On Proof Systems Behind Efficient SAT Solvers

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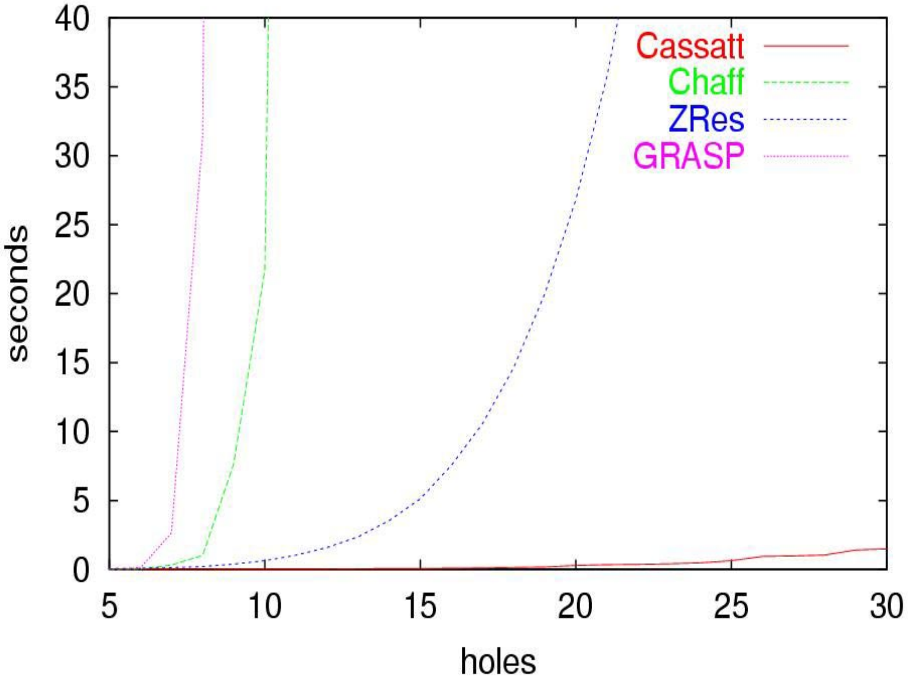
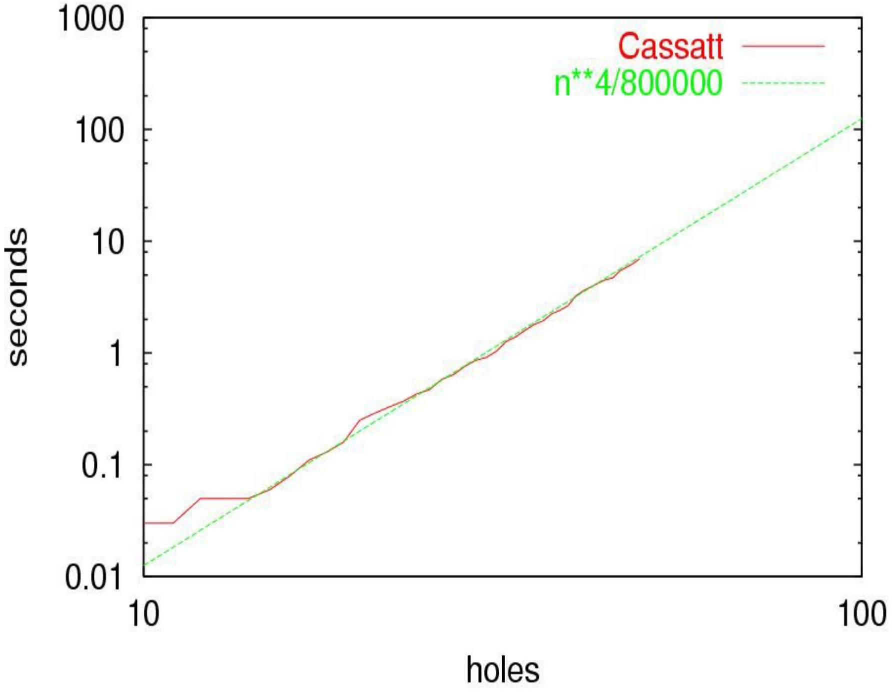
Motivation

- Best complete SAT solvers are based on DLL
  - Runtime (on unSAT instances) is lower-bounded by the length of resolution proofs
  - Exponential lower bounds for pigeonholes
- **Previous work**: we introduced the Compressed Breadth-First Search algorithm (CBFS/**Cassatt**)
  - Empirical measurements: our implementation of Cassatt spends $\Theta(n^4)$ time on $\overline{PHP}_n^{n+1}$
- **This work**: we show analytically that CBFS refutes pigeonhole instances $\overline{PHP}_n^{n+1}$ in poly time
  - Hope to find a proof system behind Cassatt

