EECS598: Prediction and Learning: It's Only a Game Fall 2013

Lecture 8: Game Theory III

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Announcements

- Moving 10/16 class. Possible candidates:
  - 9:30am - 11:00am (3 students unavailable)
  - Relocate to central campus at usual time (4 students unavailable)
- Class participation credit may be satisfied in the following ways:
  - Scribing (50% + 50% for an additional scribe)
  - Solving a challenge problem (5-20% depending on difficulty of problem and/or quality of solution)
  - Presenting homework solutions (10%)
  - Presenting final project (50%)
- On scribe notes...
  - Notes are expected back 3 business days from original lecture
  - Sign-up sheet for scribing is posted on CTools
- Notes on proof of Nash's Theorem posted on course website

An Observation on Nash's Theorem

Given a bimatrix game (A, B), we would like to define the map $(x, y) \mapsto (x', y')$ such that

$$x' = \arg\max_x x^\top Ay \qquad \text{and} \qquad y' = \arg\max_y x^\top By. \tag{8.1}$$

We would like to use Brouwer's fixed point theorem to prove that there exists some (x, y) which maps to itself. However, the mapping we have defined is not continuous. Yet, for some small $\epsilon > 0$, we may define a mapping $(x, y) \mapsto (x', y')$ such that

$$x' = \arg\max_x x^\top Ay - \epsilon\|x\|^2 \qquad \text{and} \qquad y' = \arg\max_y x^\top By - \epsilon\|y\|^2.$$

In this case, the mapping is continuous and Brouwer applies. We may then regard any fixed point, (x^ϵ, y^ϵ), of this mapping as an "ϵ-optimal" Nash equilibrium pair. Now, we should be able to take some sequence of ϵ values which converge to zero and argue that there exists a fixed point in the limiting case of equation **??**.

Exercise: Formalize and prove (or disprove) the arguments given above.

Von Neumann's Minimax Theorem

Let p take values in Δ_n, let q take values in Δ_m, and let $M \in [0,1]^{n\times m}$. Then

$$\min_p \max_q p^\top M q = \max_q \min_p p^\top M q. \tag{8.2}$$

8.1 Proof of "≥"

This part of the proof is trivial. To see this, consider the right hand side of ??. Let us regard p as a function of q where

$$p(q) = \min_p p^\top M q. \tag{8.3}$$

Informally, we may regard the value of ?? as the value obtained from p's best strategy given q's strategy. Similarly, we may regard the left hand side of ?? as the value achieved from q's best strategy given p's strategy. Now, since q can do no worse upon observing p's strategy than it could if it had to choose a strategy beforehand, it follows that the LHS of equation ?? meets or exceeds the RHS.

8.2 Alternate Formulation of "≥" Proof

The LHS of equation ?? is the smallest c_1 such that

$$\exists p \forall q : p^\top M q \leq c_1. \tag{8.4}$$

However, the RHS of equation ?? is the smallest c_2 such that

$$\forall q \exists p : p^\top M q \leq c_2. \tag{8.5}$$

Let $p*$ be any strategy which achieves the minimum value in ??. Then we must have $c_1 \geq c_2$ since we may always play $p*$ in response to any q in ??.

Linear Programming Connections

Consider a linear program which seeks to minimize c over variables c and $p \in \Delta_n$ subject to the constraints:

$$p^\top M e_j \leq c \qquad j = 1, \ldots, m$$

where $M \in [0,1]^{n\times m}$. This is equivalent to finding the value of c_1 in ??. Note that we do not need explicit constraints for every $q \in \Delta_m$; we only need to consider the corners of the simplex. One can show that the dual LP is equivalent to the max-min formulation.

8.3 General LP's

For given vectors d and b and matrix A, we seek the maximum value of $d^\top x$ over x subject to the constraint that $Ax \leq b$ and $x \geq 0$. If we allow these constraints to be broken with infinite cost, then we may instead seek

$$\max_{x\geq 0} \min_{y \geq 0} d^\top x + y^\top (b - Ax).$$

Thus, if any entry of Ax exceeds b, then y may be arbitrarily large such that the sum diverges to $-\infty$. Now, in cases where strong duality holds, we may write

$$\begin{aligned}\max_{x\geq 0}\min_{y\geq 0} d^\top x + y^\top(b-Ax) &= \min_{y\geq 0}\max_{x\geq 0} d^\top x + y^\top(b-Ax)\\ &= \min_{y\geq 0}\max_{x\geq 0}\left(d - y^\top A\right)x + y^\top b\end{aligned}$$

This invokes the dual LP which seeks to minimize $y^\top b$ subject to the constraint that $y^\top A \geq d$ and $y \geq 0$.

