Solutions to Math 416 Homework due Sept 24, Chapter 3

September 26, 2004

1 Problem 3-2

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

Justifications: a. (page 54). b. (3.9). c. $n^{\sin n}$ is infinitely often as big as n and infinitely often as small as $1/n$. The result follows from facts about powers. d. $2^n = 2^{n/2} \cdot 2^{n/2}$ and $2^{n/2}$ grows faster than any constant. e. $n^{\lg c} = c^{\lg n}$ exactly, for *all* $n \geq 1$. (Take the lg of both sides.) f. $\ln(n!) = n \ln n + O(n)$ and $\ln(n^n) = n\ln(n)$.

2 Problem 3-4

a. False. Put $f(n) = n$ and $g(n) = n^2$; then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
b. False. Put $f(n) = n$ and $g(n) = n^2$; then $f(n) + g(n) = \Theta(n^2)$ but $\min(f, g) = n$.
c. True. Fix n_0 and c such that, for $n \geq n_0$, $f(n) \leq cg(n)$. Then, for $n \geq n_0$, we have $\lg(f(n)) \leq \lg(cg(n)) = \lg(c) + \lg(g(n)) \leq (\lg(c) + 1)g(n)$, by hypothesis about $g(n)$. (We need $f(n) \geq 1$ so that $\lg(f(n)) \geq 0$, a general precondition.)
d. False. Put $f(n) = 2n$ and $g(n) = n$. Then $f(n) \leq O(g(n))$. Also, $2^{g(n)} = 2^n$ but $2^{f(n)} = 2^{2n} = 2^n \cdot 2^{g(n)}$.
e. False. Put $f(n) = 1/n$. (Note: For runtimes, we usually consider only increasing functions. But, for failure probabilities, we will usually consider decreasing functions. So this example actually is fairly common.)
f. True. (page 49 and immediate from definition.)
g. False. Put $f(n) = 2^n$ and argue as above.
h. True. (Assuming we make sense out of the two occurrences of asymptotic notation.) Let $g(n)$ be any function in $o(f(n))$. Then, for asymptotically positive functions, $f(n) \leq f(n) + g(n)$. Conversely, fix $c = 1$ and find n_0 such that, for all $n \geq n_0$, we have $g(n) \leq f(n)$. Then, for such n, we have $f(n) + g(n) \leq 2f(n)$. Following the convention on page 47, we've shown that for any $g \in o(f(n))$ there is some $h \in \Theta(f(n))$ such that $f + g = h$.
Note that it is also true that $f(n) - o(f(n)) = \Theta(f(n))$, and this is a sensible thing to write. But it is *not* true that $f(n) - \Theta(f(n)) = o(f(n))$. This part will not be worth any points and, in the future, we will avoid more than a single occurrance of asymptotic notation. (But it is valuable to know how to read such things, since it occurs in the literature.)