Empirical Performance





Related Work

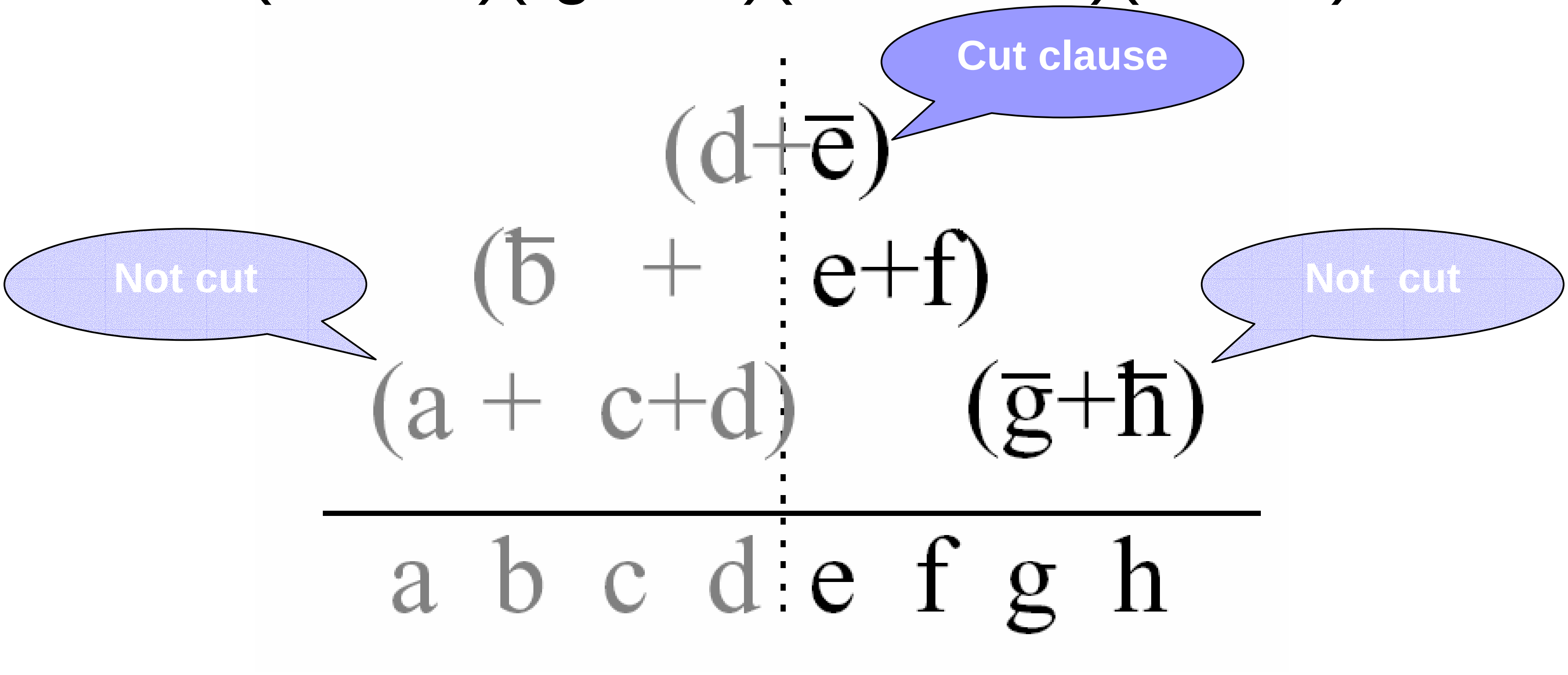
- We are pursuing novel algorithms for SAT facilitated by data structures with compression
  - Zero-suppressed Binary Decision Diagrams (ZDDs)
- Existing algorithms can be implemented w ZDDs
  - The DP procedure: Simon and Chatalic, *[ICTAI 2000]*
  - DLL: Aloul, Mneimneh and Sakallah, *[DATE 2002]*
- We use the union-with-subsumption operation
- Details of the Cassatt algorithm are in
  - Motter and Markov, *[ALENEX 2002]*

Outline

- Background
- Compressed BFS
    - Overview
    - Example
    - Algorithm
- Pigeonhole Instances
- Outline of Proof
    - Some bounds
- Conclusions and Ongoing Work

Background

(a+c+d)(-g + -h)(-b + e + f)(d + -e)



Cut clause

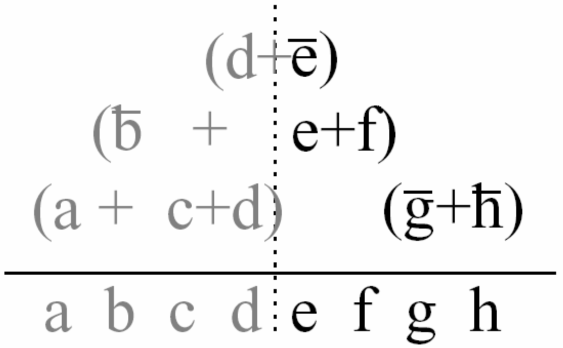
Not cut

Not cut

Background: Terminology

- Given partial truth assignment
- Classify all clauses into:
  - **Satisfied**
    - At least one literal assigned true
  - **Violated**
    - ✗ All literals assigned, and not satisfied
  - **Open**
    - 1+ literal assigned, and no literals assigned true
    - Open clauses are activated but not satisfied
  - **Activated**
    - Have at least one literal assigned some value
  - **Unit**
    - Have all but one literal assigned, and are *open*
- A valid partial truth assignment ⇔ no violated clauses

