7.1 Nash Equilibrium and Existence

In the context of game theory, the concept of Nash Equilibrium is introduced. The Existence of a Nash Equilibrium pair is proven.

7.1.1 Nash Equilibrium

In the context of game theory, a pair \((x^*, y^*)\) is a Nash equilibrium pair if \(x^*\) is a best response to \(y^*\) and if \(y^*\) is a best response to \(x^*\).

This is equivalent to an event where Player 1 has no regret to any other action, that is
\[
x^{i^*}\top A y^* \geq e_i\top A y^* \quad \forall i,
\]
and Player 2 has no regret to any other action, that is
\[
x^{i^*}\top A y^* \geq x^{i^*}\top A e_j \quad \forall j.
\]

7.1.2 Existence of a Nash Equilibrium

We will prove that in any game in this context, there is always at least one Nash equilibrium:

**Theorem 7.1.** (Nash). For any bimatrix game \((A, B)\) there exists at least one Nash equilibrium pair.

The proof uses Brouwer’s fix point theorem of topology:

**Theorem 7.2.** (Brouwer’s fix point). For any convex compact \(B \subset \mathbb{R}^n\) and for any continuous function \(f : B \rightarrow B\), there exists \(x^* \in B\) such that \(f(x^*) = x^*\).

Proof (Nash):

Define a map \(f : \Delta_n \times \Delta_m \rightarrow \Delta_n \times \Delta_m\) by \(f(x, y) = (x', y')\) where
\[
c_i(x, y) := \max(0, e_i\top A y - x\top A y) \quad \text{and} \quad d_j(x, y) := \max(0, x\top A e_j - x\top A y)
\]
and
\[
x_i' = \frac{x_i + c_i(x, y)}{1 + \sum_{i'=1}^n c_i'(x, y)} \quad \text{and} \quad y_j' = \frac{y_j + d_j(x, y)}{1 + \sum_{j'=1}^n d_j'(x, y)}.
\]

Observe that \((x, y)\) is a Nash equilibrium if and only if \(c_i(x, y) = 0 \quad \forall i\) and \(d_j(x, y) = 0 \quad \forall j\) (observe too that such \((x, y)\) will then be a fix point of \(f\)).
Using Brouwer's fix point theorem, there is a fix point \((x, y)\) of \(f\), that is for each \(i\) and \(j\):

\[
x_i = \frac{x_i + c_i(x, y)}{1 + \sum_{i'=1}^n c_{i'}(x, y)} \quad \text{and} \quad y_i = \frac{y_i + d_i(x, y)}{1 + \sum_{j'=1}^m d_{j'}(x, y)}.
\]

We want to show that such \((x, y)\) is a Nash equilibrium pair. By way of contradiction, assume there exists an \(i\) such that \(c_i(x, y) > 0\).

Since the average payoff can't be worst than each and all payoffs of choices \(c_i\), there must exist some \(k \in [n]\) such that \(c_k(x, y) = 0\).

But then

\[
x_k = \frac{x_k + c_k(x, y)}{1 + \sum_{i'=1}^n c_{i'}(x, y)} \leq \frac{x_k}{1 + c_i(x, y)} < x_k.
\]

This is a contradiction, so we were wrong in assuming that \(c_i(x, y) > 0\).

The same argument holds for \(y\) and \(d_j(x, y)\). It follows that \((x, y)\) is a Nash equilibrium pair.

### 7.2 Von Neumann Minimax Theorem

Von Neumann Minimax Theorem is stated and proven in two different ways.

#### 7.2.1 Von Neumann Minimax Theorem

In the context of game theory, consider a zero sum game where Player 1 has a payoff matrix \(A\) and Player 2 has a payoff matrix \(B\) (note that \(B = -A\)). Player 1 plays \(x \in \Delta_n\) and Player 2 plays \(y \in \Delta_m\).

Player 1 wants to play an strategy \(x\) so that no matter what Player 2 does, Player 1 can be ensured a certain minimum payoff, that is

\[
\max_x \min_y x^\top Ay.
\]

Player 2 wants also an strategy \(y\) so that no matter what Player 1 does, Player 2 can be ensured a certain minimum payoff, that is

\[
\max_y \min_x x^\top By = -\min_y \max_x x^\top Ay.
\]

**Theorem 7.3.** (von Neumann Minimax Theorem).

\[
\max_x \min_y x^\top Ay = \min_y \max_x x^\top Ay
\]

From this theorem it follows that Player 1’s strategy guarantees him a payoff of \(\max_x \min_y x^\top Ay\) regardless of Player 2’s strategy, and similarly Player 2 can guarantee himself a payoff of \(-\max_x \min_y x^\top Ay\) regardless of Player 1’s strategy.
7.2.2 Proof 1

In the game \((A, -A)\), we prove that

\[
\max_x \min_y x^T Ay \geq \min_y \max_x x^T Ay
\]

(the other side \(\leq\) is trivial and left as an exercise).

Choose any Nash equilibrium \((x^*, y^*)\). It follows that

\[
x^*^T Ay^* \geq x^T Ay^* \quad \forall x \quad \text{and} \quad x^*^T Ay^* \leq x^T Ay^* \quad \forall y.
\]

We get

\[
x^*^T Ay \geq x^T Ay^* \quad \forall x, y.
\]

This implies that

\[
\min_y x^*^T Ay \geq \max_x x^T Ay^*
\]

which in turn implies that

\[
\max_{x'} \min_y x'^T Ay \geq \min_x \max_{y'} x^T Ay'.
\]

The theorem follows.

7.2.3 Proof 2

Using the Exponential Weights Algorithm, given a sequence of loss vectors \(l^1, \ldots, l^T\), we generate distributions \(p^1, \ldots, p^T\), so that

\[
\sum_{t=1}^T p^t \cdot l^t \leq \min \sum_{t=1}^T \rho \cdot l^t + R(T).
\]

We can also talk about payoffs \(a^1, \ldots, a^T\) so that

\[
\sum_{t=1}^T p^t \cdot a^t \geq \max \sum_{t=1}^T \rho \cdot a^t - R(T).
\]

We put this in the context of game theory, with Player 1 playing \(x_t = p^t\) and setting \(a^t = Ay^t\) so that Player 2 plays \(y_t = A^{-1} a^t\). The previous equations become

\[
\sum_{t=1}^T x_t^T ay_t \leq \min_y \sum_{t=1}^T x^T Ay_t + R(T) \quad \text{and} \quad \sum_{t=1}^T x_t^T ay_t \geq \max_x \sum_{t=1}^T x^T Ay_t - R(T).
\]

Set \(v(T) = \sum_{t=1}^T x_t^T Ay_t\) and \(\bar{x} = \frac{1}{T} \sum_{t=1}^T x_t, \quad \bar{y} = \frac{1}{T} \sum_{t=1}^T y_t\), to obtain

\[
\frac{v(T)}{T} + \frac{R(T)}{T} \geq \max_x x^T \bar{A} \bar{y} \geq \bar{x} \bar{A} \bar{y} \quad \text{and} \quad \frac{v(T)}{T} - \frac{R(T)}{T} \leq \min_y x^T \bar{A} \bar{y} \leq \bar{x} \bar{A} \bar{y}.
\]
It follows that
\[
\max_x x^\top A \bar{y} \geq \bar{x} A \bar{y} + 2 \frac{R(T)}{T}
\quad \text{and} \quad
\min_y \bar{x}^\top A y \geq \bar{x} A \bar{y} - 2 \frac{R(T)}{T}.
\]

Therefore, we have an optimal strategy and a Nash equilibrium.