Announcements

- Class by Guest Lecturer Ambuj Tewari

1 Convex Optimization

\[
\min_{x \in C} f(x)
\]  \hspace{1cm} (1.1)

\( C \subseteq \mathbb{R}^d, f : \mathbb{R}^d \rightarrow \mathbb{R}, C \) is a convex set and \( f \) is a convex function.

\( C \) is convex if: \( x, y \in C \Rightarrow \lambda x + (1 - \lambda)y \in C, \forall \lambda \in [0, 1] \)

\( f \) is convex if: \( f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \forall x, y, \lambda \in [0, 1] \)

Rockafeller: "Watershed in optimization is not between linearity and non-linearity but it is between convexity and inconvexity".

Seminal work in online convex optimization: Zinkevich (2003) "Online convex prog. and generalized infinitesimal gradient descent".

1.1 Online Convex Optimization (OCO) "protocol"

FOR \( t = 1 \) to \( T \)

learner/player plays \( x_t \in C \)

nature/adversary reveals \( f_t \) (convex)

learner suffers a cost/loss \( f_t(x_t) \)

END

Remarks:

- "Converging to a solution " does not really make sense in this setting
- Redefine goal to be "achieve low regret"

\[
\text{regret} = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in C} \sum_{t=1}^{T} f_t(x)
\]  \hspace{1cm} (1.2)
2 Interesting Special Cases of OCO

2.1 Experts Problems

\[ C = \Delta^d = \{ \vec{p} \in \mathbb{R}^d : p_j \geq 0, \sum_{j=1}^{d} p_j = 1 \} \]  

\[ f_t(x) = l_t \cdot x \text{ where } l_t \in [0,1]^d \]  

Think of player’s move \( x_t \in C = \Delta^d \) as a probability distribution over experts. \( l_t \in \mathbb{R} \) encodes loss suffered by the experts at time \( t \).

\( l_t \cdot x = \) expected loss of the player where he picks expert \( j \) with probability \( x_j \).

\[ \text{regret} = \sum_{t=1}^{T} l_t \cdot x_t - \min_{x \in C} \sum_{t=1}^{T} l_t \cdot x = \text{expected loss of player} - \text{loss of best expert in hindsight} \]  

2.2 Online Classification/Regression using linear prediction and convex loss

for \( t=1 \) to \( T \)

learner/player receives \( z_t \in \mathbb{R}^d \)

classifier/predictor encoded by \( w_t \in \mathbb{R}^d \) is used to output \( w_t \cdot z_t \)

learner receives true value label \( y_t \)

learner suffers \( l[w_t \cdot z_t; y_t] \) (loss function)

end

Let’s say we only want to consider linear predictors \( z \to w \cdot t \) such that \( \|w\|_2 \leq W \), where \( \|w\|_2 = \sqrt{\sum_{j=1}^{d} w_j^2} \).

\[ \text{Convex: } C = \{ w \in \mathbb{R}^d, \|w\|_2 \leq w \} \]  

\[ f_t(w) = l(w \cdot z_t, y_t) \]  

If \( l(\cdot, y) \) is convex \( \forall y \) then \( f_t \) is convex in \( w \)

Example of convex losses:

\[ l(t, y) = \frac{1}{2} (t - y)^2 \text{(regression)} \]  

\[ l(t, y) = \max(0,1 - t \cdot y) \text{("hinge loss")} \]  

\[ t(t, y) = \log(1 + e^{-yt}) \text{("logistic loss")} \]

3 Algorithms for OCO

Idea1: Just "follow the leader"

\[ x_t = \arg \min_{x \in C} \sum_{s=1}^{t-1} f_s(x) \]
Counter example

\[ c = [-1, 1] \in \mathbb{R} \]  \hspace{1cm} (3.2)

\[ f_t(x) = c_t x \]  \hspace{1cm} (3.3)

