Typical Steps for Solving Optimization Problems, v. 3

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Here is a list of steps that are often useful in solving optimization problems. A similar list appears in the text on page 200.

1. Draw picture.
2. Label picture with variables. It is often easiest if you allocate a new variable name for each quantity you want to label. We will later eliminate some variables.
3. Name the objective with a variable. (The *objective* is a technical term that refers to the quantity being optimized or the associated function.)
4. Find the domains for all the labeled variables. (Finding the domain for x is a special case of finding a constraint among x and other variables.)
5. Write objective in terms of other variables. At this point, you may find that you have not labeled enough quantities. Label more, if necessary.
6. Which labeled quantities change and which are constant? Which are allowed to be present in the final answer? Which labeled quantities may only change in a dependent way?
7. Find relationships among variables, from given constraints. If there are n variables including the objective, we generally need $n - 2$ equations. It may help at this stage to identify tension among intermediate goals.
8. Write objective in terms of just *one* variable. Usually, this means solving for all the auxiliary variables in terms of just one variable.
9. Optimize the objective. Use the machinery from Chapter 4.3:
   (a) Find the derivative of the objective function.
   (b) Identify critical points—where the derivative is zero or undefined.
   (c) Identify the endpoints of intervals in the domain of the objective function. Critical points and endpoints are candidates for global extrema.
   (d) Use the second derivative test or another such test to classify candidate points as local minima, local maxima, or neither.
   (e) If the domain is not closed (has some holes or excludes some endpoints) or goes off to $-\infty$ and/or $+\infty$, then investigate the behavior of the function *near* the holes/endpoints/infinities. Usually this means taking a limit, either algebraically or visually from a graph.
   (f) The candidate point with the largest function value is the global maximum *unless* a bigger function value can be gotten *near* an endpoint of the domain. (In the latter case, there is no global maximum.)

10. Solve the problem asked. Which quantity is requested?

11. Sanity check. Try a few other points to make sure that your point is optimum. Check units/dimensions.

We now repeat the list pointing out how these steps play out in 4.5 Example 1, a 40 in^3 can of minimal material.

1. Draw picture. (See page 198.)

2. Label picture with variables. It is often easiest if you allocate a new variable name for each quantity you want to label. We will later eliminate some variables. (See page 198. The variables are r and h.)

3. Name the objective with a variable. (The objective is the amount of material, labeled by M.)

4. Find the domains for all the labeled variables. (Here $r \geq 0$ and $h \geq 0$. The domain may be further restricted below.)

5. Write objective in terms of other variables. At this point, you may find that you have not labeled enough quantities. Label more, if necessary. (We have $M = 2\pi r^2 + 2\pi rh$; see page 199.)

6. Which labeled quantities change and which are constant? Which are allowed to be present in the final answer? Which labeled quantities may only change in a dependent way? (The quantites r, h, and M may change. We can control r and h (in a dependent way); the objective M depends on these, as usual.)

7. Find relationships among variables, from given constraints. If there are n variables including the objective, we generally need $n - 2$ equations. It may help at this stage to identify tension among intermediate goals. (The constraint that the volume is 40 in^3 tells us that $40 = \pi r^2 h$. Note that there are $n = 3$ variables, including the objective, namely, M, r, h. We have $n - 2 = 1$ relationship at this step. The objective is a second relationship and the optimization itself substitutes for a third relation among the three variables M, r, h.

   The tension here is that, to minimize material, we want to make both r and h small. But, if we make r smaller, we need to *increase* h in order to keep the volume at 40 in^3. See discussion on page 198. Sometimes one can conclude from the tension alone that there must be a unique local minimum. See also the discussion on line -3 of page 198 and the discussion on limits, below.)

   An alternative approach my have a fourth variable, V, for the volume of the can and a second equation, $V = 40\ \text{in}^3$.)

8. Write objective in terms of just *one* variable. Usually, this means solving for all the auxiliary variables in terms of just one variable. (First we solve for h in terms of r, getting $h = \frac{40}{\pi r^2}$. Then substitue this expression for h into the expression for M, getting $M = 2\pi r^2 + \frac{80}{r}$.)

