Cleaning Up Coffee

January 10, 2007

1 Formula for Coffee Temperature

Suppose we observe a cup of coffee at the following temperatures at the following times:

Time	11:36:31	11:42:07	11:46:48
Temperature (°F)	206	180	162

Newton's law of heating/cooling implies that the temperature difference between the coffee and the surrounding air decays exponentially. Also, we are told that the surrounding air is 66°F. We want to find a formula for the temperature of the coffee as a function of the elapsed time, in seconds.

One way is as follows. First, it is convenient to convert the time into elapsed seconds:

Time (elapsed)	0	336	617
Time (clock)	11:36:31	11:42:07	11:46:48
Temperature (°F)	206	180	162

Newton's law implies

$$F(t) - 66 = (F_0 - 66)a^t,$$

where F_0 is the coffee temperature (in °F) at time $t = 0$ and $F(t)$ is the temperature at elapsed time t in seconds. We need to find a and F_0.

Plugging in two of the given points, we get

$$\begin{aligned} 206 - 66 &= (F_0 - 66)a^0 \\ 180 - 66 &= (F_0 - 66)a^{336}. \end{aligned}$$

Dividing, we get

$$\frac{180 - 66}{206 - 66} = a^{336},$$

or $a = \left(\frac{180-66}{206-66}\right)^{1/336} \approx 0.999389 = 1 - .000610$. It is also immediate that $F_0 = 206$. (In other situations, we could find F_0 easily after finding a.)

We can use the third point to check. We need to check that

$$162 - 66 = (206 - 66)(1 - .000610)^{617},$$

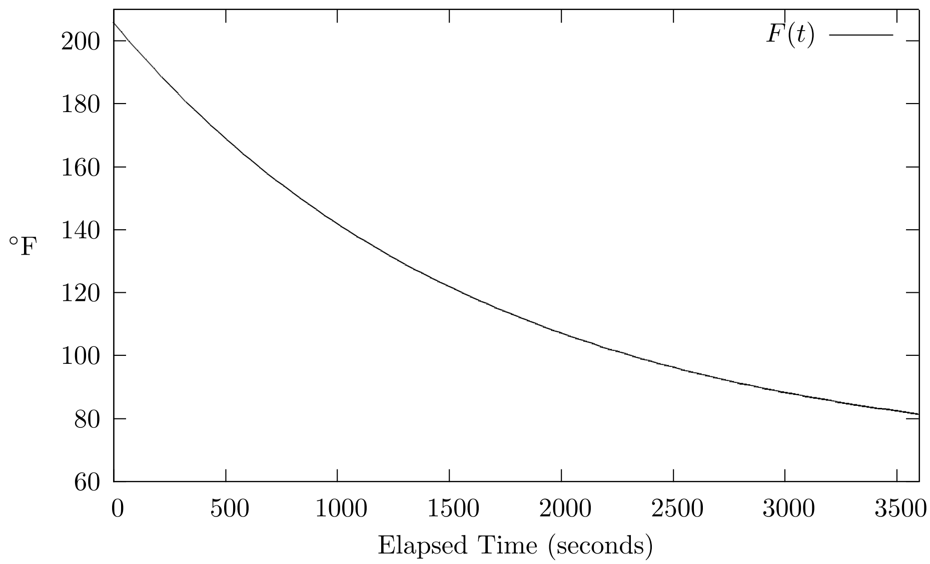
which is correct to a few decimal places.

Putting it all together, we get

$$F(t) = 140(1 - .000610)^t + 66,$$

in °F as function of elapsed time, in seconds.

The above measurements were done using a ceramic mug and nearly boiling coffee. Below is a plot of the resulting temperature over an hour.



1.1 Remarks

1. Here and in similar problems, the base is close to 1. It is ok to round the *percentage rate* to, say, three decimal places, getting $-.000610$, but *do not* round the base .999389 to 1.000—all information would then be lost.

2. In class Jan 10, I might have instead used the formula $(F_0 - 66)b^{-t}$ for the temperature as a function of time. (Note the minus sign on the t.) Then $b = 1/a \approx 1.000612$. We further discussed the thickness of the cup. If we use a double-wall cup instead of a single wall cup, then the temperature decreases more slowly. That is, both a and b get closer to 1, so that a gets bigger and b gets smaller.

2 Comparing Temperature

Suppose Alice mixes a fresh cup of coffee with an equal-sized 10-minute-old cup of coffee. Bob has a 5-minute-old cup of coffee. Whose cup is hotter?

