EECS598: Prediction and Learning: It's Only a Game Fall 2013

Lecture 7: Game Theory 2

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7.1 Nash Equilibrium and Existence

In the context of game theory, the concept of Nash Equilibrium is introduced. The Existence of a Nash Equilibrium pair is proven.

7.1.1 Nash Equilibrium

In the context of game theory, a pair (x^*, y^*) is a *Nash equilibrium pair* if x^* is a best response to y^* and if y^* is a best response to x^*.

This is equivalent to an event where Player 1 has no regret to any other action, that is

$$\mathbf{x}^{*\top} A \mathbf{y}^* \geq \mathbf{e_i}^\top A \mathbf{y}^* \quad \forall i\ ,$$

and Player 2 has no regret to any other action, that is

$$\mathbf{x}^{*\top} A \mathbf{y}^* \geq \mathbf{x}^{*\top} A \mathbf{e_j} \quad \forall j\ .$$

7.1.2 Existence of a Nash Equilibrium

We will prove that in any game in this context, there is always at least one Nash equilibrium:

Theorem 7.1. (Nash). *For any bimatrix game (A, B) there exists at least one Nash equilibrium pair.*

The proof uses Brouwer's fix point theorem of topology:

Theorem 7.2. (Brouwer's fix point). *For any convex compact $B \subset \mathbb{R}^n$ and for any continuous function $f : B \to B$, there exists $x^* \in B$ such that $f(x^*) = x^*$.*

Proof (Nash):

Define a map $f : \Delta_n \times \Delta_m \to \Delta_n \times \Delta_m$ by $f(\mathbf{x}.\mathbf{y}) \to (\mathbf{x}'.\mathbf{y}')$ where

$$c_i(\mathbf{x}, \mathbf{y}) := \max(0, \mathbf{e_i}^\top A \mathbf{y} - \mathbf{x}^\top A \mathbf{y}) \qquad \text{and} \qquad d_j(\mathbf{x}, \mathbf{y}) := \max(0, \mathbf{x}^\top A \mathbf{e_j} - \mathbf{x}^\top A \mathbf{y})$$

and

$$x_i' = \frac{x_i + c_i(\mathbf{x}.\mathbf{y})}{1 + \sum_{i'=1}^n c_{i'}(\mathbf{x}.\mathbf{y})} \qquad \text{and} \qquad y_i' = \frac{y_j + d_j(\mathbf{x}.\mathbf{y})}{1 + \sum_{j'=1}^n d_{j'}(\mathbf{x}.\mathbf{y})}\ .$$

Observe that $(\mathbf{x}.\mathbf{y})$ is a Nash equilibrium if and only if $c_i(\mathbf{x}.\mathbf{y}) = 0\ \forall i$ and $d_j(\mathbf{x}.\mathbf{y}) = 0\ \forall j$ (observe too that such $(\mathbf{x}.\mathbf{y})$ will then be a fix point of f).

Using Brouwer's fix point theorem, there is a fix point $(\mathbf{x}.\mathbf{y})$ of f, that is for each i and j:

$$x_i = \frac{x_i + c_i(\mathbf{x}.\mathbf{y})}{1 + \sum_{i'=1}^n c_{i'}(\mathbf{x}.\mathbf{y})} \qquad \text{and} \qquad y_i = \frac{y_j + d_j(\mathbf{x}.\mathbf{y})}{1 + \sum_{j'=1}^n d_{j'}(\mathbf{x}.\mathbf{y})} .$$

We want to show that such $(\mathbf{x}.\mathbf{y})$ is a Nash equilibrium pair. By way of contradiction, assume there exists an i such that $c_i(\mathbf{x}.\mathbf{y}) > 0$.

Since the average payoff can't be worst than each and all payoffs of choices e_i, there must exist some $k \in [n]$ such that $c_k(\mathbf{x}.\mathbf{y}) = 0$.

But then

$$x_k = \frac{x_k + c_k(\mathbf{x}.\mathbf{y})}{1 + \sum_{i'=1}^n c_{i'}(\mathbf{x}.\mathbf{y})} \le \frac{x_k}{1 + c_i(\mathbf{x}.\mathbf{y}))} < x_k .$$

This is a contradiction, so we were wrong in assuming that $c_i(\mathbf{x}.\mathbf{y}) > 0$.

