms are the most commonly used approaches for solving MVIPs. While earlier approaches to solving variational inequality problems show asymptotic convergence of proximal point algorithms (see, e.g., [16, 19]), some of the more recent works have shown exponential convergence to the equilibrium points (see, e.g., [13, 22]).

The function g in (1) is not necessarily differentiable, for instance, if it is required that the solution of the MVIP belongs to a non-empty, closed convex set $\mathscr{C} \subseteq \mathbb{R}^n$, then g can be chosen as the indicator function, i.e., $g = \delta_{\mathscr{C}}$, where

$$\delta_{\mathscr{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathscr{C}; \\ \infty, & \text{otherwise} \end{cases}$$

and the MVIP reduces to a variational inequality problem (VIP) of the form:

$$\text{Find } x^* \in \mathscr{C} \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \mathscr{C}. \tag{2}$$

The VIP is succinctly represented by $\mathrm{VI}(F, \mathscr{C})$. Projection methods can be used to solve the VIP, particularly when the projection mapping is easier to compute in a closed-form, and are quite popular in the literature.

While the treatment of MVIPs is done in full generality in this paper, special emphasis is placed on convex optimization problems. Note that a convex optimization problem of the form:

$$\min_{x \in \mathbb{R}^n} f(x) + h(x),$$

with $f : \operatorname{dom} h \to \mathbb{R}$ being a differentiable convex function and $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ being a proper, lower semi-continuous convex function, is equivalent to an MVIP with the operator

[1] The notion of fixed-time stability is defined formally in Definition 1.

$F = \nabla f$ and the function $g = h$ in (1). Similarly, also consider the *convex-concave* saddle point problem given by:

$$\min_{x_1 \in \mathbb{R}^{n_1}} \max_{x_2 \in \mathbb{R}^{n_2}} f(x_1, x_2) + h_1(x_1) - h_2(x_2),$$

where $f : \operatorname{dom} h_1 \times \operatorname{dom} h_2 \to \mathbb{R}$ is a differentiable convex-concave function, $h_1 : \mathbb{R}^{n_1} \to \mathbb{R} \cup \{\infty\}$ and $h_2 : \mathbb{R}^{n_2} \to \mathbb{R} \cup \{\infty\}$ are proper, lower semi-continuous convex functions. This saddle point problem can be re-cast as an MVIP, by letting $x = [x_1^\mathsf{T}\ x_2^\mathsf{T}]^\mathsf{T}$, the operator $F = [\nabla_{x_1} f^\mathsf{T}\ \ -\nabla_{x_2} f^\mathsf{T}]^\mathsf{T}$ and the function $g = h_1 + h_2$ in (1). Hence, a large class of optimization problems can equivalently be re-formulated as MVIPs.

The use of dynamical systems has emerged as a viable alternative for solving VIPs with a particular focus on optimization problems (see, e.g., [3, 8, 20, 22, 23, 28]). This viewpoint allows tools from Lyapunov theory to be employed for the design and analysis of novel dynamical systems that converge to the solution of a VIP. Under the assumptions of monotonicity and strong monotonicity on the operator F in (2), it is shown in [41, 42] that the solution of the VIP is globally asymptotically stable and globally exponentially stable, respectively, for the corresponding projected dynamical system. The authors in [20] relax the assumption of strong monotonicity by showing exponential convergence under the assumptions of strong pseudomonotonicity and Lipschitz continuity on the operator F in (2). The exponential convergence results are further generalized in the context of non-smooth convex optimization problems, in [22], under the assumptions of strong monotonicity and Lipschitz continuity on the operator F in (1).

In contrast to the aforementioned results with asymptotic or exponential stability guarantees, that pertain to convergence to an equilibrium point in an infinite time, in this paper, novel proximal dynamical systems are introduced so that the convergence is guaranteed in a fixed time. In [7], the authors introduced the notion of finite-time stability of an equilirbrium point, where the convergence of the solutions to the equilibrium point, is guaranteed in a finite time. The authors also give sufficient conditions for the finite-time stability of an equilirbrium point, in terms of the existence of a Lyapunov function. Under this notion, the settling-time, or time of convergence, depends upon the initial conditions and can grow unbounded with the distance of the initial condition from an equilibrium point. A stronger notion, called fixed-time stability, is developed in [32], where the settling-time has a finite upper bound for all initial conditions.

While there is some work on finite- or fixed-time stable schemes for certain classes of convex optimization problems, to the best of the authors' knowledge, this is the first paper proposing fixed-time stable proximal dynamical systems for MVIPs or general non-smooth convex optimization problems. In [12], authors show finite-time convergence of solutions of the normalized gradient flow to a minimizer of the unconstrained convex optimization problem. The authors in [9] consider convex optimization problems with equality constraints, and design a dynamical system with finite-time convergence guarantees to the minimizer of the convex optimization problem under the assumption of strong convexity of the objective function. In [27], the authors design a modified gradient flow scheme with fixed-time convergence guarantees assuming that the objective function is strongly convex. In [17], modified gradient flow schemes are introduced for unconstrained and constrained convex optimization problems, as well as for min-max problems posed as convex-concave optimization problems. The work in [17] only considered linear equality constraints, and assumed that the objective function is continuously differentiable, and satisfies strong or strict convexity, or is gradient-dominated. The schemes proposed in this paper apply to a broader class of problems, namely, MVIPs, and non-smooth convex optimization problems arise as special cases of the general framework considered in this paper. The proposed work has two main contributions:

(i) A novel modified continuous-time proximal dynamical system for solving MVIPs is proposed, and existence and uniqueness of solutions, as well as their convergence to the unique solutions of the corresponding MVIPs are shown.
(ii) Tools from fixed-time stability theory are leveraged to demonstrate global fixed-time convergence to the equilibrium point, i.e., all solutions of the modified proximal dynamical system converge to the solution of the associated MVIP in a fixed time irrespective of the initial conditions.

Since proximal dynamical systems are generalizations of projected dynamical systems, the results naturally extend to global fixed-time stability of suitably modified projected dynamical systems. Furthermore, a large class of optimization problems, namely convex optimization problems with and without constraints, as well as convex-concave optimization problems with or without constraints, arise as special cases of MVIPs, and hence, can be solved using the proposed schemes in this paper. To the best of authors' knowledge, fixed-time stability of continuous-time dynamical systems for solving MVIPs has not been explored earlier, and hence, the results of this paper substantially improve and extend the existing results available in the literature.

The rest of the paper is organized as follows. Some useful definitions in stability theory, convexity of functions, monotonicity and Lipschitz continuity of operators are reviewed in Section 2. The nominal proximal dynamical system is described in Section 4. The modified proximal dynamical system is described in Section 4 and it is shown that the solutions of the proposed proximal dynamical system exist globally and are unique, and are globally fixed-time convergent to the equilibrium point. The proposed method is then validated through several numerical examples in Section 5. The paper is concluded with detailed discussions and directions for future work.

2 Preliminaries

Some useful definitions on the various notions of stability of an equilibrium point of a vector field, convexity of functions, monotonicity and Lipschitz continuity of operators, together with a few supplementary results are reviewed below. In what follows, an inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, and $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ denotes the induced norm.

2.1 Notions of Stability

Consider the autonomous differential equation:

$$\dot{x}(t) = f(x(t)), \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f(0) = 0$, i.e., the origin is an equilibrium point of (3).[2]

Definition 1. *The origin of* (3) *is said to be:*

(i) ***Lyapunov stable*** *or simply, stable, if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t \geq 0$.*

[2] It is assumed that solutions of (3) exist in the classical sense and are uniquely determined.

(ii) ***Globally asymptotically stable****, if it is stable, and for any* $x(0) \in \mathbb{R}^n$*, the solution of* (3) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

(iii) ***Globally exponentially stable****, if it is globally asymptotically stable, and there exists* $a, \gamma > 0$ *such that*

$$\|x(t)\| \le ae^{-\gamma t}\|x(0)\|.$$

(iv) ***Globally finite-time stable****, if it is stable, and there exists a function* $T : \mathbb{R}^n \to [0,\infty)$ *such that for any* $x(0) \in \mathbb{R}^n$*, the solution of* (3) *satisfies* $\lim_{t\to T(x(0))} x(t) = 0$.