disjoint

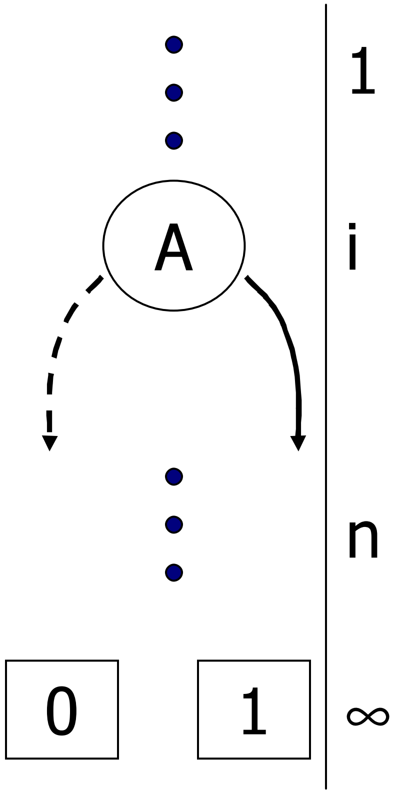


Open Clauses

- Straightforward **Breadth-First Search**
  - Maintain all valid partial truth assignments of a given depth; increase depth in steps
- **Valid partial truth assignments** → **sets of open clauses**
  - **No** literals assigned ⇒ Clause is **not activated**
  - **All** literals assigned ⇒ Clause must be **satisfied**
    - Because: assignment is valid ⇒ no clauses are violated
- "**Cut**" clause = **some**, but not all **literals assigned**
  - Must be either **satisfied** or **open**
  - This is determined by the partial assignment

Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level index
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal
- BDDs are characterized by reduction rules
  - If two nodes have the same level index and children
    - Merge these nodes



Zero-Supressed BDDs (ZDDs)

- Zero-supression rule
  - Eliminate nodes whose T-Child is **0**
  - No node with a given index ⇒ assume a node whose T-child is **0**
- ZDDs can store a collection of subsets
  - Encoded by the collection’s characteristic function
  - **0** is the empty collection ∅
  - **1** is the one-collection of the empty set {∅}
- Zero-suppression rule enables compact representations of sparse or regular collections

Compressed BFS: Overview

- Maintain collection of subsets of open clauses
  - Analogous to maintaining all "promising" partial solutions of increasing depth
  - Enough information for BFS on the solution tree
- This collection of sets is called **the front**
  - Stored and manipulated in compressed form (ZDD)
  - Assumes a clause ordering (global indices)
    - Clause indices correspond to node levels in the ZDD
- Algorithm: expand one variable at a time
  - When all variables are processed two cases possible
    - The front is $\emptyset$ $\Rightarrow$ Unsatisfiable
    - The front is $\{\emptyset\}$ $\Rightarrow$ Satisfiable

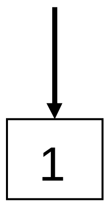
Compressed BFS

```
Front ← 1     # assign {∅} to front
foreach v ∈ Vars
  Front2 ← Front
  Update(Front,  v ← 1)
  Update(Front2, v ← 0)
  Front ← Front $\cup_s$ Front2
if Front == 0 return Unsatisfiable
if Front == 1 return Satisfiable
```

Compressed BFS: An Example

$$\underbrace{(b + c + d)}_{1}\underbrace{(-b + c + -d)}_{2}\underbrace{(a + c + d)}_{3}\underbrace{(a + b + -c)}_{4}\underbrace{(-a + -c + d)}_{5}\underbrace{(-a + b + d)}_{6}$$

- Process variables in the order {a, b, c, d}
- Initially the front is set to 1
  - The collection should contain one “branch”
  - This branch should contain no open clauses $\Rightarrow \{\emptyset\}$



Compressed BFS: An Example

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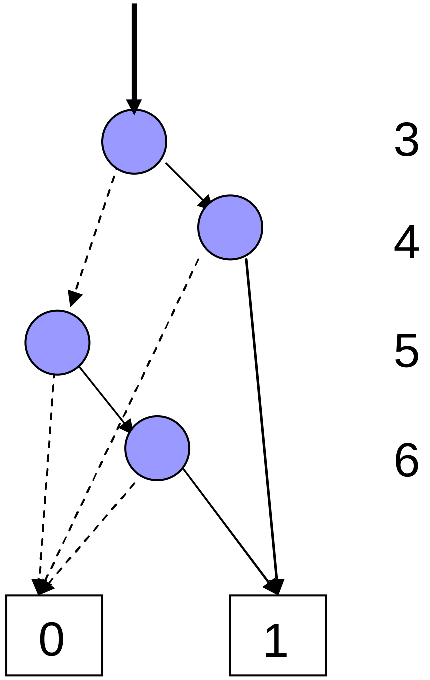
- Processing variable a
  - Activate clauses {3, 4, 5, 6}
    - Cut clauses: {3, 4, 5, 6}
  - a = 0
    - Clauses {3, 4} become open
  - a = 1
    - Clauses {5, 6} become open
- ZDD contains { {3, 4}, {5, 6} }



