1 Problem 5.2-4

Hat check problem. Let $X_i = I\{i\text{'th customer gets the proper hat}\}$. Then, by the arguments used to analyze random permutations in the text, $E[X_i] = 1/n$. Thus $\sum_i X_i$ is the total number of customers getting the correct hat, and $E[\sum_i X_i] = \sum_i E[X_i] = n(1/n) = 1$.

2 Problem 5-1

We are implementing an approximate counter data structure, that supports two operations, INCORRECT and VALUE?. For our purposes, there will be a sequence of INCORRECT’s followed by a single VALUE? that, ideally, returns the number of INCORRECT’s. The idea is that our counter is internally bumped up only some of the time. In the example of part b, the value represented is increased by 100 with probability $1/100$ each time an INCORRECT operation is applied.

a. Let $V_n$, a random variable, denote the value represented by the counter after the $n$’th increment. So, for some $i$, we have $V_n = n_i$. Suppose that the raw value of the counter stored after the $j$’th increment is $i_j$.

Then $X_j = V_j - V_{j-1}$, for $j > 0$, denotes the change to the value represented by the counter as a result of the $j$’th increment. Put another way, $V_n = V_0 + \sum_{j=1}^{n} X_j$ where $V_0 = 0$.

Thus $X_j = n_{i(j)+1} - n_{i(j)}$ with probability $1/(n_{i(j)+1} - n_{i(j)})$ and $X_j = 0$ with the remaining probability, so that $E[X_j] = 1$. By linearity of expectation, $E[V_n] = E[\sum_{1\leq j \leq n} X_j] = \sum_{1\leq j \leq n} E[X_j] = n$.

b. In this case, each $X_j$ is $100 = n_{i+1} - n_i$ with probability $1/100$ and zero otherwise. (This example is special because the distribution of $X_j$ does not depend on $j$. In the other “interesting” examples, e.g., $n_i = 2^i$, the distribution on $X_j$ depends on $i(j)$.) Thus $E[X_j] = 1$ and $E[(X_j - 1)^2] = E[X_j^2] - 1^2 = 99$, since $X_j^2 = 100^2$ with probability $1/100$ and zero otherwise. Since the $X_j$’s are independent, $\text{var}(V_n) = 99n$. 

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