(v) ***Globally fixed-time stable****, if it is globally finite-time stable, and there exists* $\bar{T} < \infty$*, independent of the choice of the initial condition such that for any* $x(0) \in \mathbb{R}^n$*, the solution of* (3) *satisfies* $\lim_{t\to\bar{T}} x(t) = 0$.

Lemma 1 (Lyapunov condition for fixed-time stability [32]**).** *Suppose that there exists a continuously differentiable function* $V : \mathscr{D} \to \mathbb{R}$*, where* $\mathscr{D} \subseteq \mathbb{R}^n$ *is a neighborhood of the origin for* (3) *such that*

$$V(0) = 0,\ V(x) > 0$$

for all $x \in \mathscr{D}\setminus\{0\}$ *and*

$$\dot{V}(x) \le -\left(a_1 V(x)^{\gamma_1} + a_2 V(x)^{\gamma_2}\right)^{\gamma_3}$$

for all $x \in \mathscr{D}\setminus\{0\}$*, with* $a_1, a_2, \gamma_1, \gamma_2, \gamma_3 > 0$ *such that* $\gamma_1\gamma_3 < 1$ *and* $\gamma_2\gamma_3 > 1$*. Then, the origin of* (3) *is fixed-time stable such that*

$$T(x(0)) \le \frac{1}{a_1^{\gamma_3}(1-\gamma_1\gamma_3)} + \frac{1}{a_2^{\gamma_3}(\gamma_2\gamma_3 - 1)}$$

for any $x(0) \in \mathbb{R}^n$*. Furthermore, if the function* V *is radially unbounded and* $\mathscr{D} = \mathbb{R}^n$*, then the origin of* (3) *is globally fixed-time stable.*

Remark 1. *Lemma 1 provides characterization of fixed-time stability in terms of a Lyapunov function* V*. The existence of such a Lyapunov function for a suitably modified proximal dynamical system constitutes the foundation for rest of the analysis in the paper, where the above result is used with* $\gamma_3 = 1$.

2.2 Convexity of Functions

Some well-known definitions on the various notions of convexity of functions are given below (see, e.g., [24] for more details).

Definition 2. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a non-empty, open convex set. A differentiable function* $f : \Omega \to \mathbb{R}$ *is called:*

(i) ***Convex****, if*

$$f(x) \ge f(y) + \langle \nabla f(y), x-y\rangle \text{ for all } x, y \in \Omega.$$

(ii) ***Strongly convex*** *with modulus* μ*, if there exists* $\mu > 0$ *such that*

$$f(x) \ge f(y) + \langle \nabla f(y), x-y\rangle + \frac{\mu}{2}\|x-y\|^2 \text{ for all } x, y \in \Omega.$$

(iii) ***Pseudoconvex****, if*

$$\langle \nabla f(y), x-y\rangle \ge 0 \text{ implies } f(x) \ge f(y) \text{ for all } x, y \in \Omega.$$

(iv) ***Strongly pseudoconvex*** *with modulus* μ*, if there exists* $\mu > 0$ *such that*

$$\langle \nabla f(y), x-y\rangle \ge 0 \text{ implies } f(x) \ge f(y) + \frac{\mu}{2}\|x-y\|^2 \text{ for all } x, y \in \Omega.$$

2.3 Monotonicity of Operators

Some well-known definitions on the various notions of monotonicity of operators are given below (see, e.g., [20, 24] for more details).

Definition 3. *A mapping $F : \Omega \to \mathbb{R}^n$, where Ω is a non-empty subset of $\mathbb{R}^n$, is called:*

(i) ***Monotone****, if*

$$\langle F(x) - F(y), x - y\rangle \geq 0 \text{ for all } x, y \in \Omega.$$

(ii) ***Strongly monotone*** *with modulus μ, if there exists $\mu > 0$ such that*

$$\langle F(x) - F(y), x - y\rangle \geq \mu\|x - y\|^2 \text{ for all } x, y \in \Omega.$$

(iii) ***Pseudomonotone****, if*

$$\langle F(y), x - y\rangle \geq 0 \text{ implies } \langle F(x), x - y\rangle \geq 0 \text{ for all } x, y \in \Omega.$$

(iv) ***Strongly pseudomonotone*** *with modulus μ, if there exists $\mu > 0$ such that*

$$\langle F(y), x - y\rangle \geq 0 \text{ implies } \langle F(x), x - y\rangle \geq \mu\|x - y\|^2 \text{ for all } x, y \in \Omega.$$

Remark 2. *It is clear that in the definition above, (ii) implies (i) and (iv); (i) implies (iii); and (iv) implies (iii).*

Proposition 1 (Relationship between convexity of functions and monotonicity of operators [24]**).** *Let $f : \Omega \to \mathbb{R}$ be a differentiable function on a non-empty, open convex set $\Omega \subseteq \mathbb{R}^n$. Then the function f is convex (respectively, strongly convex with modulus μ and pseudoconvex) if and only if its gradient mapping $\nabla f : \Omega \to \mathbb{R}^n$ is monotone (respectively, strongly monotone with modulus μ and pseudomonotone). Furthermore, the function f is strongly pseudoconvex with modulus μ, if the mapping ∇f is strongly pseudomonotone with modulus μ.*

2.4 Lipschitz Continuity of Operators

The Lipschitz continuity of an operator is defined as follows:

Definition 4. *A mapping $F : \Omega \to \mathbb{R}^n$, where Ω is a non-empty subset of $\mathbb{R}^n$, is said to be Lipschitz continuous with Lipschitz constant L, if there exists $L > 0$ such that*

$$\|F(x) - F(y)\| \leq L\|x - y\| \text{ for all } x, y \in \Omega.$$

3 Proximal Operator and the Proximal Dynamical System

This section provides the definition of a proximal mapping (which are frequently used in algorithms used to solve non-smooth convex optimization problems), and lays out the foundation for the proximal dynamical system, proposed in the next section. Recall that the *proximal operator* associated with a proper, lower semi-continuous convex function $w : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is given by:

$$\text{prox}_w(x) := \arg\min_{y \in \mathbb{R}^n} \left(w(y) + \frac{1}{2}\|x - y\|^2 \right). \tag{4}$$

For solving the MVIP, first consider the following nominal proximal dynamical system:

$$\dot{x} = -\kappa\left(x - \text{prox}_{\lambda g}(x - \lambda F(x))\right), \tag{5}$$

where $\kappa, \lambda > 0$. In what follows, for the sake of brevity, set $y(x) := \text{prox}_{\lambda g}(x - \lambda F(x))$, where $x \in \mathbb{R}^n$. The following lemma establishes the relationship between an equilibrium point of the nominal proximal dynamical system and a solution of the associated MVIP.

Lemma 2. *A point $\bar{x} \in \mathbb{R}^n$ is an equilibrium point of* (5) *if and only if it solves* $\text{MVI}(F, \text{dom}\, g)$.

Proof. From [6, Proposition 12.26], it follows that

$$\begin{aligned} \bar{x} = y(\bar{x}) \text{ if and only if } & \langle (\bar{x} - \lambda F(\bar{x})) - \bar{x}, z - \bar{x}\rangle + \lambda g(\bar{x}) \le \lambda g(z), \\ \text{if and only if } & \lambda \langle F(\bar{x}), z - \bar{x}\rangle + \lambda g(z) - \lambda g(\bar{x}) \ge 0, \\ \text{if and only if } & \langle F(\bar{x}), z - \bar{x}\rangle + g(z) - g(\bar{x}) \ge 0 \end{aligned}$$

for all $z \in \mathbb{R}^n$. Hence, $\bar{x} \in \mathbb{R}^n$ is an equilibrium point of (5) if and only if it solves $\text{MVI}(F, \text{dom}\, g)$. ☐

Remark 3. *It is shown in [22] that an equilibrium point of* (5) *is exponentially stable for a strongly monotone and Lipschitz continuous operator F. Hence, under a suitable stability property of the equilibrium point of* (5)*, it follows that the nominal proximal dynamical system can be used to solve the associated MVIP.*[3]

4 Modified Proximal Dynamical System

This section describes a novel proximal dynamical system such that its equilibrium point is *fixed-time* stable, and solves the MVIP. In what follows, the following assumptions are always in place, unless otherwise explicitly stated:

Standing Assumptions. The operator F is:

(i) Strongly monotone with modulus μ.
(ii) Lipschitz continuous with Lipschitz constant L.