Leader plays \( x_1 \in [-1, 1] \) \( c_1 = -0.5, x_2 = 1, c_2 = 1, x_3 = -1, c_3 = -1, x_4 = 1, c_4 = 1, \ldots \). For the adversary defined by \( c_1 = -0.5 \) and for \( t > 1 \)

\[ c_t = \begin{cases} 
1 & \text{t is even} \\
-1 & \text{t is odd} 
\end{cases} \]  \hspace{1cm} (3.4)

We will now argue that FTL suffers \( O(T) \) regret. Player’s loss:

\[ \sum_{t=1}^{T} c_t x_t = c_1 x_1 + 1 + 1 + 1 + \ldots + 1 = O(T) \]  \hspace{1cm} (3.5)

Best loss in hindsight = \( \min_{x \in [-1,1]} \sum c_t x \leq \sum c_t 0 = 0 \) \( \Rightarrow \) Regret is also \( O(T) \)

**Good news about FTL:** *If \( f_t \)'s are not just convex but strongly convex then FTL has \( O(\log T) \) regret.*

*If we slightly change FTL to include a strongly convex function we can improve the regret bound.*

**FTRL: Follow the Regularized leader**

\[ x_t \in \arg \min_{x \in c} \eta \sum_{s=1}^{t-1} f_s(t) + \frac{1}{2} \|x\|_2^2 \]  \hspace{1cm} (3.6)

We will prove a regret guarantee for FTRL assuming:

- There are no constraints i.e., \( C = \mathbb{R}^d \)
- \( f_t \) is a linear function, \( f_t(x) = c_t \cdot x \)

**BTL (Beat the Leader) (illegal algorithm)**

\[ x_t^{BTL} \in \arg \min_{x \in c} \sum_{s=1}^{t} f_s(x) \]  \hspace{1cm} (3.7)

**Lemma about BTL**

\[ \sum_{t=1}^{T} f_t(x^{BTL}) \leq \sum_{t=1}^{T} f_t(x), \forall x \in c \]  \hspace{1cm} (3.8)
Proof by backward induction

\[ f_1(x_1^{BTL}) + f_2(x_2^{BTL}) + \ldots + f_T(x_T^{BTL}) \leq f_1(x) + f_2(x) + \ldots + f_T(x) \]  \hfill (3.9)

\[ f_1(x_T^{BTL}) + \ldots + f_{T-1}(x_{T-1}^{BTL}) \leq f_1(x_T^{BTL}) + f_2(x_T^{BTL}) + \ldots + f_{T-1}(x_{T-1}^{BTL}) \]  \hfill (3.10)

Counting down \( T - 2, T - 3, \ldots, 1 \) proves the lemma.

- \( x_1^{FTRL} \) is BTL output on \( \frac{1}{2} \|x\|^2 \)
- \( x_2^{FTRL} \) is BTL output on \( \frac{1}{2} \|x\|^2, f_1 \)
- \( x_3^{FTRL} \) is BTL output on \( \frac{1}{2} \|x\|^2, f_1, f_2 \ldots \)

Using BTL Lemma

\[ \frac{1}{2} \|x^{FTRL}\|^2 + \sum f_t(x_{t+1}^{FTRL}) \leq \frac{1}{2} \|x\|^2 + \sum f_t(x) \]  \hfill (3.11)

\[ \Rightarrow \sum_{t=1}^{T} (f_t(x_t^{FTRL}) - f_t(x)) \leq \frac{1}{2} \|x\|^2 - \frac{1}{2} \|x_1^{FTRL}\|^2 + \sum_{t=1}^{T} (f_t(x_t^{FTRL}) - f_t(x_{t+1}^{FTRL})) \]  \hfill (3.12)

Because of assumptions \( x_t^{FTRL} = -\eta \sum_{s=1}^{t-1} c_s \), and hence \( f_t(x_t^{FTRL}) - f_t(x_{t+1}^{FTRL}) = \eta \|c_t\|^2 \)

Therefore regret against \( x \leq \frac{1}{2} \frac{\|x\|^2}{\eta} + \eta \sum_{t=1}^{T} \|c_t\|^2 \)