Inflection points of the bell-shaped curve

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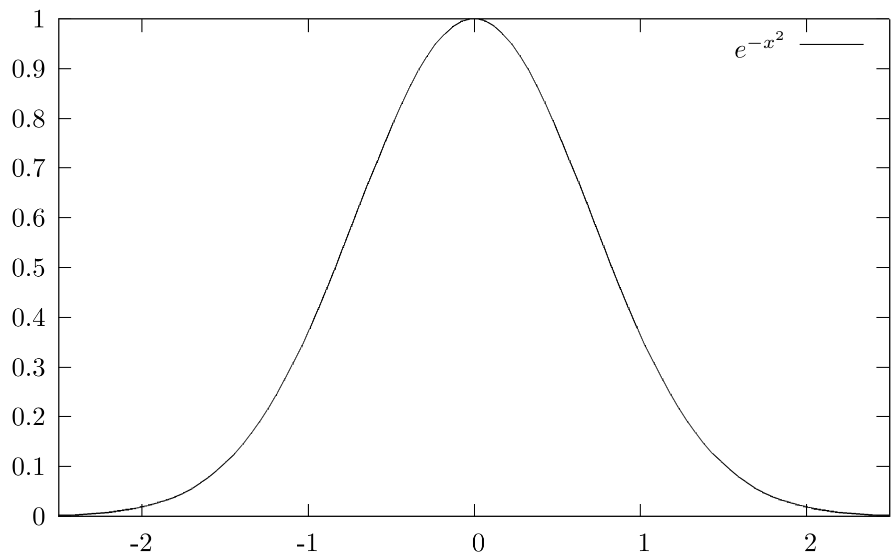
Let $f(x) = e^{-x^2}$, whose graph is below. Then the maximum is at $(0, 1)$ and **the inflection points are at** $\pm\frac{1}{\sqrt{2}}$. (In class I may have said something else about the inflection points.)



This document gives the completion of the problem we started in class.

1 Paradigm

For the paradigm function $f(x) = e^{-x^2}$, we have

$$\begin{aligned} f(x) &= e^{-x^2} \\ f'(x) &= -2xe^{-x^2} \\ f''(x) &= (-2x)^2 e^{-x^2} - 2e^{-x^2} \\ &= 2(2x^2 - 1)e^{-x^2}. \end{aligned}$$

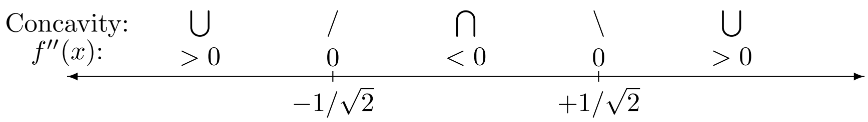
Since $2e^{-x^2} > 0$, we conclude that

$$\begin{cases} f''(x) > 0, & 2x^2 - 1 > 0 \\ f''(x) = 0, & 2x^2 - 1 = 0 \\ f''(x) < 0, & 2x^2 - 1 < 0 \end{cases}$$

The equation $2x^2 - 1 = 0$ has roots at $x = \pm\frac{1}{\sqrt{2}}$. Since $2x^2 - 1$ has positive end behavior for $x \to \pm\infty$, we see that

$$\begin{cases} f''(x) > 0, & x > \frac{1}{\sqrt{2}} \text{ or } x < \frac{-1}{\sqrt{2}} \\ f''(x) = 0, & x = \pm\frac{1}{\sqrt{2}} \\ f''(x) < 0, & \frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}. \end{cases}$$

We can diagram this as follows:



That is, the function is concave up for $x < \frac{-1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$ and concave down for $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. (Note that this computation matches a computation in the text for $b = 1$.)

2 Moving inflection points using horizontal stretches and shifts

Suppose we want a "Bell curve with inflection points at 5 and 13." This means "Find the formula for a horizontally stretched and shifted Bell curve whose inflection points are at 5 and 13."

In the paradigm $f(x) = e^{-x^2}$, the inflection points are at $\pm\frac{1}{\sqrt{2}}$, or $\frac{2}{\sqrt{2}}$ apart. So our strategy is

1. Stretch the graph horizontally by $\frac{13-5}{2/\sqrt{2}} = 4\sqrt{2}$. This moves the inflection points from $\frac{1}{\sqrt{2}}$ to $4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 4$ and from $\frac{-1}{\sqrt{2}}$ to $4\sqrt{2} \cdot \frac{-1}{\sqrt{2}} = -4$. The distance between inflection points is now $4 - (-4) = 8 = 13 - 5$.

2. Shift the graph rightward by 9. This moves the inflection points from -4 and $+4$ to $-4 + 9 = 5$ and $+4 + 9 = 13$.

We accomplish this as follows. To stretch, $f(x) = e^{-x^2}$ beccomes $g(x) = e^{-\left(\frac{x}{4\sqrt{2}}\right)^2}$, by replacing x with $\left(\frac{x}{4\sqrt{2}}\right)$. Next, to shift, we further transform by replacing x with $x - 9$ in $g(x)$, getting $\boxed{h(x) = e^{-\left(\frac{x-9}{4\sqrt{2}}\right)^2}}$. This is the answer.

3 Moving inflection points using calculus

Again, suppose we want a "Bell curve with inflection points at 5 and 13." We use the parametrization from the textbook: $f(x) = e^{-\frac{(x-a)^2}{b}}$, in which, it turns out, a represents the horizontal shift and b, which must be positive, is the **square** of the horizontal stretch. We have

$$\begin{aligned} f(x) &= e^{-\frac{(x-a)^2}{b}} \\ f'(x) &= \frac{-2(x-a)}{b} e^{-\frac{(x-a)^2}{b}} \\ f''(x) &= \left(\frac{-2(x-a)}{b}\right)^2 e^{-\frac{(x-a)^2}{b}} - \frac{2}{b} e^{-\frac{(x-a)^2}{b}} \\ &= \frac{1}{b}\left[\frac{4(x-a)^2}{b} - 2\right] e^{-\frac{(x-a)^2}{b}}. \end{aligned}$$

Since $\frac{1}{b}e^{-\frac{(x-a)^2}{b}}$ is always positive, the second derivative is negative, zero, or positive when $\frac{4(x-a)^2}{b} - 2$ is negative, zero, or positive, respectively. This means that the inflection points occur when the second derivative is zero, or $\frac{4(x-a)^2}{b} - 2 = 0$. We are told that this occurs when $x = 5$ and $x = 13$, or $\frac{4(5-a)^2}{b} - 2 = 0$ and $\frac{4(13-a)^2}{b} - 2 = 0$; we want to solve for a and b.

The equations can be rewritten as

$$\begin{cases} 4(5-a)^2 = 2b \\ 4(13-a)^2 = 2b \end{cases}$$

Subtracting, we get $4(13-a)^2 - 4(5-a)^2 = 0$. Dividing by 4 and then expanding, we get $(169 - 26a + a^2) - (25 - 10a + a^2) = 0$, or $144 - 16a = 0$, so that $a = 144/16 = 9$. Plugging this back into one of the equations, we get $4(5-9)^2 = 2b$, or $64 = 2b$, or $b = 32$. (Note that the horizontal stretch is $\sqrt{b} = \sqrt{32} = 4\sqrt{2}$.) Thus the formula is $\boxed{e^{-\frac{(x-9)^2}{32}}}$.