The following theorem will be required in the proof of the main result of the paper.

Theorem 1. *For every $\lambda \in \left(0, \frac{2\mu}{L^2}\right)$, there exists $c \in (0,1)$ such that*

$$\|y(x) - x^*\| \le c\|x - x^*\|$$

for all $x \in \mathbb{R}^n$, where $x^* \in \mathbb{R}^n$ is a solution of $\text{MVI}(F, \text{dom}\, g)$.

Proof. For any given $x \in \mathbb{R}^n$, from [6, Proposition 12.26], it follows that

$$\langle y(x) - (x - \lambda F(x)), z - y(x)\rangle \ge \lambda\left(g(y(x)) - g(z)\right) \tag{6}$$

for all $z \in \mathbb{R}^n$. In particular, for $z = x^*$ and after making some re-arrangements, (6) reads:

$$\langle y(x) - x, x^* - y(x)\rangle \ge \lambda\left(g(y(x)) - g(x^*)\right) + \lambda\left\langle F(x), y(x) - x^*\right\rangle. \tag{7}$$

[3] The existence and uniqueness of a solution of the MVIP holds for a strongly monotone and Lipschitz continuous operator F in (1) (see [2, Theorem 3.1]).

Furthermore, from (1), it follows that

$$\lambda\left(g(y(x))-g(x^*)\right)\geq\lambda\left\langle F(x^*),x^*-y(x)\right\rangle. \tag{8}$$

Using (8), (7) reads:

$$\langle x-y(x),x^*-y(x)\rangle\leq\lambda\left\langle F(x^*)-F(x),y(x)-x^*\right\rangle,$$

which can re-written as

$$\langle x-y(x),x^*-y(x)\rangle\leq\lambda\left\langle F(x^*)-F(y(x)),y(x)-x^*\right\rangle+\lambda\left\langle F(y(x))-F(x),y(x)-x^*\right\rangle. \tag{9}$$

From standing assumption (i), the first term in the right hand side of (9) can be upper bounded as follows:

$$\lambda\left\langle F(x^*)-F(y(x)),y(x)-x^*\right\rangle\leq-\lambda\mu\|x^*-y(x)\|^2. \tag{10}$$

Using the Cauchy–Schwarz inequality and the standing assumption (ii), the second term in the right hand side of (9) can be upper bounded as follows:

$$\lambda\left\langle F(y(x))-F(x),y(x)-x^*\right\rangle\leq\lambda L\|x-y(x)\|\|x^*-y(x)\|. \tag{11}$$

Using Cauchy's inequality, the right hand side of (11) can further be upper bounded as follows:

$$\lambda L\|x-y(x)\|\|x^*-y(x)\|\leq\frac{1}{2}\|x-y(x)\|^2+\frac{\lambda^2L^2}{2}\|x^*-y(x)\|^2$$

and so, (11) reads:

$$\lambda\left\langle F(y(x))-F(x),y(x)-x^*\right\rangle\leq\frac{1}{2}\|x-y(x)\|^2+\frac{\lambda^2L^2}{2}\|x^*-y(x)\|^2. \tag{12}$$

Using (10) and (12), the right hand side of (9) can be upper bounded as follows:

$$\langle x-y(x),x^*-y(x)\rangle\leq-\lambda\mu\|x^*-y(x)\|^2+\frac{1}{2}\|x-y(x)\|^2+\frac{\lambda^2L^2}{2}\|x^*-y(x)\|^2. \tag{13}$$

Furthermore, the left hand side of (13) can be re-written as

$$\langle x-y(x),x^*-y(x)\rangle=\frac{1}{2}\|x-y(x)\|^2+\frac{1}{2}\|x^*-y(x)\|^2-\frac{1}{2}\|x-x^*\|^2. \tag{14}$$

Using (14), (13) reads:

$$\begin{aligned}\|x-y(x)\|^2+\|x^*-y(x)\|^2-\|x-x^*\|^2\leq&-2\lambda\mu\|x^*-y(x)\|^2+\|x-y(x)\|^2\\&+\lambda^2L^2\|x^*-y(x)\|^2,\end{aligned}$$

which simplifies to

$$\|y(x)-x^*\|^2\leq\bar{c}\|x-x^*\|^2, \tag{15}$$

where $\bar{c}:=\frac{1}{1+2\lambda\mu-\lambda^2L^2}$. Note that $\bar{c}\in(0,1)$, since by the assumption of the theorem, $\lambda\in\left(0,\frac{2\mu}{L^2}\right)$ and so, (15) can be re-written as

$$\|y(x)-x^*\|\leq c\|x-x^*\|,$$

where $c:=\sqrt{\bar{c}}\in(0,1)$, which completes the proof of theorem, since c is independent of the choice of $x\in\mathbb{R}^n$. □

A novel proximal dynamical system is now introduced, such that its equilibrium point is fixed-time stable. Let $\mathrm{Fix}(y) := \{\bar{x} \in \mathbb{R}^n : y(\bar{x}) = \bar{x}\}$ and consider the modification of (5) given by:

$$\dot{x} = -\rho(x)\,(x - y(x))\,, \tag{16}$$

where

$$\rho(x) := \begin{cases} \kappa_1 \frac{1}{\|x-y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{1}{\|x-y(x)\|^{1-\alpha_2}}, & \text{if } x \in \mathbb{R}^n \setminus \mathrm{Fix}(y); \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

with $\kappa_1, \kappa_2 > 0$, $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$.

The following lemma establishes the relationship between equilibrium points of the modified and nominal proximal dynamical systems.

Lemma 3. *A point $\bar{x} \in \mathbb{R}^n$ is an equilibrium point of* (16) *if and only if it is an equilibrium point of* (5)*.*

Proof. Using (17), it is clear that if $\bar{x} \in \mathbb{R}^n$ is an equilibrium point of (16), then it is also an equilibrium point of (5). To show the other implication, it suffices to note that $\rho(x) = 0$ for any $x \in \mathrm{Fix}(y)$. □

Remark 4. *Lemma 3 shows that the equilibrium points of* (5) *are same as those of* (16) *and also from Lemma 2, it follows that an equilibrium point of* (5) *coincides with a solution of the associated MVIP. Hence, the modified proximal dynamical system can be used to solve the associated MVIP, under the standing assumptions on the operator F in* (1)*, which guarantee that a solution of the MVIP exists and is unique, and also a suitable stability property of the equilibrium point of* (16)*, which will be shown in Theorem 2.*

The following proposition establishes that the solutions of (16) exist in the classical sense and are uniquely determined.

Proposition 2. *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz continuous vector field such that*

$$X(\bar{x}) = 0 \text{ and } \langle x - \bar{x}, X(x) \rangle > 0$$

for all $x \in \mathbb{R}^n \setminus \{\bar{x}\}$. Consider the following autonomous differential equation:

$$\dot{x}(t) = -\sigma(x(t))X(x(t)), \tag{18}$$

where

$$\sigma(x) := \begin{cases} \kappa_1 \frac{1}{\|X(x)\|^{1-\alpha_1}} + \kappa_2 \frac{1}{\|X(x)\|^{1-\alpha_2}}, & \text{if } X(x) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

with $\kappa_1, \kappa_2 > 0$, $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$. Then, starting from any given initial condition, the solution of (18) *exists in the classical sense and is uniquely determined for all $t \geq 0$.*

The proof of the above proposition is given in Appendix A.