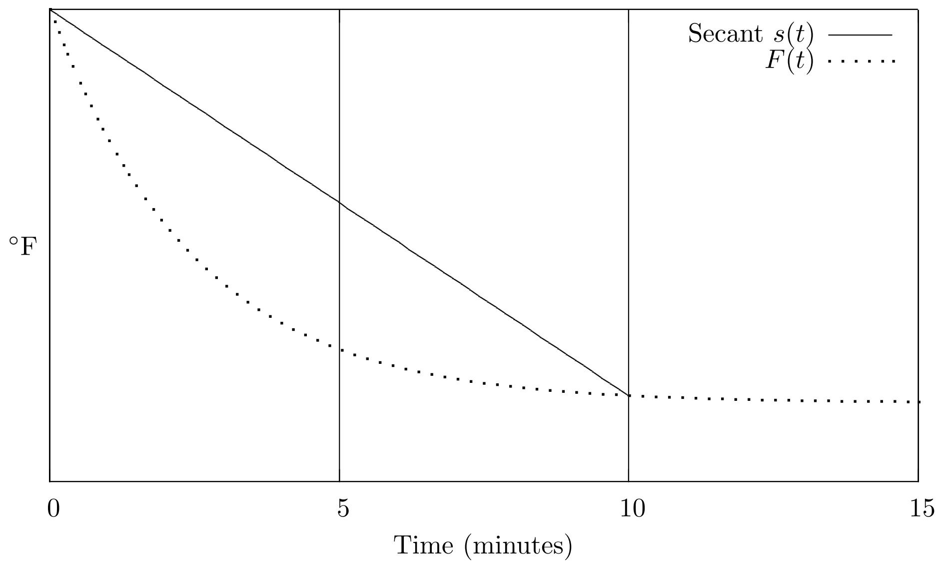
For this problem, ignore the temperature measurements made above. We want a conceptual analysis, in which we assume only the following:

1. Coffee starts out warmer than the air.

2. Newton's law (see above).

3. If we mix two equal-sized cups of coffee at different temperatures, the resulting cup has the average temperature of the given cups.

In the following graph, $F(t)$ is the graph of the temperature of a cooling cup of coffee against elapsed time in *minutes*. The line $s(t)$ is a secant line to $F(t)$. It is an auxiliary line that we draw to help solve the problem at hand. (Knowing which auxiliary lines to draw is a matter of practice and some trial and error.)



From the graph, one can read off the following:

1. The height $F(0)$ is the temperature of a fresh cup of coffee.
2. $F(5)$ is the temperature of 5-minute-old cup. This is the temperature of Bob's cup.
3. $F(10)$ is the temperature of 10-minute-old cup.

From our assumptions, Alice's cup has temperature $\frac{F(0)+F(10)}{2}$. Since s and F agree at 0 and 10, Alice's cup has temperature $\frac{s(0)+s(10)}{2}$. From the graph, this is $s(5)$, since the secant is linear.

Finally, from the graph, we can read off that Alice's cup is hotter than Bob's, *i.e.*, $s(5) > F(5)$. This relies on the concavity of $F(t)$. (We only need concavity. Otherwise, Newton's law and the exact shape of $F(t)$ don't matter.)

2.1 Remarks

One can solve the problem intuitively. (The graph is easiest, but, once you understand the graph, think about the intuition here.) Think about the statement "the coffee temperature decreases at a decreasing rate." This means that in cooling from "fresh" to "10 minutes old," most of the cooling takes place at the beginning. Bob's cup experiences the first five minutes of cooling whereas Alice's cup experiences the average of the first and second five minutes worth of cooling. Since more cooling occurs in the first five minutes than the second, Bob's cup is cooler.

3 Relation with Team Homework 1.1.36

Team homework problem 1.1.36 also talks about Newton's law of heating/cooling, and says that the *rate* of heating is proportional to the temperature difference. For cooling coffee instead of a warming yam, this becomes "the rate of cooling is proportional to the temperature difference between the coffee and air." Note that we had said that the temperature decreases at a decreasing rate, *i.e.*, the temperature (difference) is decreasing and that the rate of cooling is decreasing; Newton's law in fact gives the more precise statement that the temperature difference and rate of cooling are *proportional*. Everything is consistent between the coffee setup and team homework, but we won't fully see the connection until we study the derivative of the exponential function, later in this class. For now, please note that the coffee problem asks for a temperature as a function of time and the team homework problem asks for a different sort of function—pay attention to units! The coffee formula is exponential, but the formula in the Team homework from Section 1.1 is unlikely to be an exponential.