Math 416, Midterm I

October 20, 2004

General Instructions

50 minutes. Two problems: 1a, 1b, 1c, 2a, 2b, 2c **[sic]**. Show all work. Write your name on each page you are submitting.

No collaboration, no notes, and no appearance of same (no headphones, no unapproved scratch paper, etc.) No calculators.

Problem 1

Solve the following recurrences and prove your answer without simply appealing to the master method. Your answer should be of the form $\Theta(f(n))$, where $f(n)$ is a "simple" function of n. You may assume n is a power of 2.

(a)

$$T(n) = \begin{cases} 8T(n/2) + n^2, & n > 1; \\ 1, & n = 1. \end{cases}$$

(b)

$$T(n) = \begin{cases} 8T(n/2) + n^3, & n > 1; \\ 1, & n = 1. \end{cases}$$

(c)

$$T(n) = \begin{cases} 8T(n/2) + n^4, & n > 1; \\ 1, & n = 1. \end{cases}$$

Solution.

We will start with the general recurrence

$$T(n) = \begin{cases} 8T(n/2) + c(n), & n > 1; \\ 1, & n = 1, \end{cases}$$

where $c(n)$ is n^2, n^3, or n^4.

The recursion tree for a computation with cost $T(n)$ is an 8-ary tree, so there are 8^d problems at depth d. Each problem at depth d has size $n/2^d$, and cost $c(n/2^d)$. Thus the total cost attributable to depth d is $8^d c(n/2^d)$. There are $\log_2(n)$ levels, since the problem size decreases by the factor 2 per level. There are $8^d = n^3$ leaves. Thus the total cost (possibly double-counting the leaves, which is immaterial) is $T(n) = n^3 + \sum_{d=0}^{\log_2(n)} 8^d c(n/2^d)$.

If $c(n) = n^2$, the cost is $T(n) = n^3 + \sum_{d=0}^{\log_2(n)} 8^d (n/2^d)^2 = n^2 \sum_{d=0}^{\log_2(n)} 2^d$. The sum of the increasing geometric series is proportional to the largest term, $2^{\log_2(n)} = n$, so the cost is $\Theta(n^3)$.

If $c(n) = n^3$, the cost is $T(n) = n^3 + \sum_{d=0}^{\log_2(n)} 8^d (n/2^d)^3 = n^3 \sum_{d=0}^{\log_2(n)} 1$. This is $\Theta(n^3 \log(n))$.

If $c(n) = n^4$, the cost is $T(n) = n^3 + \sum_{d=0}^{\log_2(n)} 8^d (n/2^d)^4 = n^4 \sum_{d=0}^{\log_2(n)} 1/2^d$. The sum of the decreasing geometric series is proportional to its largest term, the first, 1, so the cost is $\Theta(n^4)$.

Problem 2

Consider the following algorithm to approximate π:

For each i, independently pick a pair (x_i, y_i) of real numbers, where $0 \le x_i \le 1$ and $0 \le y_i \le 1$ uniformly at random. Define

$$X_i = \begin{cases} 1, & x_i^2 + y_i^2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Pr(X_i = 1) = \pi/4$. (This has to do with the areas of circles and squares. You may *assume* this fact.) Let $X = \frac{4}{n} \sum_{1 \le i \le n} X_i$.

(a) What is the expectation of X, as a function of n and π? Justify your answer. (You may use general theorems about expectations.)

Solution.

For each i, $E[X_i] = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) = \pi/4$.

By linearity of expectation, $E[X] = \frac{4}{n} \sum_{1 \le i \le n} E[X_i] = \pi$.

(b) Assuming $3 < \pi < 3.5$, give the variance of X as $\Theta(g(n))$, where $g(n)$ is a "simple" function. (You may use general theorems about variances.)

Solution.

For each i, $E[X_i^2] = 1^2 \cdot \Pr(X_i = 1) + 0^2 \cdot \Pr(X_i = 0) = \pi/4$. So $\text{var}(X_i) = \pi/4 - (\pi/4)^2 = \pi/4(1 - \pi/4)$.

Since $3 < \pi < 3.5$, it follows that $0 < \text{var}(X_i) < 1$ (independent of n). Since the (x_i, y_i) pairs are independent, it follows that the X_i's are independent. Thus the variance of $\sum X_i$ is the sum of the variances, which is $\Theta(n)$. Finally, the variance of X is $(4/n)^2$ times the variance of $\sum X_i$, so the variance of X is $\Theta(1/n)$.

(Note that if $\pi = 0$ or $\pi = 4$, then the variance of each X_i would be zero. It would follow that the variance of X is zero.)

Also, note that, if $X = \sum_i X_i$ for independent X_i's, then $\text{var}(X) = \sum_i \text{var}(X_i)$. But it is *not* true in general that $E[X^2] = \sum_i E[X_i^2]$.

As a check of your work, it should be apparent that the variance decreases as a function of n. Also, variances are non-negative.