Remark 5. *In the case, when the vector field X is chosen to be the one in* (5)*, i.e.,* $X(x) := x - y(x)$ *for any* $x \in \mathbb{R}^n$*, then it can be shown that the vector field X has the property:*

$$\langle x - \bar{x}, X(x) \rangle > 0 \tag{20}$$

for all $x \in \mathbb{R}^n \setminus \{\bar{x}\}$*. To see this, first note that from [2, Theorem 3.1] and Lemma 2, it follows that the vector field in* (5) *has a unique equilibrium point* $\bar{x} = x^*$*, where* $x^* \in \mathbb{R}^n$ *is the solution of* $\mathrm{MVI}(F, \mathrm{dom}\, g)$*, i.e., the set* $\mathrm{Fix}(y)$ *consists only of a single element* $\bar{x} = x^*$.[4] *Furthermore, the following equality:*

$$\langle x - \bar{x}, x - y(x) \rangle = \|x - \bar{x}\|^2 + \langle x - \bar{x}, \bar{x} - y(x) \rangle, \tag{21}$$

holds for all $x \in \mathbb{R}^n$*. Using the Cauchy–Schwarz inequality and Theorem 1 (keeping in mind the fact that* $\bar{x} = x^*$*), the second term in the right hand side of* (21) *can be lower bounded and so,* (20) *reads:*

$$\langle x - \bar{x}, x - y(x) \rangle \geq (1 - c)\|x - \bar{x}\|^2,$$

where $c \in (0, 1)$*, from which, it follows that* (20) *holds for all* $x \in \mathbb{R}^n \setminus \{\bar{x}\}$.

The following lemma will also be required in the proof of the main result of the paper.

Lemma 4. *For every* $c \in (0, 1)$*, there exists* $\varepsilon(c) = \frac{\log(c)}{\log\left(\frac{1-c}{1+c}\right)} > 0$ *such that*

$$\left(\frac{1 - c}{1 + c}\right)^{1-\alpha} > c, \tag{22}$$

for any $\alpha \in (1 - \varepsilon(c), 1)$*. Furthermore,* (22) *holds for any* $c \in (0, 1)$ *and* $\alpha > 1$.

Proof. The proof for the first claim of the lemma is shown as follows. For any given $c \in (0, 1)$ and $\alpha \in (1 - \varepsilon(c), 1)$, it is clear that the following strict inequality:

$$(1 - \alpha) \log\left(\frac{1 - c}{1 + c}\right) > \log(c),$$

holds, from which, it follows that the following strict inequality:

$$\left(\frac{1 - c}{1 + c}\right)^{1-\alpha} > c,$$

also holds.

The proof for the second claim of the lemma is shown as follows. First note that the ratio $\left(\frac{1-c}{1+c}\right)^{1-\alpha}$ can be re-written as $\left(\frac{1+c}{1-c}\right)^{\alpha-1}$ for any $c \in (0, 1)$ and $\alpha > 1$. Furthermore, it is clear that the following strict inequality:

$$\left(\frac{1 + c}{1 - c}\right)^{\alpha-1} > 1,$$

holds for any $c \in (0, 1)$ and $\alpha > 1$, from which, it follows that the following strict inequality:

$$\left(\frac{1 - c}{1 + c}\right)^{1-\alpha} > c,$$

also holds. □

[4] Alternatively, let $x_1^* \in \mathbb{R}^n$ and $x_2^* \in \mathbb{R}^n$ be two distinct solutions of $\mathrm{MVI}(F, \mathrm{dom}\, g)$, where their existence follows from [2, Theorem 3.1]. Then, from Theorem 1, it follows that $\|x_1^* - x_2^*\| \leq c\|x_1^* - x_2^*\|$ and since $c \in (0, 1)$, it further follows that a solution $x^* \in \mathbb{R}^n$ of $\mathrm{MVI}(F, \mathrm{dom}\, g)$ is unique. Furthermore, from Lemma 2, it follows that the vector field in (5) has a unique equilibrium point $\bar{x} = x^*$, i.e., the set $\mathrm{Fix}(y)$ consists only of a single element $\bar{x} = x^*$.

The following theorem establishes the main result of the paper.

Theorem 2. *For every* $\lambda \in \left(0, \frac{2\mu}{L^2}\right)$*, there exists* $\varepsilon > 0$ *such that the solution* $x^* \in \mathbb{R}^n$ *of* $\mathrm{MVI}(F, \mathrm{dom}\, g)$ *is a globally fixed-time stable equilibrium point of* (16) *for any* $\alpha_1 \in (1-\varepsilon, 1) \cap (0,1)$ *and* $\alpha_2 > 1$.

Proof. First note that the vector field in (5) is Lipschitz continuous on $\mathbb{R}^n$, which follows from the Lipschitz continuity of the proximal operator (see, e.g., [6, Proposition 12.28]) and the standing assumption (ii), with a unique equilibrium point $\bar{x} = x^*$ (see Remark 5). Furthermore, it also satisfies the required property, which is assumed in Proposition 2 (see Remark 5). Hence, from Proposition 2, it follows that starting from any given initial condition, the solution of (16) exists in the classical sense and is uniquely determined for all $t \geq 0$. Consider now the candidate Lyapunov function:

$$V(x) := \frac{1}{2}\|x - x^*\|^2,$$

where from Lemma 3, it follows that $x^* \in \mathbb{R}^n$ is also the unique equilibrium point of the vector field in (16). The time-derivative of the candidate Lyapunov function V along the solution of (16), starting from any $x(0) \in \mathbb{R}^n \setminus \{x^*\}$, reads:

$$\begin{aligned}\dot{V}(x) &= -\left\langle x - x^*, \kappa_1 \frac{x - y(x)}{\|x - y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{x - y(x)}{\|x - y(x)\|^{1-\alpha_2}} \right\rangle \\ &= -\left\langle x - x^*, \kappa_1 \frac{x - x^*}{\|x - y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{x - x^*}{\|x - y(x)\|^{1-\alpha_2}} \right\rangle \\ &\quad - \left\langle x - x^*, \kappa_1 \frac{x^* - y(x)}{\|x - y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{x^* - y(x)}{\|x - y(x)\|^{1-\alpha_2}} \right\rangle. \end{aligned} \tag{23}$$

Using the Cauchy–Schwarz inequality, the second term in the right hand side of (23) can be upper bounded and so, (23) reads:

$$\begin{aligned}\dot{V}(x) \leq &-\left(\kappa_1 \frac{\|x - x^*\|^2}{\|x - y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{\|x - x^*\|^2}{\|x - y(x)\|^{1-\alpha_2}}\right) \\ &+ \left(\kappa_1 \frac{\|x - x^*\|\|x^* - y(x)\|}{\|x - y(x)\|^{1-\alpha_1}} + \kappa_2 \frac{\|x - x^*\|\|x^* - y(x)\|}{\|x - y(x)\|^{1-\alpha_2}}\right). \end{aligned} \tag{24}$$

Note that by the assumption of the theorem, $\lambda \in \left(0, \frac{2\mu}{L^2}\right)$ and so, Theorem 1 can be invoked. Using Theorem 1 and the triangle inequality, the following inequality:

$$\|x - y(x)\| \leq (1 + c)\|x - x^*\|, \tag{25}$$

holds, where $c \in (0,1)$. Similarly, using Theorem 1, the reverse triangle inequality and the fact that $c \in (0,1)$, the following inequality:

$$\|x - y(x)\| \geq (1 - c)\|x - x^*\|, \tag{26}$$

also holds. Using (25), (26) and Theorem 1, the right hand side of (24) can further be upper bounded and so, (24) reads:

$$\begin{aligned}\dot{V}(x) \leq &-\left(\frac{\kappa_1}{(1+c)^{1-\alpha_1}} \frac{\|x - x^*\|^2}{\|x - x^*\|^{1-\alpha_1}} + \frac{\kappa_2}{(1+c)^{1-\alpha_2}} \frac{\|x - x^*\|^2}{\|x - x^*\|^{1-\alpha_2}}\right) \\ &+ \left(\frac{c\kappa_1}{(1-c)^{1-\alpha_1}} \frac{\|x - x^*\|^2}{\|x - x^*\|^{1-\alpha_1}} + \frac{c\kappa_2}{(1-c)^{1-\alpha_2}} \frac{\|x - x^*\|^2}{\|x - x^*\|^{1-\alpha_2}}\right) \\ = &-p_1(\alpha_1)\|x - x^*\|^{1+\alpha_1} - p_2(\alpha_2)\|x - x^*\|^{1+\alpha_2}, \end{aligned} \tag{27}$$