The same argument holds for $\mathbf{y}$ and $d_j(\mathbf{x}.\mathbf{y})$. It follows that $(\mathbf{x}.\mathbf{y})$ is a Nash equilibrium pair.

7.2 Von Neumann Minimax Theorem

Von Neumann Minimax Theorem is stated and proven in two different ways.

7.2.1 Von Neumann Minimax Theorem

In the context of game theory, consider a zero sum game where Player 1 has a payoff matrix A and Player 2 has a payoff matrix B (note that $B = -A$). Player 1 plays $x \in \Delta_n$ and Player 2 plays $y \in \Delta_m$.

Player 1 wants to play an strategy x so that no matter what Player 2 does, Player 1 can be ensured a certain minimum payoff, that is

$$\max_x \min_y x^\top A y .$$

Player 2 wants also an strategy y so that no matter what Player 1 does, Player 2 can be ensured a certain minimum payoff, that is

$$\max_y \min_x x^\top B y = -\min_y \max_x x^\top A y .$$

Theorem 7.3. (von Neumann Minimax Theorem).

$$\max_x \min_y x^\top A y = \min_y \max_x x^\top A y$$

From this theorem it follows that Player 1's strategy guarantees him a payoff of $\max_x \min_y x^\top A y$ regardless of Player 2's strategy, and similarly Player 2 can guarantee himself a payoff of $-\max_x \min_y x^\top A y$ regardless of Player 1's strategy.

7.2.2 Proof 1

In the game $(A, -A)$, we prove that

$$\max_x \min_y x^\top Ay \geq \min_y \max_x x^\top Ay$$

(the other side $\leq$ is trivial and left as an exercise).

Choose any Nash equilibrium (x^*, y^*). It follows that

$$x^{*\top} Ay^* \geq x^\top Ay^* \quad \forall x \qquad \text{and} \qquad x^{*\top} Ay^* \leq x^{*\top} Ay \quad \forall y\,.$$

We get

$$x^{*\top} Ay \geq x^\top Ay^* \quad \forall x, y\,.$$

This implies that

$$\min_y x^{*\top} Ay \geq \max_x x^\top Ay^*$$

which in turn implies that

$$\max_{x'} \min_y x'^\top Ay \geq \min_{y'} \max_x x^\top Ay'\,.$$

The theorem follows.

7.2.3 Proof 2

Using the Exponential Weights Algorithm, given a sequence of loss vectors $l^1, \ldots, l^T$, we generate distributions $p^1, \ldots, p^T$, so that

$$\sum_{t=1}^T p^t \cdot l^t \leq \min_\rho \sum_{t=1}^T \rho \cdot l^t + R(T)\,.$$

We can also talk about payoffs $a^1, \ldots, a^T$ so that

$$\sum_{t=1}^T p^t \cdot a^t \geq \max_\rho \sum_{t=1}^T \rho \cdot a^t - R(T)\,.$$

We put this in the context of game theory, with Player 1 playing $x_t = p^t$ and setting $a^t = Ay^t$ so that Player 2 plays $y^t = A^{-1}a^t$. The previous equations become

$$\sum_{t=1}^T x_t^\top a y_t \leq \min_y \sum_{t=1}^T x^\top Ay_t + R(T) \qquad \text{and} \qquad \sum_{t=1}^T x_t^\top a y_t \geq \max_x \sum_{t=1}^T x^\top Ay_t - R(T)\,.$$

Set $v(T) = \sum_{t=1}^T x_t^\top Ay_t$ and $\bar{x} = \frac{1}{T}\sum_{t=1}^T x_t$, $\bar{y} = \frac{1}{T}\sum_{t=1}^T y_t$, to obtain

$$\frac{v(T)}{T} + \frac{R(T)}{T} \geq \max_x x^\top A\bar{y} \geq \bar{x}A\bar{y} \qquad \text{and} \qquad \frac{v(T)}{T} - \frac{R(T)}{T} \leq \min_y \bar{x}^\top Ay \leq \bar{x}A\bar{y}\,.$$

It follows that

$$\max_x x^\top A\bar{y} \geq \bar{x}A\bar{y} + 2\frac{R(T)}{T} \qquad \text{and} \qquad \min_y \bar{x}^\top Ay \geq \bar{x}A\bar{y} - 2\frac{R(T)}{T}\,.$$

Therefore, we have an optimal strategy and a Nash equilibrium.