Fitting

EECS 598-08 Fall 2014
Foundations of Computer Vision

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Readings: FP 10; SZ 4.3, 5.1
Date: 10/8/14

Materials on these slides have come from many sources in addition to myself; individual slides reference specific sources.
Plan

• Problem Formulation
• Least Squares Methods
• RANSAC
• Hough Transform
• Multi-model Fitting
• Expectation-Maximization
• Examples of Uses of Fitting
What is Fitting?

Goals:

• Choose a parametric model to fit a certain quantity from data
• Estimate model parameters

- Lines
- Curves
- Homographic transformation
- Fundamental matrix
- Shape model
Example: fitting lines
(for computing vanishing points)

Source: S. Savarese slides.
Example: Estimating an homographic transformation

Source: S. Savarese slides.
Example: Estimating F

Source: S. Savarese slides.
Example: fitting a 2D shape template

Source: S. Savarese slides.
Example: fitting a 3D object model

Source: S. Savarese slides.
Fitting

- Critical issues:
  - Noisy data
  - Outliers
  - Missing data

Source: S. Savarese slides.
Critical issues: noisy data

Source: S. Savarese slides.
Critical issues: noisy data (intra-class variability)

Source: S. Savarese slides.
Critical issues: outliers

Source: S. Savarese slides.
Critical issues: missing data (occlusions)

Source: S. Savarese slides.
Fitting

Goal: Choose a parametric model to fit a certain quantity from data

Techniques:
- Least square methods
- RANSAC
- Hough transform
- EM (Expectation Maximization)

Source: S. Savarese slides.
Least squares methods
- fitting a line -

• Data: \((x_1, y_1), \ldots, (x_n, y_n)\)

• Line equation: \(y_i = mx_i + b\)

• Find \((m, b)\) to minimize

\[
E = \sum_{i=1}^{n} (y_i - mx_i - b)^2
\]

Source: S. Savarese slides.
Least squares methods
- fitting a line -

\[ E = \sum_{i=1}^{n} (y_i - mx_i - b)^2 \]

Source: S. Savarese slides.
Least squares methods
- fitting a line -

\[ E = \sum_{i=1}^{n} (y_i - mx_i - b)^2 \]

\[ E = \sum_{i=1}^{n} \left( y_i - \begin{bmatrix} x_i & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} \right)^2 = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ x_n & \cdots & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} \right\|^2 = \|Y - XB\|^2 \]

\[ = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB) \]

Find \((m, b)\) that minimize \(E\)

\[ \frac{dE}{dB} = -2X^TY + 2X^TXB = 0 \]

\[ X^TXB = X^TY \]

**Normal equation**

\[ B = \left( X^TX \right)^{-1} X^TY \]

Source: S. Savarese slides.
Least squares methods
- fitting a line -

\[ E = \sum_{i=1}^{n} (y_i - mx_i - b)^2 \]

\[ B = \begin{pmatrix} m \\ b \end{pmatrix} = \left( X^T X \right)^{-1} X^T Y \]

Limitations

- Fails completely for vertical lines

Source: S. Savarese slides.
Least squares methods
- fitting a line -

- Distance between point \((x_n, y_n)\) and line \(ax + by = d\)

- Find \((a, b, d)\) to minimize the sum of squared perpendicular distances

\[
E = \sum_{i=1}^{n} (ax_i + by_i - d)^2
\]

\[
\begin{bmatrix}
U & N
\end{bmatrix} = 0
\]

data model parameters

Source: S. Savarese slides.
Least squares methods
- fitting a line -

\[ A h = 0 \]

Minimize \( \| A h \| \) subject to \( \| h \| = 1 \)

\[ A = UDV^T \]

\( h = \text{last column of } V \)
Least squares methods
– fitting an homography –

\[ U N = 0 \]

data \hspace{1cm} model parameters

Source: S. Savarese slides.
Least squares: Robustness to noise

Source: S. Savarese slides.
Least squares: Robustness to noise

Source: S. Savarese slides.
Critical issues: outliers

CONCLUSION: Least square is not robust w.r.t. outliers

Source: S. Savarese slides.
Least squares: Robust estimators

Instead of minimizing

\[ E = \sum_{i=1}^{n} (a x_i + b y_i - d)^2 \]

We minimize

\[ E = \sum_i \rho(u_i; \sigma) \]

\[ u_i = a x_i + b y_i - d \]

- \( u_i \) = error (residual) of \( i^{th} \) point w.r.t. model parameters \( \beta = (a,b,d) \)
- \( \rho \) = robust function of \( u_i \) with scale parameter \( \sigma \)

\[ \rho(u; \sigma) = \frac{u^2}{\sigma^2 + u^2} \]