where $p_1(\alpha_1) := \frac{\kappa_1}{(1-c)^{1-\alpha_1}}\left(\left(\frac{1-c}{1+c}\right)^{1-\alpha_1} - c\right)$ and $p_2(\alpha_2) := \frac{\kappa_2}{(1-c)^{1-\alpha_2}}\left(\left(\frac{1-c}{1+c}\right)^{1-\alpha_2} - c\right)$. Furthermore, from Lemma 4, it follows that there exists $\varepsilon(c) = \frac{\log(c)}{\log\left(\frac{1-c}{1+c}\right)} > 0$ such that (27) reads:

$$\dot{V}(x) \leq -\left(a_1(\alpha_1)V(x)^{\gamma_1(\alpha_1)} + a_2(\alpha_2)V(x)^{\gamma_2(\alpha_2)}\right),$$

with $a_1(\alpha_1) := 2^{\gamma_1(\alpha_1)}p_1(\alpha_1) > 0$ for any $\alpha_1 \in (1-\varepsilon(c), 1) \cap (0,1)$, where $\gamma_1(\alpha_1) := \frac{1+\alpha_1}{2} \in (0.5, 1)$ and $a_2(\alpha_2) := 2^{\gamma_2(\alpha_2)}p_2(\alpha_2) > 0$ for any $\alpha_2 > 1$, where $\gamma_2(\alpha_2) := \frac{1+\alpha_2}{2} > 1$. Finally, the proof can be concluded using Lemma 1. □

Remark 6. *Theorem 2 establishes fixed-time convergence of the modified proximal dynamical system to the solution of the MVIP. Furthermore, from Lemma 1 (keeping in mind the final inequality given in the proof of Theorem 2), it also follows that for any given* $\alpha_1 \in (1-\varepsilon(c), 1) \cap (0,1)$ *and* $\alpha_2 > 1$, *the following inequality:*

$$T(x(0)) \leq \frac{1}{a_1(\alpha_1)(1-\gamma_1(\alpha_1))} + \frac{1}{a_2(\alpha_2)(\gamma_2(\alpha_2)-1)},$$

holds for any $x(0) \in \mathbb{R}^n$. *Hence, for any given time budget* $\bar{T} < \infty$, *the parameters* $\kappa_1, \kappa_2, \alpha_1$ *and* α_2 *in* (16) *can always be chosen in a suitable way so as to achieve convergence under the given time budget* $\bar{T}$, *irrespective of any given initial condition.*

4.1 Modified Projected Dynamical System

In the special case, when the function w in (4), is chosen to be the indicator function of a non-empty, closed convex set $\mathscr{C} \subseteq \mathbb{R}^n$, the proximal operator reduces to the projection operator, i.e., $\mathsf{P}_{\mathscr{C}} = \text{prox}_{\delta_{\mathscr{C}}}$, where the projection operator is given by:

$$\mathsf{P}_{\mathscr{C}}(x) := \arg\min_{y \in \mathscr{C}} \|x - y\|$$

and so, the nominal proximal dynamical system reduces to a nominal projected dynamical system:

$$\dot{x} = -\kappa\left(x - \mathsf{P}_{\mathscr{C}}(x - \lambda F(x))\right), \tag{28}$$

which can be used to solve VIPs (see, e.g., [8,20,31,41]).[5] Furthermore, the modified proximal dynamical system now reduces to a novel projected dynamical system:

$$\dot{x} = -\rho(x)\left(x - \mathsf{P}_{\mathscr{C}}(x - \lambda F(x))\right). \tag{29}$$

It is shown in [8,20,31,41] that the equilibrium point of (28) is globally exponentially stable for a strongly monotone/pseudomonotone and Lipschitz continuous operator F. The following corollary of Theorem 2 establishes the global fixed-time stability of the equilibrium point of the modified projected dynamical system.

Corollary 1. *For every* $\lambda \in \left(0, \frac{2\mu}{L^2}\right)$, *there exists* $\varepsilon > 0$ *such that the solution* $x^* \in \mathbb{R}^n$ *of* VI$(F, \mathscr{C})$ *is a globally fixed-time stable equilibrium point of* (29) *for any* $\alpha_1 \in (1-\varepsilon, 1) \cap (0,1)$ *and* $\alpha_2 > 1$.

[5] The existence and uniqueness of a solution of the VIP holds for a strongly monotone/pseudomonotone and Lipschitz continuous operator F in (2) (see [40, Theorem 2.1]).

Remark 7. *In the special case of a projection operator, Theorem 1 continues to hold, even when the standing assumption (i) is relaxed to that of strong pseudomonotonicity (see the proof of [20, Theorem 2]). Hence, Corollary 1 continues to hold, even when the standing assumption (i) is relaxed to that of strong pseudomonotonicity. Furthermore, by following the steps given in the proof of Theorem 2, with $\kappa_2 = 0$ and $\alpha_1 = 1$, it can be seen that [20, Theorem 2] is now a special case of Corollary 1, from which only the exponential stability (instead of fixed-time stability) of the equilibrium point can be concluded now.*

4.2 Application to Convex Optimization Problems

Consider the unconstrained convex optimization problem of the form:

$$\min_{x \in \mathbb{R}^n} f(x) + h(x),$$

where $f : \operatorname{dom} h \to \mathbb{R}$ is a differentiable convex function and $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semi-continuous convex function. Note that the above unconstrained convex optimization problem subsumes the constrained convex optimization problem of the form:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} f(x) \\ &\text{ s.t.}\, p_i(x) = 0,\ i = 1, \ldots, l, \\ &\qquad p_j(x) \le 0,\ j = l+1, \ldots, m, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function, $p_i : \mathbb{R}^n \to \mathbb{R}$ and $p_j : \mathbb{R}^n \to \mathbb{R}$ are convex functions for every $i \in \{1, \ldots, l\}$ and $j \in \{l+1, \ldots, m\}$, by letting $\mathscr{C} := \{x \in \mathbb{R}^n : p_1(x) = 0, \ldots, p_l(x) = 0, p_{l+1}(x) \le 0, \ldots, p_m(x) \le 0\}$, which is assumed to be non-empty and the function $h = \delta_{\mathscr{C}}$. Furthermore, from [4, Lemma 2.1] or [30, Theorem 1-5.1], it follows that $x^* \in \mathbb{R}^n$ is a minimizer of the above unconstrained convex optimization problem if and only if it solves the MVIP, with the operator $F = \nabla f$ and the function $g = h$ in (1). Hence, if $f : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function such that its gradient mapping $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous, then from Theorem 2, it follows that $x^* \in \mathbb{R}^n$ is a globally fixed-time stable equilibrium point of (16), with the operator $F = \nabla f$ and the function $g = h$.[6]

5 Numerical Examples

The fixed-time convergent behavior of the modified proximal dynamical system is illustrated through several examples, which range from solving VIPs to an MVIP and a convex-concave saddle-point problem. The simulations are performed in `MATLAB` using the "`ode23s`" solver and the results are shown in `log-lin` plots, so as to have a better visualization of the rate of convergence.

[6] Note that from Proposition 1, it follows that the mapping ∇f is strongly monotone.

5.1 Variational Inequality Problems

Example 1 ([38,39]). Let $\mathscr{C} := \{x \in \mathbb{R}^{10} : x_1 \geq 0, \ldots, x_{10} \geq 0\}$ and consider now the operator F in (2) to be given by:

$$F(x) = [f_1(x) \ldots f_{10}(x)]^\mathsf{T} + Ax + b,$$

where

$$f_i(x) = x_i^2 + x_{i-1}^2 + x_{i+1}x_i + x_i x_{i-1},\ i = 1, \ldots, 10,$$

with $x_0 = 0$ and $x_{11} = 0$, the elements of the matrix $A \in \mathbb{R}^{10\times 10}$ are given by:

$$a_{ij} = \begin{cases} 4, & \text{if } i = j; \\ 1, & \text{if } i = j+1; \\ -2, & \text{if } i = j-1; \\ 0, & \text{otherwise} \end{cases}$$

and $b = [-1 \ldots -1]^\mathsf{T} \in \mathbb{R}^{10}$. It is not known, if the standing assumptions hold for this example, nonetheless, the modified projected dynamical system is still used to find a solution of the VIP. Figure 1 shows some sample results, with $k_1 = 10$, $k_2 = 10$, $\alpha_1 = 0.85$, $\alpha_2 = 1.5$ and $\lambda = 0.25$. Figure 2 shows some sample results for various values of $\alpha_1 \in [0.85, 1]$ and $\alpha_2 \in [1, 1.5]$, where the color in the plots shift from blue to red as α_1 decreases from 1 to 0.85 and α_2 increases from 1 to 1.5.