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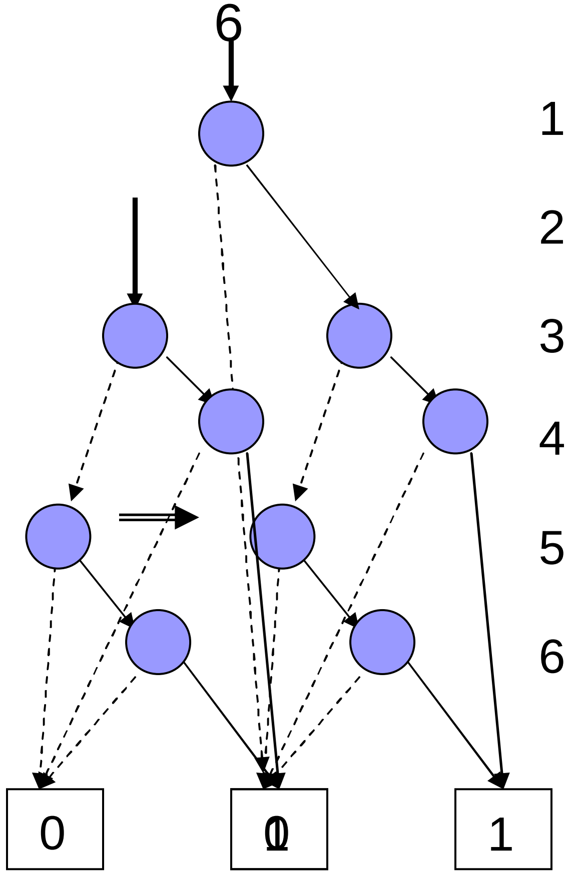
- Processing variable b
  - Activate clauses {1, 2}
    - Cut clauses: {1, 2, 3, 4, 5, 6}
  - b = 0
    - No clauses can become violated
      - b is not the end literal for any clause
    - Clause 2 is satisfied
      - Don’t need to add it
    - Clause 1 first becomes activated



Compressed BFS: An Example

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- Processing variable b
  - Activate clauses {1, 2}
    - Cut clauses: {1, 2, 3, 4, 5, 6}
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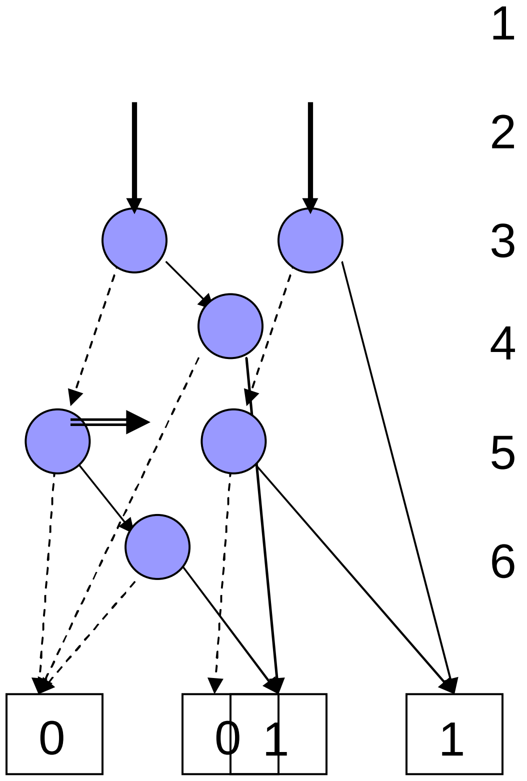
      - b is not the end literal for any clause
    - Existing clauses 4, 6 are satisfied
    - Clause 1 is satisfied
      - Don’t need to add it
    - Clause 2 first becomes activated



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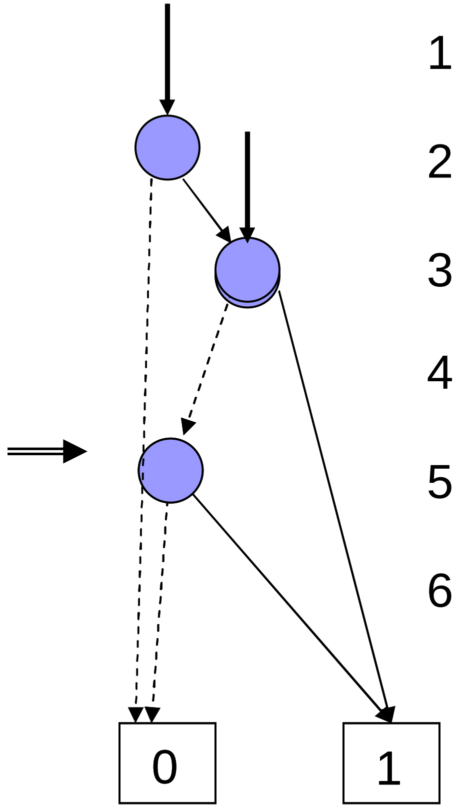