9. Optimize the objective. Use the machinery from Chapter 4.3:

   (a) Find the derivative of the objective function. (We have $\frac{dM}{dr} = 4\pi r - \frac{80}{r^2}$.)

   (b) Identify critical points—where the derivative is zero or undefined. (The derivative is defined for all $r \neq 0$. It is zero at $r = \left(\frac{20}{\pi}\right)^{1/3}$.)

   (c) Identify the endpoints of intervals in the domain of the objective function. (The domain is $r > 0$.) Critical points and endpoints are candidates for global extrema.

   (d) Use the second derivative test or another such test to classify candidate points as local minima, local maxima, or neither. (We have $\frac{d^2M}{dr^2} = 4\pi + \frac{160}{r^3}$. Since $r > 0$, we conclude $\frac{d^2M}{dr^2} > 0$.)
   Note that, to perform the second derivative test, we need to find the second derivative, but we don't need to evaluate it. Avoiding the evaluation is probably good, because it might be a source of errors.

(e) If the domain is not closed (has some holes or excludes some endpoints) or goes off to $-\infty$ and/or $+\infty$, then investigate the behavior of the function *near* the holes/endpoints/infinities. Usually this means taking a limit, either algebraically or visually from a graph. (Here, the domain is $0 < r < +\infty$. Then the limit as $r \to 0$ of $M = 2\pi r^2 + \frac{80}{r}$ is $+\infty$, since $2\pi r^2 = 0$ at $r = 0$ and $\frac{80}{r} \to \infty$. Similarly, the limit as $r \to +\infty$ of $M = 2\pi r^2 + \frac{80}{r}$ is $+\infty$, since $\frac{80}{r} \to 0$ and $2\pi r^2 \to +\infty$.)

(f) The candidate point with the largest function value is the global maximum *unless* a bigger function value can be gotten *near* an endpoint of the domain. (There is one interior critical point, at $r = \left(\frac{20}{\pi}\right)^{1/3}$. The value of M is finite there. As $r \to 0$ or $r \to +\infty$, the value of M goes to $+\infty$. So the global minimum is at $r = \left(\frac{20}{\pi}\right)^{1/3}$ and there is no global maximum.)

We now discuss some other techniques that may be useful and less error prone. (This is an aside from the main task of Chapter 4.5.)

First, note that $2\pi r^2$ and $\frac{80}{r}$ are both concave up. (Try sketching the graph of $y = x^2$ and $y = \frac{1}{x}$, which you can maybe do without a calculator.) A curve is concave up if it lies below secant lines (or, if differentiable, if the curve lies above its tangent lines). It follows that the sum of the two concave-up functions is concave up, and our local minimum is a global minimum.

Alternatively, note that if there is a single critical point in the interior of the domain, to distinguish maxima and minima it suffices to evaluate the function at endpoints or to look at end behavior. End behavior is conceptually more difficult than evaluation, but is *much* easier and less error prone in the execution (see above). Note that we didn't have to perform any evaluation more complicated than $2\pi \cdot 0 = 0$ and $\frac{80}{0^+} = +\infty$, which may be hard to do on a calculator but easy to do in one's head. It follows that our interior critical point of $r = \left(\frac{20}{\pi}\right)^{1/3}$ is a global minimum. Investigation of end behavior often needs to be done anyway—here it needs to be done to seek a global maximum (there is no global maximum), but not to find a global minimum if we use the second derivative test instead.

Finally, note that the sum of a positive and negative integer power of the independent variable with positive coefficients comes up fairly often. That is, we have something like $f(r) = 2r^3 + 4r^{-5}$, where the 2, 3, 4, and 5 may be replaced by other numbers. Look for this pattern and expect the graph to be concave up. (But check using one of the techniques above.)

10. Solve the problem asked. Which quantity is requested? (Amount of material is 64.7 in^2. See page 199.)

11. Sanity check. Try a few other points to make sure that your point is optimum. (See Table 4.3. This table was constructed by trying a few values of r and using the formula $h = \frac{40}{\pi r^2}$ to get h. Make sure that each area is the square of a length and each volume is the cube of a length.)