Homework 3, due Thurs, Feb 9, 2006

To be done in groups.

1 Unequal Divide and Conquer (revised from last time)

1.1 Guidelines

In these problems, you are not required to find \( c \) and \( n_0 \) and prove the result by induction. It is sufficient to give a proof that appeals to results in the chapter. For example, you can say, without further comment, \( 5n^2 + 7n \leq O(n^2) \) or \( \sum_{j=1}^n j^3 = \Theta \left( \int_0^n x^3 \, dx \right) = \Theta(n^4) \) or \( \) a \( k \)-ary tree with height \( h \) has \( \Theta(k^h) \) leaves and nodes \( (k > 1) \). In fact, I’d prefer that you get some practice in this “bigger picture” way of thinking, and not always get bogged down with induction details.

1.2 The Problem

Exercises CLRS 4.2-4 and CLRS 4.2-5 illustrate that, if we use a divide-and-conquer approach to a problem, it is typically better to divide the problem into nearly equal-sized subproblems. The above and the last example of CLRS Section 4.2 also illustrate that it is not necessary that the subproblems be exactly the same size.

We now switch from analyzing to designing algorithms. Given a problem of size \( n \), we will break it into two problems, of size \( a(n) \) and \( n - a(n) \), where \( a(n) \) is a “common” function satisfying \( 1 \leq a(n) \leq n/2 \). We assume the Divide and Recombine cost is linear, so we have recurrence

\[
T(n) = T(a(n)) + T(n - a(n)) + cn.
\]

All other things being equal, we’d want \( a(n) = n/2 \) to minimize \( T(n) \). But sometimes there is a separate (direct or indirect) cost involved in making \( a(n) \) exactly \( n/2 \) and it’s easier to choose other \( a()’s \). Below we investigate which \( a()’s \) are acceptable while still meeting certain overall cost requirements. Problem:

- How slowly-growing can \( a(n) \) be and still make \( T(n) \leq O(n \log^2(n)) \)?

Restrict attention to functions of the form \( n^r(\log(n))^s \), where \( r \) and \( s \) are constant real numbers, and you only need to find \( s \) up to 1, additively. That is, your answer should be numbers \( r \) and \( s \) with

\[
n^r(\log(n))^s \leq a(n) \leq n^r(\log(n))^{s+1}.
\]

(If you’ve already found exactly the right \( s \), please turn it in, but don’t attempt this if you haven’t started.)

2 New Problems

- CLRS 5.1-3, on biased coins. You may assume that different tosses of the coin are independent.
- CLRS 5.2-5, expected number of inversions