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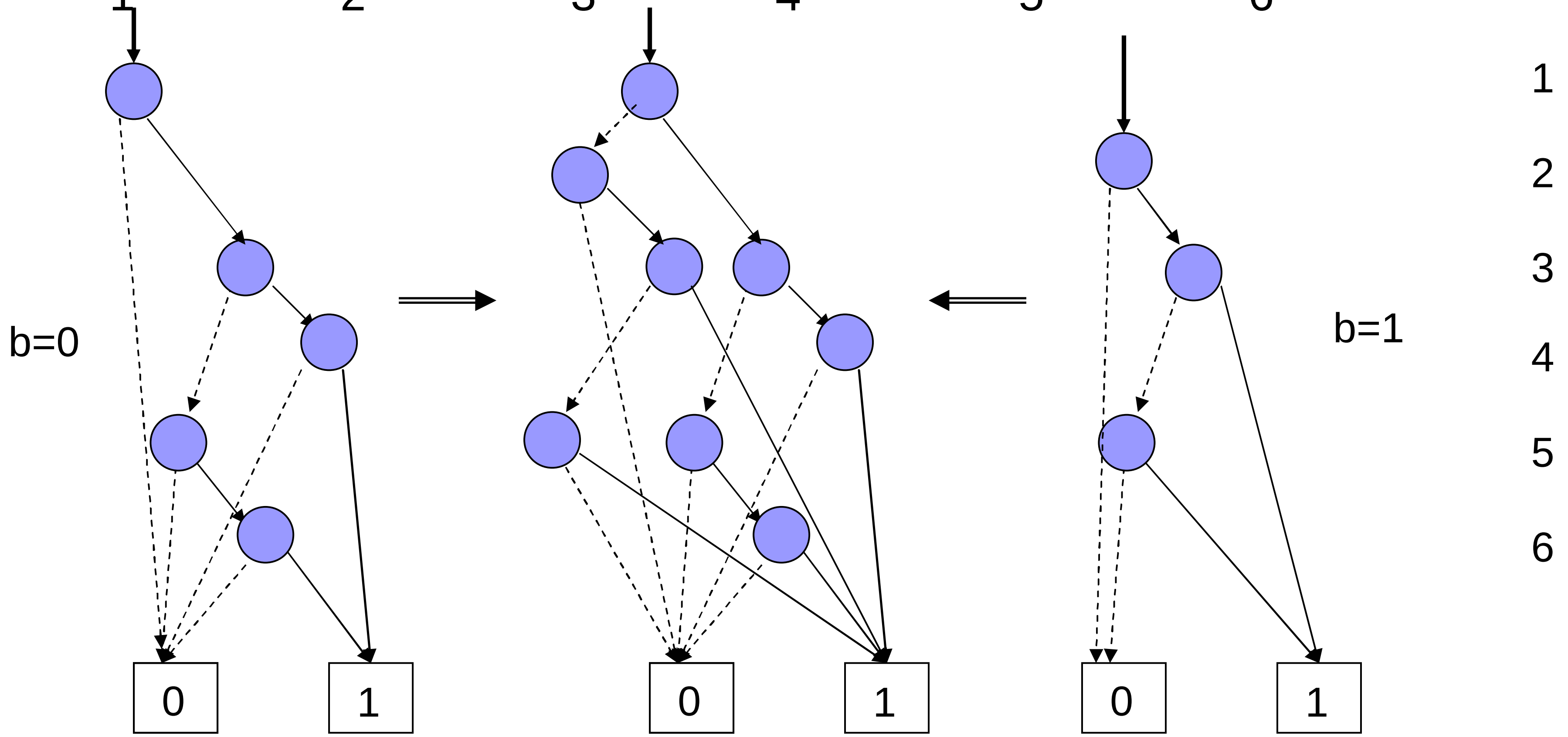
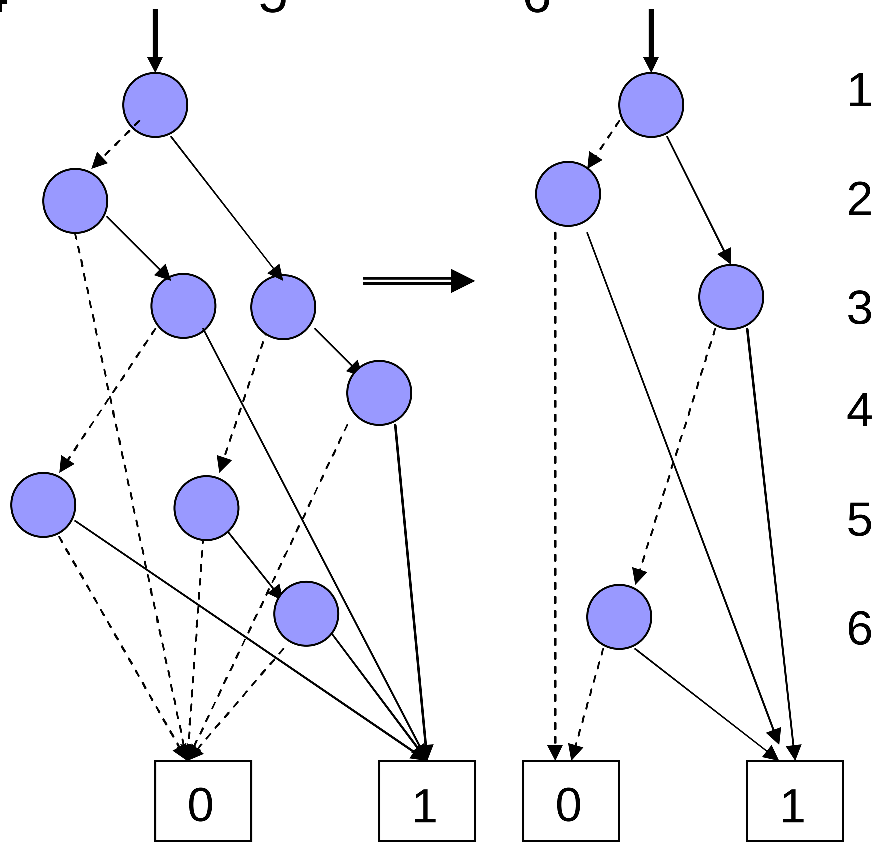


Compressed BFS: An Example

$$\underbrace{(b + c + d)}_{1}\underbrace{(-b + c + -d)}_{2}\underbrace{(a + c + d)}_{3}\underbrace{(a + b + -c)}_{4}\underbrace{(-a + -c + d)}_{5}\underbrace{(-a + b + d)}_{6}$$