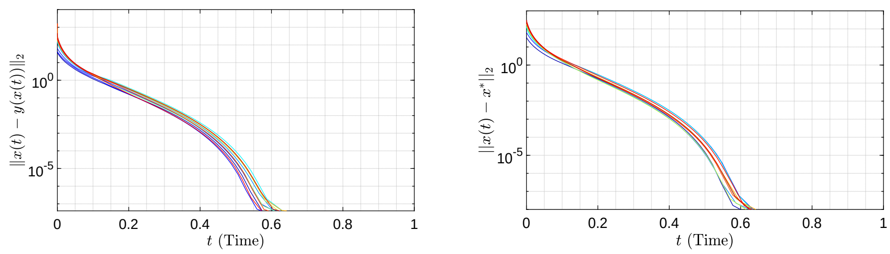

Fig. 1: Plots of $\|x(t) - y(x(t))\|_2$ vs. t (Time) and $\|x(t) - x^*\|_2$ vs. t (Time) for various initial conditions $x(0) \in \mathbb{R}^{10}$.

Example 2 ([23]). Let $\mathscr{C} := \{x \in \mathbb{R}^2 : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\}$ and consider now the operator F in (2) to be given by:

$$F(x) = [0.5x_1x_2 - 2x_2 - 10^7 \quad 0.1x_2^2 - 4x_1 - 10^7]^\mathsf{T},$$

which according to [23] can safely be assumed to be strongly pseudomonotone with modulus $\mu = 11$ (using a Monte Carlo approach) and also can be shown to be Lipschitz continuous, with Lipschitz constant $L = 5$. Since the standing assumptions hold for this example, from Corollary 1, it follows that for any $\lambda \in (0, 0.88)$, the solution of the VIP is a globally fixed-time stable equilibrium point of the modified projected dynamical system. Figure 3 shows some sample results, with $k_1 = 20$, $k_2 = 20$, $\alpha_1 = 0.8$, $\alpha_2 = 1.5$ and $\lambda = 0.44$. Figure 4 shows some sample results for various values of $\alpha_1 \in [0.8, 1]$ and $\alpha_2 \in [1, 1.2]$, where the color in the plots shift from blue to red as α_1 decreases from 1 to 0.8 and α_2 increases from 1 to 1.2.

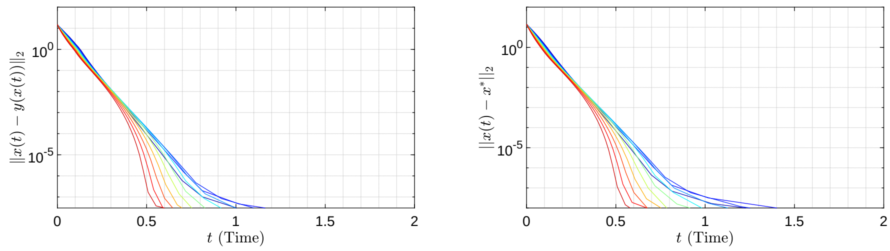

Fig. 2: Plots of $\|x(t)-y(x(t))\|_2$ vs. t (Time) and $\|x(t)-x^*\|_2$ vs. t (Time) for various values of $\alpha_1 \in [0.85, 1]$ and $\alpha_2 \in [1, 1.5]$.

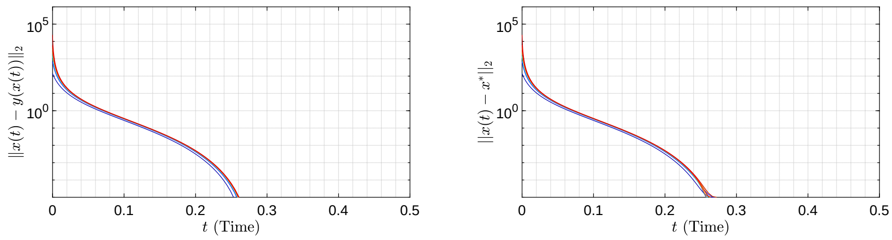

Fig. 3: Plots of $\|x(t)-y(x(t))\|_2$ vs. t (Time) and $\|x(t)-x^*\|_2$ vs. t (Time) for various initial conditions $x(0) \in \mathbb{R}^2$.

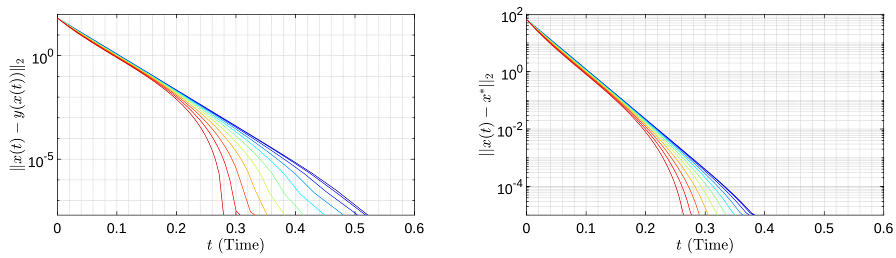

Fig. 4: Plots of $\|x(t)-y(x(t))\|_2$ vs. t (Time) and $\|x(t)-x^*\|_2$ vs. t (Time) for various values of $\alpha_1 \in [0.8, 1]$ and $\alpha_2 \in [1, 1.2]$.

5.2 Mixed Variational Inequality Problem

Example 3. Consider the following logistic regression problem with an L^1-regularization term:

$$\min_{x \in \mathbb{R}^{10}} \sum_{i=1}^{100} \log\left(1 + \exp(-a_i b_i^\top x)\right) + \eta \|x\|_1,$$

where $a_i \in \{-1, 1\}$, $b_i \in \mathbb{R}^{10}$, $i = 1, \ldots, 100$ are chosen randomly using the "`rand`" command in `MATLAB`, $\eta > 0$ and the non-smooth L^1-regularization term is added to prevent overfitting on the given data. It is not known, if the standing assumptions hold for this example, nonetheless, the modified proximal dynamical system is still used to find a solution of the above logistic regression problem with an L^1-regularization term. Figure 5 shows some sample results, with $k_1 = 10$, $k_2 = 10$, $\alpha_1 = 0.9$, $\alpha_2 = 1.1$ and $\lambda = 0.01$.

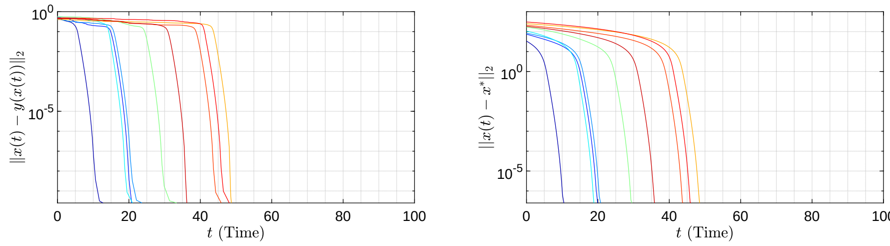

Fig. 5: Plots of $\|x(t) - y(x(t))\|_2$ vs. t (Time) and $\|x(t) - x^*\|_2$ vs. t (Time) for various initial conditions $x(0) \in \mathbb{R}^{10}$.

