Math 416, Midterm II

November 22, 2004

General Instructions

50 minutes. Show all work. Write your name on each page you are submitting.

No collaboration, no notes, and no appearance of same (no headphones, no unapproved scratch paper, etc.) No calculators.

Let $\mathbb{Z}_n$ denote the integers mod n.

Problem 1

Let $g_{a,b,c,d} : \mathbb{Z}_2^3 \to \mathbb{Z}_2$ be defined by $g_{a,b,c,d}(x_0, x_1, x_2) = a + bx_0 + cx_1 + dx_2 \bmod 2$, where each of a, b, c, d is a random element of $\mathbb{Z}_2$.

Suppose the vectors $\vec{x} = (x_0, x_1, x_2)$ and $\vec{y} = (y_0, y_1, y_2)$ differ in at least one coordinate. Give a tight upper bound on $\Pr_{a,b,c,d}(g_{a,b,c,d}(x_0, x_1, x_2) = g_{a,b,c,d}(y_0, y_1, y_2))$ and briefly justify your answer.

Solution.

We want the probability that $g_{a,b,c,d}(x_0, x_1, x_2) - g_{a,b,c,d}(y_0, y_1, y_2) = 0$. Suppose, wlog, that $x_0 \neq y_0$, so that $x_0 - y_0 = 1$. Then we want the probability that $(a + bx_0 + cx_1 + dx_2) - (a + by_0 + cy_1 + dy_2) = b + c(x_1 - y_1) + d(x_2 - y_2)$ is 0. Conditioned on any values of c and d, and no matter what values are taken by x_1, y_1, x_2, y_2, the distribution is uniform over 0 and 1 as b varies. It follows that the probability is exactly 1/2.

Problem 2

Consider the problem of making change using the fewest number of coins. For example, the $k = 4$ common US coins have denominations 1¢, 5¢, 10¢, and 25¢. We could express $n = 55$¢ as $55¢ = 3 \cdot 10¢ + 25¢$ using four coins or (optimally) as $55¢ = 2 \cdot 25¢ + 5¢$ using three coins. In general, the input to this problem is a desired change value, n, and a set $S = \{c_1, c_2, \ldots, c_k\}$ of k integer-valued denominations where $c_1 = 1$ so that we can always make change. The output is the number of coins of each denomination that are used to express n.

a. Show how to solve the general problem using dynamic programming in time $O(nk)$. That is, describe a table of computed values. Explain the index(es) to the table and the values stored in the table. Show how to compute the values in the table. Show how to read the solution from the completed table. Finally, analyze the runtime. (Hint: to express n, first express $n - c_1$ and add c_1, or express $n - c_2$ and add c_2, or...)

   Solution.[We are assuming all coin denominations are non-negative. I should have said so, but no one seemed to stumble over this.]

   Let $C[i]$, for $0 \leq i \leq n$, be a table that stores the minimum number of coins needed to express i. Then $C[0] = 0$ and, from the hint, $C[i] = \min(C[i - c_1], C[i - c_2], \ldots, C[i - c_k]) + 1$. There is a natural algorithm for computing the $C[i]$'s in order of increasing i, since $i - c_1, i - c_2, \ldots, i - c_k$ are all strictly less than i. The total number of coins is $C[n]$. To read off the number of each denomination, keep a parallel table $B[i]$ that stores a denomination j that witnesses the minimum above. (Alternatively, find a coin to use to express i by checking $C[i - c_1], C[i - c_2], \ldots, C[i - c_k]$ in the completed table to see which is minimal.) The time for this $O(nk)$, since there are n cells to compute and each k-wise minimum takes time $O(k)$.

b. Consider the greedy algorithm for US coins that first uses as many 25¢ coins as possible, then uses as many 10¢ coins as possible on the remainder, then 5¢, then 1¢. Show that this greedy algorithm always gives an optimal solution. (Hint: First, for partial credit, show the statement for $n < 25$¢. For the general statement, at most how many 1¢, 5¢, and 10¢ coins can be used in an optimal solution? What combinations of 10¢ and 5¢ can be used in an optimal solution?)

Solution.

In an optimal solution, at most 4 pennies are used since replacing 5 pennies with one nickel leaves everything else alone and improves by four coins. Similarly, at most one nickel is used. So the pennies and nickels together make up at most 9¢. Similarly, at most 2 dimes are used, but 2 dimes and 1 nickel are not used together. It follows that pennies, nickels, and dimes together account for at most 24¢. To achieve this, we need to use as many quarters as possible, then represent n mod 25 optimally by pennies, nickels, and dimes—if we use fewer quarters, the remainder will be at least 25¢, so the solution will be suboptimal. Similarly, we should use as many dimes as possible on the remainder, then a nickel as possible.

c. Now suppose that 5¢ coins are not allowed, only 1¢, 10¢, and 25¢. Show that the corresponding greedy algorithm (25¢ then 10¢ then 1¢) will no longer work, by considering $n = 30$¢.

Solution. The optimal solution for $n = 30$¢ is three dimes (three coins), whereas the greedy algorithm gives $25¢ + 5 \cdot 1¢$ (six coins).