Note: It has been shown that LP strong duality is equivalent to minimax duality [**?**].

Exponential Weights Algorithm with Gains

Consider EWA in the setting where money can always be made. We observe a sequence of gain vectors $g^1,\ldots,g^T \in [0,1]^N$. Let g_i^t = money earned for action i on round t. In this setting, we have

$$\text{regret}_T = \max_j \sum_{t=1}^T g_j^t - \sum_{t=1}^T p^t \cdot g^t \tag{8.6}$$

where p^t is our strategy on round t. We assume that we can achieve some bound on regret which is sub-linear in T. In particular, we assume that our familiar bound $O(\sqrt{T\log N})$ holds. We then assign weights sequentially such that $w_i^{t+1} = w_i^t \exp(\eta g_i^t)$.

8.4 Proof of Strong Duality for Minimax Theorem

We shall now prove that $\min_p \max_q p^\top M q \leq \max_q \min_p p^\top M q + \epsilon$. However, we will see that this ϵ is inconsequential. Now, imagine that both players P_1 and P_2 play a game, each learning from the actions of the other. Then P_1 chooses $p^1, p^2, \ldots$ and P_2 chooses $q^1, q^2, \ldots$ according to the information available at each time step. Each p^t is chosen by learning from loss vectors $\ell^1,\ldots,\ell^{t-1}$, where $\ell^s = Mq^s$. Now, suppose at each time step, P_2 announces his strategy, q^t, to P_1. Our results will hold even when this is not the case, but this assumption will simplify our analysis. Now, we will have q^t chosen according to $g^1,\ldots,g^{t-1}$, where $g^s = p^s M$. Let $\hat{q}^T = \frac{1}{T}\sum_{t=1}^T q^t$, the average strategy of P_2. We may now analyze the average payoffs

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T \left(p^t\right)^\top Mq^t &= \frac{1}{T}\sum_{t=1}^T p^t \cdot \ell^t\\ &\leq \frac{1}{T}\min_p \sum_{t=1}^T p \cdot \ell^t + \frac{\text{regret}_T}{T}\\ &= \min_p p^\top M \hat{q}^T + \epsilon_T\\ &\leq \max_q \min_p p^\top M q + \epsilon_T.\end{aligned}$$

Here we have used $\epsilon_T := \frac{\text{regret}_T}{T}$ as a term that we can make vanish by increasing T.

Following a similar argument, with $\hat{p}^T = \frac{1}{T}\sum_{t=1}^T p^t$, we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left(p^t\right)^\top Mq^t &\geq \frac{1}{T}\max_q \sum_{t=1}^{T} g^t \cdot q - \frac{\text{regret}_T}{T} \\ &= \max_q \left(\hat{p}^T\right)^\top Mq - \epsilon_T \\ &\geq \min_p \max_q p^\top Mq - \epsilon_T.\end{aligned}$$

Combining these two chains of inequalities yields

$$\min_p \max_q p^\top Mq - 2\epsilon_T \leq \max_q \min_p p^\top Mq.$$

However, T was arbitrary and so we may increase T to make ϵ_T arbitrarily small. Taking the limit as $\epsilon_T \to 0$ implies that

$$\min_p \max_q p^\top Mq \leq \max_q \min_p p^\top Mq.$$

Observations:

1. $\hat{p}^T$ and $\hat{q}^T$ form a $2\epsilon_T$-optimal Nash equilibrium pair. Let v denote the value of the game. We then have

$$\begin{aligned}\max_q \hat{p}^T Mq &\leq \max_q \min_p p^\top Mq + 2\epsilon_T \\ &= v + 2\epsilon_T\end{aligned}$$

and

$$\begin{aligned}\min_p p^\top M\hat{q}^T &\geq \min_p \max_q p^\top Mq - 2\epsilon_T \\ &= v - 2\epsilon_T\end{aligned}$$

and $2\epsilon_T$-optimality holds.

2. Our proof had both players learning. Instead, if we have $q^t = \arg\max_q p^t Mq$, then we get

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left(p^t\right)^\top Mq^t &= \frac{1}{T}\sum_{t=1}^{T}\max_{q^t}\left(p^t\right)^\top Mq^t \\ &\geq \frac{1}{T}\max_q \sum_{t=1}^{T}\left(p^t\right)^\top Mq.\end{aligned}$$

The effect is that we drop the factor of two in the $2\epsilon_T$ error bound.

References

[1] Adler, Ilan. "The equivalence of linear programs and zero-sum games." International Journal of Game Theory 42, no. 1 (February 2013): 165-77. Accessed October 4, 2013. http://dx.doi.org/10.1007/s00182-012-0328-8.