5.3 Convex-Concave Saddle Point Problem

Example 4 ([15]). Consider the following two-player continuous game problem:

$$\min_{x_1 \in \mathscr{C}} \max_{x_2 \in \mathscr{C}} \underbrace{x_1^2 - 2x_2^2 + 4x_1x_2 - 3x_1 - 2x_2 + 1}_{f(x)},$$

where f is the payoff or utility function and $\mathscr{C} := [0,1]$ is the set of actions or strategies available to each player. It can be shown that the operator $F : \mathbb{R}^2 \to \mathbb{R}^2$, where $F(x) := [\nabla_{x_1} f(x)^\top \ -\nabla_{x_2} f(x)^\top]^\top$, is strongly monotone with modulus $\mu = 2$ and also Lipschitz continuous, with Lipschitz constant $L = 4$. Since the standing assumptions hold for this example, from Corollary 1, it follows that for any $\lambda \in (0, 0.25)$, the solution of the above two-player continuous game problem is a globally fixed-time stable equilibrium point of the modified projected dynamical system. Figure 6 shows some sample results, with $k_1 = 10$, $k_2 = 10$, $\alpha_1 = 0.8$, $\alpha_2 = 1.5$ and $\lambda = 0.1$.

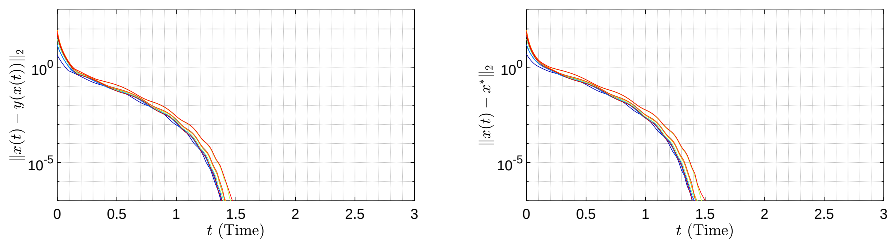

Fig. 6: Plots of $\|x(t) - y(x(t))\|_2$ vs. t (Time) and $\|x(t) - x^*\|_2$ vs. t (Time) for various initial conditions $x(0) \in \mathbb{R}^2$.

6 Discussions

Several numerical examples have been considered in Section 5, which corroborate the fixed-time convergent behavior of the proposed proximal dynamical system. While methods involving continuous-time dynamical systems to solve MVIPs or optimization problems are important and have major theoretical relevance, in general, only discrete-time iterative algorithms are of practical significance. It is still an open question as to what would be a discrete-time analogue of (16), with fixed-time convergence guarantees. In the numerical examples considered in Section 5, where the standing assumptions hold, it is observed that the convergence is *super-linear* and happens within a fixed time, irrespective of any given initial condition. Actually, this behavior holds for all the numerical examples considered in Section 5. To show that the convergence properties are still preserved after applying a suitable discretization scheme to a continuous-time dynamical system, is an active area of research (see [33–35]). In [33], the authors study a particular class of homogeneous systems and show that there exists a *consistent* discretization scheme that preserves the finite-time convergence property. They extend their results to *practically* fixed-time stable systems in [34], where they show that the trajectories of the discrete-time system reach to an arbitrary small neighborhood of the equilibrium point in a fixed number of time-steps, irrespective of any given initial condition. The hope is that this theory can further be expanded to include a more general class of finite- and fixed-time stable systems so as to be able to show that the fixed-time convergence property is still preserved after applying a suitable discretization scheme to the proposed proximal dynamical system.

7 Conclusions and Future Work

In this paper, a novel proximal dynamical system is presented such that its solution exists in the classical sense, is uniquely determined and converges to the unique solution of the associated MVIP in a fixed time, under the standard assumptions of strong monotonicity and Lipschitz continuity on the associated operator. Furthermore, as a special case, the proposed proximal dynamical system reduces to a novel fixed-time stable projected dynamical system, where the fixed-time stability of the modified projected dynamical system continues to hold, even if the assumption of strong monotonicity is relaxed to that of strong pseudomonotonicity. Finally, even though, the formulation of the proposed proximal dynamical system is in continuous-time, through various numerical examples considered in Section 5, it is observed that the proposed proximal dynamical system also exhibits super-linear convergence properties in the discrete-time setting.

One of the directions for future work is to investigate the fixed-time stability of the proposed proximal dynamical system in the more general setting of Hilbert or Banach spaces. Also, as mentioned in Section 6, a potential direction for future work is to investigate suitable "convergence-preserving" discretization schemes for the proposed proximal dynamical system, either in the finite- or infinite-dimensional setting. Finally, another potential direction for future work is to investigate the fixed-time stability of the proposed proximal dynamical system by further relaxing the assumptions of monotonicity and Lipschitz continuity on the associated operator.

A Proof of Proposition 2

Proof. The proof for the first claim of the proposition is shown as follows. First note that the equilibrium point $\bar{x} \in \mathbb{R}^n$ of the vector field X is unique, since under its properties assumed in the proposition, it can be shown that the equilibrium point $\bar{x} \in \mathbb{R}^n$ of the vector field $-X$ is globally asymptotically stable and hence, it is unique (another way to see this, is to use the Cauchy–Schwarz inequality to upper bound the left hand side of (20)). Furthermore, the vector field in (18) is continuous on $\mathbb{R}^n$. To see this, it suffices to show that the vector field in (18) is continuous at $\bar{x} \in \mathbb{R}^n$, since the function σ is continuous on $\mathbb{R}^n \setminus \{\bar{x}\}$ and the vector field X is assumed to be locally Lipschitz continuous on $\mathbb{R}^n$. To this end, it will suffice to show that $\lim_{x\to\bar{x}} \sigma(x)X(x) = 0$, which clearly holds, since $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$. Hence, from [21, Theorem I.1.1], it follows that for any given $x(0) \in \mathbb{R}^n$, there exists a solution of (18) on some interval $[0,\tau(x(0))]$, with $\tau(x(0)) > 0$. Furthermore, from [21, Theorem I.2.1] any such solution of (18) on the interval $[0,\tau(x(0))]$ has a continuation to a maximal interval of existence $[0,\bar{\tau}(x(0)))$. Consider now the candidate Lypaunov function:

$$V(x) := \frac{1}{2}\|x-\bar{x}\|^2.$$

The time-derivative of the candidate Lyapunov function V along a solution of (18), starting from $x(0) \in \mathbb{R}^n$, reads:

$$\dot{V}(x) = -\langle x-\bar{x}, \sigma(x)X(x)\rangle . \tag{30}$$

Recalling that $\langle x-\bar{x}, X(x)\rangle \geq 0$ and $\sigma(x) \geq 0$ for all $x \in \mathbb{R}^n$, it follows that the right hand side of (30) can be upper bounded and so, (30) reads:

$$\dot{V}(x) \leq 0.$$

Hence, $V(x(t)) \leq V(x(0))$ for all $t \in [0,\bar{\tau}(x(0)))$ and it follows that a solution of (18) defined on the interval $[0,\bar{\tau}(x(0)))$ lies entirely in $K_{x(0)} := \{z \in \mathbb{R}^n : \|z-\bar{x}\| \leq \|x(0)-\bar{x}\|\}$. Since the set $K_{x(0)}$ is compact, from [7, Proposition 2.1], it follows that $\bar{\tau}(x(0)) = \infty$.

The proof for the second claim of the proposition is shown as follows. For any given $x(0) \in \mathbb{R}^n$, let γ be a solution of (18), with $\gamma(0) = x(0)$ and consider the following two cases:

(i) Let $\gamma(0) \in \mathbb{R}^n \setminus \{\bar{x}\}$ and it will be shown that a solution corresponding to the vector field in (18) is also a solution corresponding to the vector field X, under a suitable reparameterization of time (see, e.g., [10, 11, Section 1.5]). Let $T := \inf\{t \geq 0 : \gamma(t) = \bar{x}\}$ and from the continuity of γ, it follows that $T > 0$. Consider now the function $\mathsf{s} : [0,T) \to [0,\infty)$ given by:

$$\mathsf{s}(t) := \int_0^t \sigma(\gamma(\nu))d\nu. \tag{31}$$

Since the function σ is continuous on $\mathbb{R}^n$, γ is continuous on the interval $[0,T)$ and $\sigma(\gamma(\nu)) > 0$ for any $\nu \in [0,T)$, it follows that the function s is a strictly increasing continuous function, with $\frac{d\mathsf{s}}{dt} \neq 0$ for all $t \in (0,T)$. Furthermore, from the inverse function theorem, it follows that the function $\mathsf{t} := \mathsf{s}^{-1}$ exists, is strictly increasing, continuous and satisfies:

$$\left.\frac{d\mathsf{t}}{ds}\right|_{s=\mathsf{s}(t)} = \frac{1}{\sigma(\gamma(t))} \tag{32}$$

for all $t \in (0,T)$. Let $\bar{\gamma}(s) := \gamma(\mathsf{t}(s))$ and from the chain rule, it follows that

$$\frac{d\bar{\gamma}}{ds} = \left.\frac{d\gamma}{dt}\right|_{t=\mathsf{t}(s)} \frac{d\mathsf{t}}{ds}. \tag{33}$$

Using (32), (33) reads:

$$\frac{d\bar{\gamma}}{ds} = -X(\bar{\gamma}(s)).$$

Hence, a solution corresponding to the vector field in (18) is also a solution corresponding to the vector field X, under the reparameterization of time given in (31). Furthermore, by following the steps, similar to the ones given in the proof of the first claim of the proposition and recalling that the vector field X is locally Lipschitz continuous on $\mathbb{R}^n$, it can be shown that for any given initial condition, there exists a unique solution corresponding to the vector field X for all $t \geq 0$ (see, e.g., [25, Theorem 3.3]). Hence, $\bar{\gamma}$ is uniquely determined and since the function s is injective, with $\mathsf{s}(0) = 0$, it follows that γ is also uniquely determined.