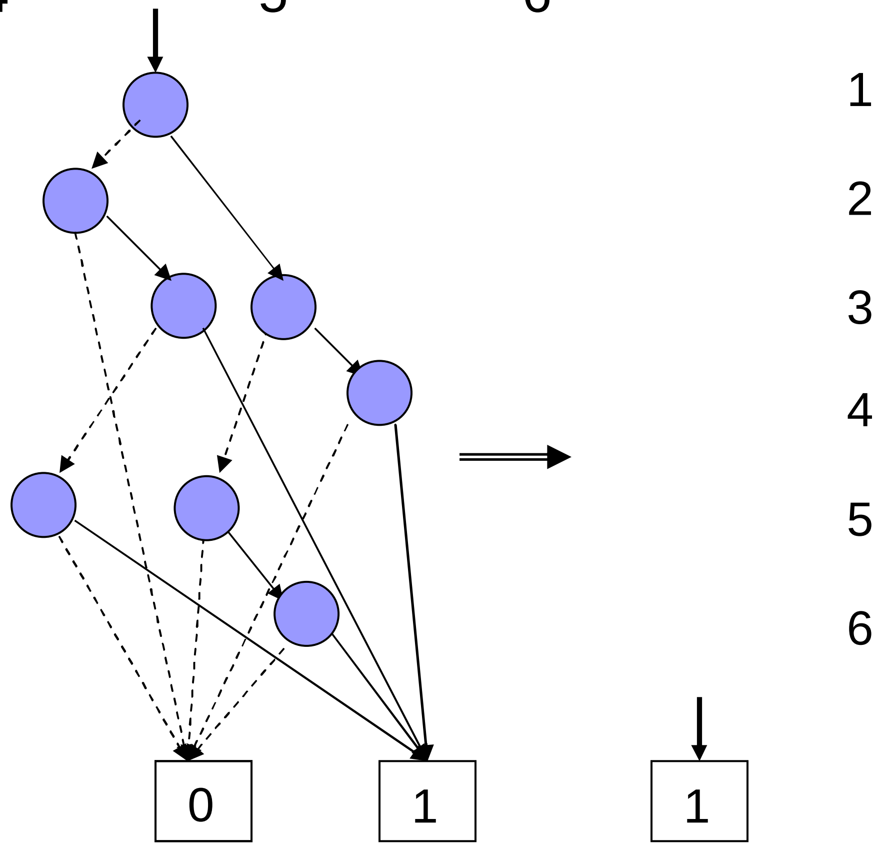
- Processing variable c
  - Finish clause 4
    - Cut clauses: {1, 2, 3, 5, 6}
  - c = 0
    - No clauses become violated
      - c ends 4, but c=0 satisfies it
    - Clauses 4,5 become satisfied
    - No clauses become activated



Compressed BFS: An Example

$$\underbrace{(b + c + d)}_{1}\underbrace{(-b + c + -d)}_{2}\underbrace{(a + c + d)}_{3}\underbrace{(a + b + -c)}_{4}\underbrace{(-a + -c + d)}_{5}\underbrace{(-a + b + d)}_{6}$$

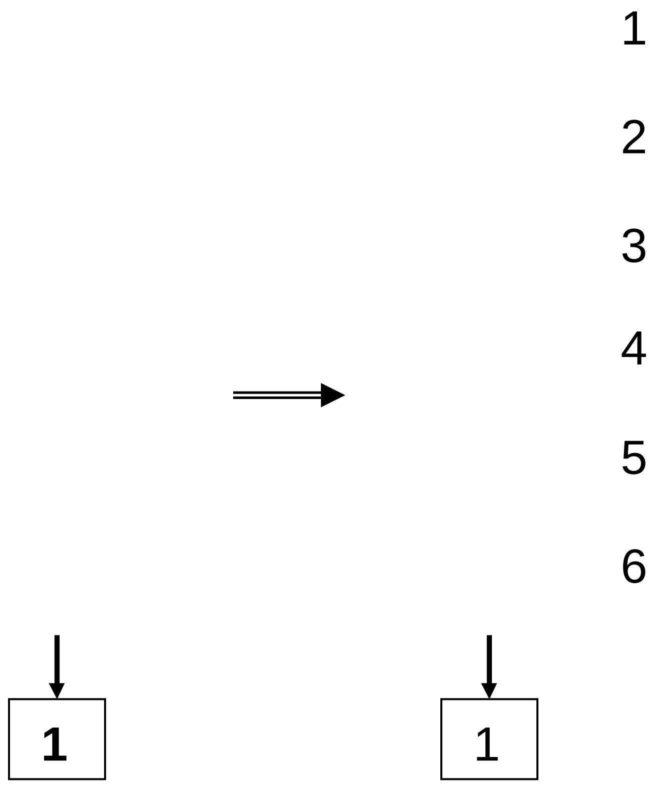
- Processing variable c
  - Finish clause 4
    - Cut clauses: {1, 2, 3, 5, 6}
  - c = 1
    - Clause 4 may be violated
      - If c appears in the ZDD, then it is still open
    - Clauses 1, 2, 3 are satisfied
    - No clauses become activated



Compressed BFS: An Example

$$\underbrace{(b + c + d)}_{1}\underbrace{(-b + c + -d)}_{2}\underbrace{(a + c + d)}_{3}\underbrace{(a + b + -c)}_{4}\underbrace{(-a + -c + d)}_{5}\underbrace{(-a + b + d)}_{6}$$

- Processing variable d
  - Finish clauses {1, 2, 3, 5, 6}
    - Cut clauses: {1, 2, 3, 5, 6}
  - d = 0, d=1
    - All clauses are already satisfied
    - Assignment doesn't affect this
    - Instance is satisfiable