The robust function \( \rho \)

- Favors a configuration with small residuals
- Penalizes large residuals

Source: S. Savarese slides.
Least squares: Robust estimators

Instead of minimizing $E = \sum_{i=1}^{n} (ax_i + by_i - d)^2$

We minimize $E = \sum \rho(u_i; \sigma)$  \hspace{1cm} $u_i = ax_i + by_i - d$

- $u_i$ = error (residual) of $i^{th}$ point w.r.t. model parameters $\beta = (a,b,d)$
- $\rho$ = robust function of $u_i$ with scale parameter $\sigma$

The robust function $\rho$
- Favors a configuration with small residuals
- Penalizes large residuals

Source: S. Savarese slides.
Least squares: Robust estimators

The effect of the outlier is eliminated

Source: S. Savarese slides.
Least squares: Robust estimators

Bad scale parameter $\sigma$ (too small!)
Fits only locally
Sensitive to initial condition

Source: S. Savarese slides.
Least squares: Robust estimators

Bad scale parameter $\sigma$ (too large!)
Same as standard LSQ

• **CONCLUSION:** Robust estimator useful if prior info about the distribution of points is known

  • Robust fitting is a nonlinear optimization problem (iterative solution)
  • Least squares solution provides good initial condition

Source: S. Savarese slides.
Fitting

Goal: Choose a parametric model to fit a certain quantity from data

Techniques:
• Least square methods
• RANSAC
• Hough transform
• EM (Expectation Maximization)

Source: S. Savarese slides.
Basic philosophy
(voting scheme)

• Data elements are used to vote for one (or multiple) models

• Robust to outliers and missing data

• Assumption 1: Noisy features will not vote consistently for any single model ("few" outliers)

• Assumption 2: there are enough features to agree on a good model ("few" missing data)

Source: S. Savarese slides.
RANSAC

(RANdom SAmple Consensus): Learning technique to estimate parameters of a model by random sampling of observed data

Fischler & Bolles in ‘81.

\[ \pi : I \rightarrow \{ P, O \} \]

such that:

\[ f(P, \beta) < \delta \]

\[ \min_{\pi} \left| O \right| \]

\[ f(P, \beta) = \left\| \beta - \left( P^T P \right)^{-1} P^T \right\| \]

Model parameters

Source: S. Savarese slides.
Algorithm:

1. Select random sample of minimum required size to fit model
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found
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Sample set = set of points in 2D

Source: S. Savarese slides.
Algorithm:

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Source: S. Savarese slides.
How many samples?

- Number of samples $S$
  - $p =$ probability at least one random sample is valid (free from outliers)
  - $e =$ outlier ratio $(1-p)$
  - $P$ is total probability of success after $S$ trials
  - Likelihood in one trial that all $s$ samples are inliers is $p^s$
  - $s =$ minimum number needed to fit the model

- Likelihood that $S$ such trials will all fail is

$$1 - P = (1 - p^s)^S$$

- Hence the required number of minimum trials is

$$S = \frac{\log(1-P)}{\log(1-p^s)}$$

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</table>

Source: S. Savarese slides.
Estimating $H$ by RANSAC

- $H \rightarrow 8$ DOF
- Need 4 correspondences

Sample set = set of matches between 2 images

Algorithm:

1. Select a random sample of minimum required size
2. Compute a putative model from these
3. Compute the set of inliers to this model from whole sample space

Repeat 1-3 until model with the most inliers over all samples is found

Source: S. Savarese slides.
Estimating F by RANSAC

Sample set = set of matches between 2 images

Algorithm:

1. Select a random sample of minimum required size [?]
2. Compute a putative model from these
3. Compute the set of inliers to this model from whole sample space
Repeat 1-3 until model with the most inliers over all samples is found

Source: S. Savarese slides.