(ii) Let $\gamma(0) = \bar{x}$ and consider now the same candidate Lypaunov function V as the one given in the first claim of the proposition. By following the steps given in the proof of the first claim of the proposition, it can be shown that the time-derivative of the candidate Lyapunov function V along a solution of (18), starting from any given intial condition is always non-positive and from [1, Theorem 3.15.1], it follows that γ is uniquely determined.

□

References

1. Agarwal, R.P., Lakshmikantham, V.: Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations. World Scientific Publishing Company (1993)
2. Aslam Noor, M.: Mixed variational inequalities. Applied Mathematics Letters **3**(2), 73–75 (1990)
3. Aslam Noor, M.: A Wiener-Hopf dynamical system for variational inequalities. New Zealand Journal of Mathematics **31**, 173–182 (2002)
4. Aslam Noor, M., Inayat Noor, K., Yaqoob, H.: On general mixed variational inequalities. Acta Applicandae Mathematicae **110**(1), 227–246 (2010)
5. Barbu, V., Neittaanmäki, P., Niemistö, A.: Approximating optimal control problems governed by variational inequalities. Numerical Functional Analysis and Optimization **15**(5-6), 489–502 (1994)
6. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer (2017)
7. Bhat, S.P., Bernstein, D.S.: Finite-time stability of continuous autonomous systems. SIAM Journal of Control and Optimization **38**(3), 751–766 (2000)
8. Cavazzuti, E., Pappalardo, M., Passacantando, M.: Nash equilibria, variational inequalities, and dynamical systems. Journal of Optimization Theory and applications **114**(3), 491–506 (2002)
9. Chen, F., Ren, W.: Convex optimization via finite-time projected gradient flows. In: Proceedings of Conference on Decision and Control, pp. 4072–4077. IEEE (2018)
10. Chicone, C.: Ordinary Differential Equations with Applications. Springer (2006)
11. Chicone, C., Sotomayor, J.: On a class of complete polynomial vector fields in the plane. Journal of Differential Equations **61**(3), 398–418 (1986)
12. Cortés, J.: Finite-time convergent gradient flows with applications to network consensus. Automatica **42**(11), 1993–2000 (2006)
13. Defazio, A.: On the curved geometry of accelerated optimization. arXiv e-Print (2018)
14. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer (2003)
15. Ferguson, T.S.: Section 7: Infinite Games. URL `https://www.math.ucla.edu/~tom/Game_Theory/mat.pdf`. Lecture Notes for Statistics 596
16. Fu, X., Wang, X., Wang, H., Zhai, Y.: An asymmetric proximal decomposition method for convex programming with linearly coupling constraints. Advances in Operations Research **2012** (2012)
17. Garg, K., Panagou, D.: Fixed-Time stable gradient-flow schemes: Applications to continuous-time optimization. arXiv e-Print (2018)
18. Giannessi, F., Maugeri, A., Pardalos, P.M.: Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models. Springer (2001)
19. Gu, G., He, B., Yuan, X.: Customized proximal point algorithms for linearly constrained convex minimization and saddle-point problems: A unified approach. Computational Optimization and Applications **59**(1-2), 135–161 (2014)
20. Ha, N.T.T., Strodiot, J.J., Vuong, P.T.: On the global exponential stability of a projected dynamical system for strongly pseudomonotone variational inequalities. Optimization Letters pp. 1–14 (2018)
21. Hale, J.K.: Ordinary Differential Equations. Krieger Publishing Company (1980)
22. Hassan-Moghaddam, S., Jovanović, M.R.: On the exponential convergence rate of proximal gradient flow algorithms. In: Proceedings of Conference on Decision and Control, pp. 4246–4251. IEEE (2018)
23. Hu, X., Wang, J.: Solving pseudomonotone variational inequalities and pseudoconvex optimization problems using the projection neural network. IEEE Transactions on Neural Networks **17**(6), 1487–1499 (2006)
24. Karamardian, S., Schaible, S.: Seven kinds of monotone maps. Journal of Optimization Theory and Applications **66**(1), 37–46 (1990)
25. Khalil, H.K.: Nonlinear Systems. Prentice Hall (2002)
26. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. SIAM (2000)

27. Li, C., Yu, X., Zhou, X., Ren, W.: A fixed time distributed optimization: A sliding mode perspective. In: Proceedings of Annual Conference of the Industrial Electronics Society, pp. 8201–8207. IEEE (2017)
28. Liang, X.B., Si, J.: Global exponential stability of neural networks with globally Lipschitz continuous activations and its application to linear variational inequality problem. IEEE Transactions on Neural Networks **12**(2), 349–359 (2001)
29. Neittaanmäki, P., Tiba, D.: A variational inequality approach to constrained control problems for parabolic equations. Applied Mathematics and Optimization **17**(1), 185–201 (1988)
30. Oden, J.T., Kikuchi, N.: Theory of variational inequalities with applications to problems of flow through porous media. International Journal of Engineering Science **18**(10), 1173–1284 (1980)
31. Pappalardo, M., Passacantando, M.: Stability for equilibrium problems: From variational inequalities to dynamical systems. Journal of Optimization Theory and Applications **113**(3), 567–582 (2002)
32. Polyakov, A.: Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Transactions on Automatic Control **57**(8), 2106–2110 (2012)
33. Polyakov, A., Efimov, D., Brogliato, B.: Consistent discretization of finite-time stable homogeneous systems. In: Proceedings of International Workshop on Variable Structure Systems, pp. 360–365. IEEE (2018)
34. Polyakov, A., Efimov, D., Brogliato, B.: consistent discretization of finite-time and fixed-time stable systems. SIAM Journal on Control and Optimization **57**(1), 78–103 (2019)
35. Polyakov, A., Efimov, D., Brogliato, B., Reichhartinger, M.: Consistent discretization of locally homogeneous finite-time stable control systems. In: Proceedings of European Control Conference (2019). To Appear
36. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pacific Journal of Mathematics **33**(1), 209–216 (1970)
37. Scutari, G., Palomar, D.P., Facchinei, F., Pang, J.S.: Convex optimization, game theory, and variational inequality theory. IEEE Signal Processing Magazine **27**(3), 35–49 (2010)
38. Sun, D.: A projection and contraction method for the nonlinear complementarity problem and its extensions. Mathematica Numerica Sinica **16**, 183–194 (1994)
39. Van Hieu, D., Cho, Y.J., Xiao, Y.B.: Golden ratio algorithms with new stepsize rules for variational inequalities. arXiv e-Print (2019)
40. Vuong, P.T., Khanh, P.D.: Qualitative properties of strongly pseudomonotone variational inequalities. Optimization Letters **10**(8), 1669–1679 (2016)
41. Xia, Y., Leung, H., Wang, J.: A projection neural network and its application to constrained optimization problems. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications **49**(4), 447–458 (2002)
42. Xia, Y., Wang, J.: A general methodology for designing globally convergent optimization neural networks. IEEE Transactions on Neural Networks **9**(6), 1331–1343 (1998)