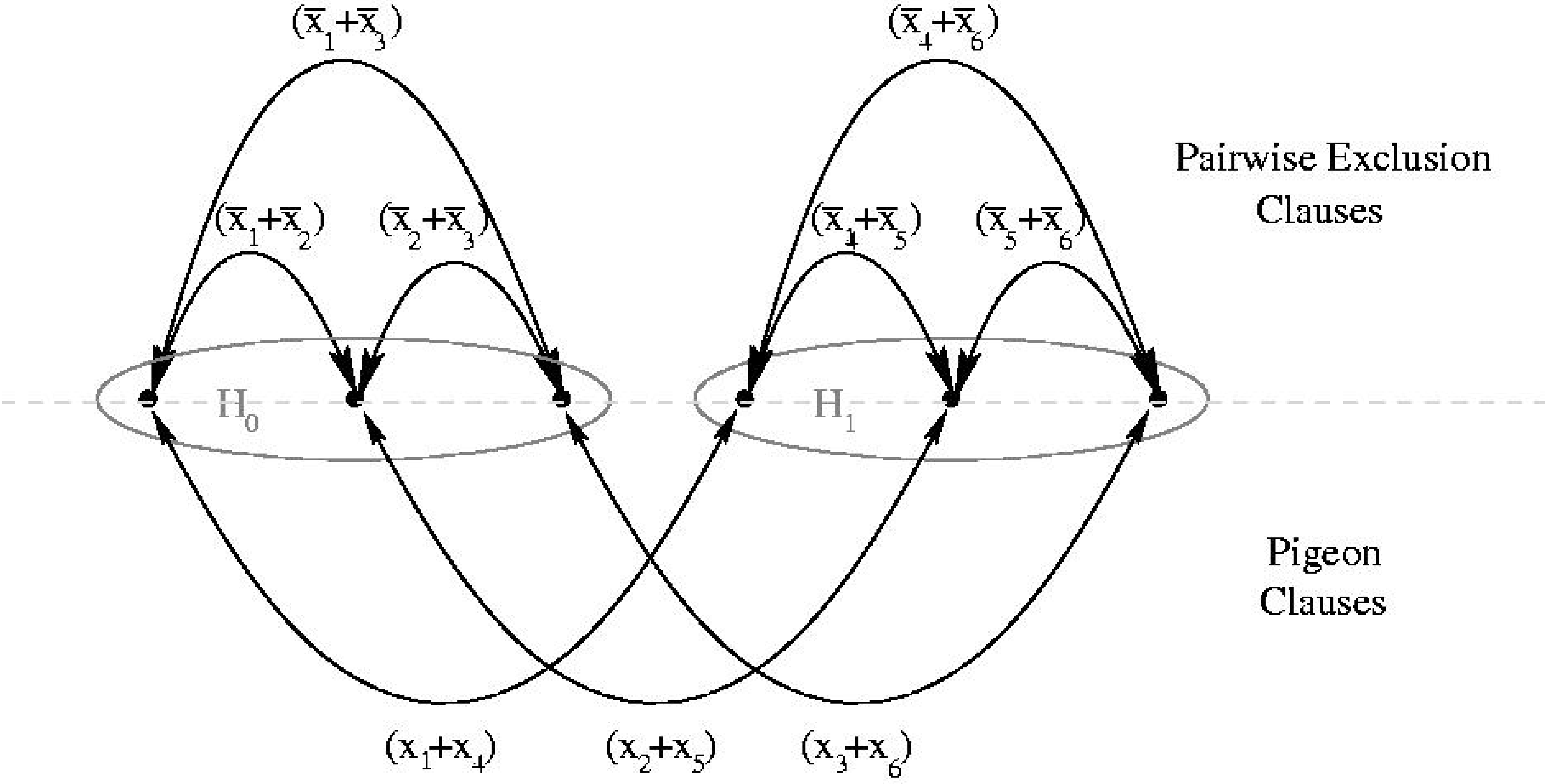
Compressed BFS: Pseudocode

```
CompressedBfs(Vars, Clauses)
    front ← 1
    for i = 1 to |Vars| do
        front' ← front
        //Modify front to reflect $x_i = 1$
        Form sets $U_{xi,1}$, $S_{xi,1}$, $A_{xi,1}$
        front ← front $\cap$ $\mathbf{2}^{Cut - Uxi,1}$
        front ← ExistAbstract(front, $S_{xi,1}$)
        front ← front $\otimes$ $A_{xi,1}$
        //Modify front' to reflect $x_i = 0$
        Form sets $U_{xi,0}$, $S_{xi,0}$, $A_{xi,0}$
        front' ← front' $\cap$ $\mathbf{2}^{Cut - Uxi,0}$
        front' ← ExistAbstract(front', $S_{xi,0}$)
        front' ← front' $\otimes$ $A_{xi,0}$
        //Combine the two branches via Union with Subsumption
        front ← front $\cup_s$ front'
    if front = 0 then
        return Unsatisfiable
    if front = 1 then
        return Satisfiable
```

