Solutions to Math 416 Homework due October 15, Chapter 5

October 15, 2004

1 Problem 5.2-4

Hat check problem. Let $X_i = I\{i\text{'th customer gets the proper hat}\}$. Then, by the arguments used to analyze random permutations in the text, $E[X_i] = 1/n$. Thus $\sum_i X_i$ is the total number of customers getting the correct hat, and $E\left[\sum_i X_i\right] = \sum_i E[X_i] = n(1/n) = 1$.

2 Problem 5-1

We are implementing an approximate counter data structure, that supports two operations, INCREMENT and VALUE?. For our purposes, there will be a sequence of INCREMENT's followed by a single VALUE? that, ideally, returns the number of INCREMENT's. The idea is that our counter is internally bumped up only some of the time. In the example of part b, the value represented is increased by 100 with probability 1/100 each time an INCREMENT operation is applied.

a. Let V_n, a random variable, denote the value represented by the counter after the n'th increment. So, for some i, we have $V_n = n_i$. Suppose that the raw value of the counter stored after the j'th INCREMENT is i_j.

Then $X_j = V_j - V_{j-1}$, for $j > 0$, denotes the *change* to the value represented by the counter as a result of the j'th increment. Put another way, $V_n = V_0 + \sum_{j=1}^{n} X_j$, where $V_0 = 0$.

Thus $X_j = n_{i(j)+1} - n_{i(j)}$ with probability $1/(n_{i(j)+1} - n_{i(j)})$ and $X_j = 0$ with the remaining probability, so that $E[X_j] = 1$. By linearity of expectation, $E[V_n] = E[\sum_{1 \le j \le n} X_j] = \sum_{1 \le j \le n} E[X_j] = n$.

b. In this case, each X_j is $100 = n_{i+1} - n_i$ with probability 1/100 and zero otherwise. (This example is special because the distribution of X_j does not depend on j. In the other "interesting" examples, e.g., $n_i = 2^i$, the distribution on X_j depends on $i(j)$.) Thus $E[X_j] = 1$ and $E[(X_j - 1)^2] = E[X_j^2] - 1^2 = 99$, since $X_j^2 = 100^2$ with probability 1/100 and zero otherwise. Since the X_j's are independent, $\mathrm{var}(V_n) = 99n$.