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Source: S. Savarese slides.
RANSAC Conclusions

Good:
• Simple and easily implementable
• Successful in different contexts

Bad:
• Many parameters to tune
• Trade-off accuracy-vs-time
• Cannot be used if ratio inliers/outliers is too small

Source: S. Savarese slides.
Fitting

**Goal:** Choose a parametric model to fit a certain quantity from data

**Techniques:**
- Least square methods
- RANSAC
- Hough transform
**Hough transform**


Given a set of points, find the curve or line that explains the data points best

Source: S. Savarese slides.
Hough transform


Given a set of points, find the curve or line that explains the data points best

\[ y = m x + n \]

Source: S. Savarese slides.
Hough transform

Source: S. Savarese slides.
Hough transform


Issue: parameter space \([m,n]\) is unbounded…

• Use a polar representation for the parameter space

\[
x \cos \theta + y \sin \theta = \rho
\]

Source: S. Savarese slides.
Hough transform - experiments

Source: S. Savarese slides.
Hough transform - experiments

How to compute the intersection point?
IDEA: introduce a grid a count intersection points in each cell
Issue: Grid size needs to be adjusted…

Source: S. Savarese slides.
Hough transform - experiments

Issue: spurious peaks due to uniform noise

Source: S. Savarese slides.
Hough transform - conclusions

Good:

• All points are processed independently, so can cope with occlusion/outliers
• Some robustness to noise: noise points unlikely to contribute consistently to any single bin

Bad:

• Spurious peaks due to uniform noise
• Trade-off noise-grid size (hard to find sweet point)

Source: S. Savarese slides.
Hough transform - experiments

Source: S. Savarese slides.
Source: S. Savarese slides.

Credit slide: K. Grauman
Generalized Hough transform

D. Ballard, Generalizing the Hough Transform to Detect Arbitrary Shapes, Pattern Recognition 13(2), 1981

- Identify a shape model by measuring the location of its parts and shape centroid

- Measurements: orientation theta, location of p
- Each measurement casts a vote in the Hough space: \( p + r(\theta) \)

Source: S. Savarese slides.
Generalized Hough transform

B. Leibe, A. Leonardis, and B. Schiele,
Combined Object Categorization and Segmentation with an Implicit Shape Model, ECCV Workshop on Statistical Learning in Computer Vision 2004

Source: S. Savarese slides.
Plan

• Problem Formulation
• Least Squares Methods
• RANSAC
• Hough Transform
• **Multi-model Fitting**
• Expectation-Maximization
• Examples of Uses of Fitting
Fitting multiple models

- Incremental fitting
- E.M. (probabilistic fitting)
- Hough transform

Source: S. Savarese slides.
Incremental line fitting

Scan data point sequentially (using locality constraints)

Perform following loop:

1. Select N point and fit line to N points
2. Compute residual $R_N$
3. Add a new point, re-fit line and re-compute $R_{N+1}$
4. Continue while line fitting residual is small enough,

> When residual exceeds a threshold, start fitting new model (line)

Source: S. Savarese slides.
Hough transform

Same cons and pros as before…

Source: S. Savarese slides.
Plan

- Problem Formulation
- Least Squares Methods
- RANSAC
- Hough Transform
- Multi-model Fitting
- Expectation-Maximization
- **Examples of Uses of Fitting**
Fitting helps matching!

Feature are matched (for instance, based on correlation)

Source: S. Savarese slides.
Idea:

• Fitting an homography $H$ (by RANSAC) mapping features from images 1 to 2
• Bad matches will be labeled as outliers (hence rejected)!

Matches bases on appearance only
Red: good matches
Green: bad matches

Source: S. Savarese slides.
Fitting helps matching!

Source: S. Savarese slides.
Recognising Panoramas


Source: S. Savarese
Fitting helps matching!

Images courtesy of Brandon Lloyd
Source: S. Savarese slides.
Source: S. Savarese slides.
Next Lecture: Moving on to Motion Module

- Readings: FP 10.6; SZ 8; TV 8
  - (TV is Trucco and Verri, which is not a required book.)
Least squares methods
- fitting a line -

\[ Ax = b \]

- More equations than unknowns

- Look for solution which minimizes \[ ||Ax - b|| = (Ax-b)^T (Ax-b) \]

- Solve \[ \frac{\partial (Ax - b)^T (Ax - b)}{\partial x_i} = 0 \]

- LS solution

\[ x = (A^T A)^{-1} A^T b \]

Source: S. Savarese slides.
Least squares methods
- fitting a line -

Solving \[ x = (A^t A)^{-1} A^t b \]

\[ A^+ = (A^t A)^{-1} A^t \] = pseudo-inverse of A

\[ A = U \sum V^t \] = SVD decomposition of A

\[ A^{-1} = V \sum^{-1} U \]

\[ A^+ = V \sum^+ U \]

with \[ \sum^+ \] equal to \[ \sum^{-1} \] for all nonzero singular values and zero otherwise

Source: S. Savarese slides.
Least squares methods
- fitting an homography -

\[
\begin{align*}
    h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - x' &= 0 \\
    h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - y' &= 0
\end{align*}
\]

From \( n \geq 4 \) corresponding points:

\[
Ah = 0
\]

Source: S. Savarese slides.