17.1 Exponential Weights Algorithm

Given convex (in $\hat{y}$) loss function $\ell(\hat{y}, y)$ (bounded in $[0, 1]$), parameter $\eta > 0$, let $w^1 = (1, \ldots, 1)$,

\begin{verbatim}
1: for $t = 1, 2, \ldots, T$ do
2:    Alg receives prediction $f^t_i \in \{0, 1\}$ from expert $i$
3:    Alg predicts $\hat{y}_t = \frac{\sum w^t_i f^t_i}{\sum_j w^t_j}$
4:    Nature reveals $y_t \in \{0, 1\}$
5:    Alg loss increases: $L_{t+1} = L_t + \ell(\hat{y}_t, y_t)$
6:    $w^{t+1}_i = w^t_i \exp(-\eta \ell(f^t_i, y_t))$
7: end for
\end{verbatim}

NOTE: $f^t_i$ and $y^t$ can be real-valued, but we are assuming for simplicity that they are binary.

**Theorem 17.1.** For any sequence of $\{y^t\}_t$, $\{f^t_i\}_{i,t}$ we have

$$L_{MA} \leq \frac{\eta L^{t+1} + \log N}{1 - \exp(-\eta)}$$

for all $i$ where $L_{i}^{t+1} = \sum_{s=1}^{t} \ell(f^s_i, y^s)$.

**Corollary 17.2.** With $\eta$ tuned appropriately

$$L_{MA} \leq L_{i^*}^{T+1} + \log N + \sqrt{2L_{i^*} T \log N}$$

where $i^*$ is the index of the "best expert". Notice that

$$\frac{L_{MA}}{T} \leq \frac{L_{i^*}^{T+1}}{T} + \epsilon_T$$

where $\epsilon_T$ is approaching 0 at a rate of about $O \left( \frac{1}{\sqrt{T}} \right)$. 
17.2 Hedge Setting (Freund and Schapire 95?)

We have \( N \) actions (or bets)

1. \( \textbf{for } t = 1, \ldots, T \textbf{ do} \)
2. Alg chooses distribution \( p^t \in \Delta_N \)
3. Alg samples \( i_t \in p^t \)
4. Nature/adversary reveals \( \ell^t \in [0, 1]^N \)
5. Alg suffers \( \ell_{i_t} \), but in expectation, \( L_{MA} = \sum_t \ell_{i_t} p^t_i \)
6. \textbf{end for}

**Theorem 17.3.** The hedge setting gives the same bound as the exponential weights algorithm when you choose

\[
p^t = \frac{w^t}{\sum_j w^t_j}.
\]

**Proof:** For this proof, we will need to call on the following inequality that holds for all \( s \in \mathbb{R} \):

\[
\log \mathbb{E} \exp(sX) \leq (e^s - 1) \mathbb{E}X.
\]

Assume \( X \) is a random variable taking values in \([0, 1]\) on round \( t \). Let \( X^t = \ell(f^t_i, y^t) \) with probability

\[
\frac{w^t_i}{\sum_{j=1}^N w^t_j}.
\]

Let

\[
\Phi_t = -\log \sum_{i=1}^N w^t_i = -\log \sum_{i=1}^N \exp(-\eta L^t_i).
\]

Then

\[
\Phi_{t+1} - \Phi_t = -\log \left( \frac{\sum_i w^{t+1}_i}{\sum_j w^t_j} \right)
\]

\[
= -\log \left( \frac{\sum_i w^t_i \exp(-\eta \ell(f^t_i, y^t))}{\sum_j w^t_j} \right)
\]

\[
\geq -(e^{-\eta} - 1) \mathbb{E}X^t
\]

\[
= (1 - e^{-\eta}) \sum_i w^t_i \ell(f^t_i, y^t)
\]

\[
\geq (1 - e^{-\eta}) \ell(\hat{y}^t, y^t)
\]

\[
= (1 - e^{-\eta}) \ell(\hat{y}^t, y^t)
\]
Recall that the loss of the algorithm on $t$ is $\ell(\frac{\sum_i w_i f_i}{\sum_j w_j}, y^t)$. This is required for the last step of the sequence of inequalities and equations above.

Whence,

$$(1 - e^{-\eta})^{LT+1} = \sum_{t=1}^{T} (\Phi_{t+1} - \Phi_t)$$

$$= -\log \sum_i \exp (-\eta L_i^{T+1}) + \log N$$

$$\leq -\log (\exp (-\eta L_i^{T+1})) + \log N$$

$$= \eta L_i^{T+1} + \log N,$$

which implies that

$$L_{MA} \leq \frac{\eta L_i^{T+1} + \log N}{1 - e^{-\eta}}$$

### 17.3 Zero-sum games

We are given $n$ strategies/actions for $P1$ and $m$ for $P2$, and the payoff matrix $M \in [-1, +1]^{n \times m}$. Simulta-

neously,

$P1$ chooses $i \in [n]$

$P2$ chooses $j \in [m]$.

As a result, $P1$ earns $M_{ij}$, and $P2$ earns $-M_{ij}$.

**Example:** Rock-Paper-Scissors

$$M = \begin{bmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{bmatrix}$$

**Definition 17.4.** A **pure strategy** Nash equilibrium is a pair $i, j$ such that $M_{ij} \geq M_{i'j}$ for all $i' \in [n]$ and $M_{ij} \geq M_{ij'}$ for all $j' \in [m]$.

**Definition 17.5.** A **mixed strategy** is a distribution $p \in \Delta_n$ for $P1$ and $q \in \Delta_m$ for $P2$. Expected payoff
to $P1$ is $\sum_{i,j} p_i q_j M_{ij} = p^\top M q$ and expected payoff for $P2$ is the negation of this.

### 17.4 Von Neumann’s Minimax Theorem

$$\min_q \max_p p^\top M q = \max_p \min_q p^\top M q$$

The minimizer gets to see the maximizer’s strategy before picking his/her own, so the right side will clearly be less than or equal to the left. The other way is more difficult.