The Instances $\overline{\mathrm{PHP}_n^{n+1}}$

- Negation of the pigeonhole principle
  - *“If n+1 pigeons are placed in n holes then some hole must contain more than one pigeon”*
- Encoded as a CNF
  - n(n+1) Boolean variables
    - v_{ij} represents that pigeon *i* is in hole *j*
  - n+1 “Pigeon” clauses: $(v_{i1} + v_{i2} + \dots + v_{in})$
    - Pigeon i must be in some hole
  - n(n+1) “Pairwise Exclusion” clauses (per hole): $(\overline{v_{i1j}} + \overline{v_{i2j}})$
    - No two pigeons can be in the same hole
- Unsatisfiable CNF instance
- Use the “hole-major” variable ordering
  - $\{x_1, x_2, \dots x_{n(n+1)}\} \Leftrightarrow \{v_{11}, v_{21}, \dots, v_{(n+1)1}, v_{12}, v_{22}, \dots\}$

The Instances $\overline{PHP_n^{n+1}}$

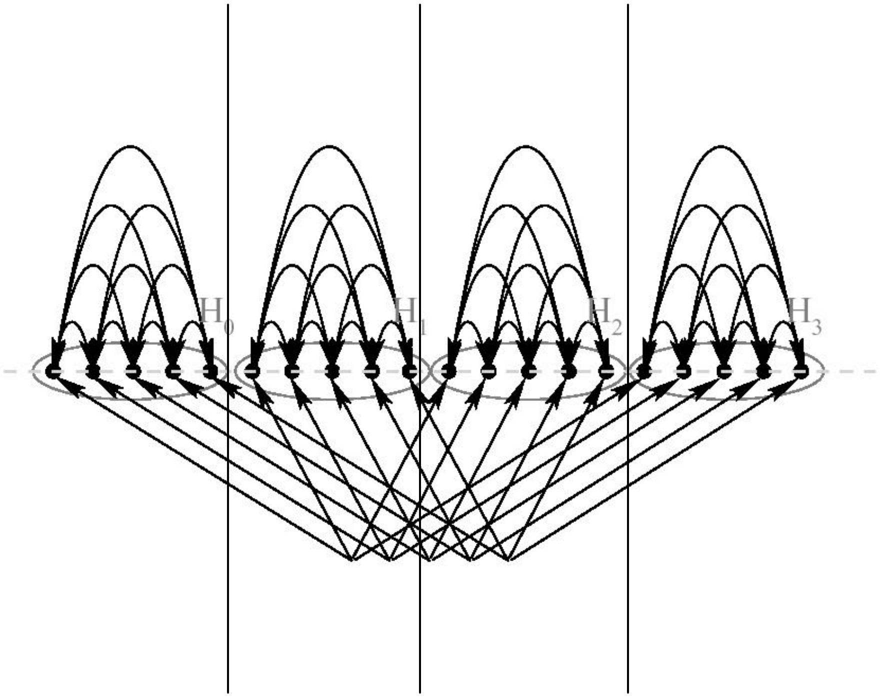


Outline of Proof

- Bound the size of the ZDD-based representation throughout execution
  - With most ZDD operations:
    - h = zdd_op(ZDD f, ZDD g)
    - h is built during a traversal of ZDDs f, g
    - The execution time is bounded by poly(|f|, |g|)
- Do not consider all effects of reduction rules
  - These obscure underlying structure of the ZDD
  - Reduction rules can only eliminate nodes
    - This will still allow an upper bound on ZDD size

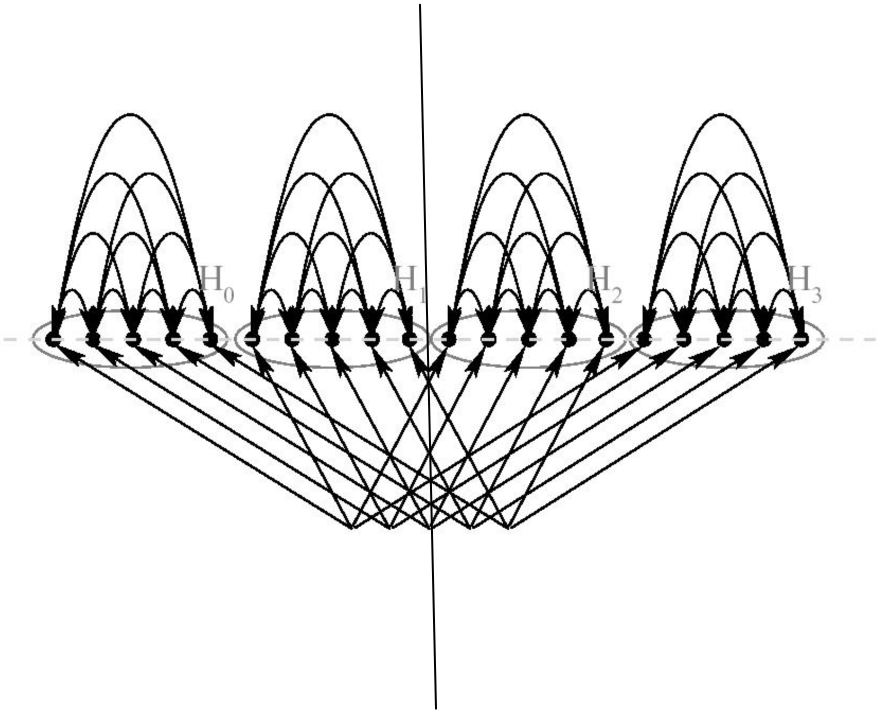
Outline of Proof

- Main idea: Bound the size of the partially reduced ZDD
  - First compute a simple bound between "holes"
  - Prove that the size does not grow too greatly inside "holes"
- Show the ZDD at given step has a specific structure

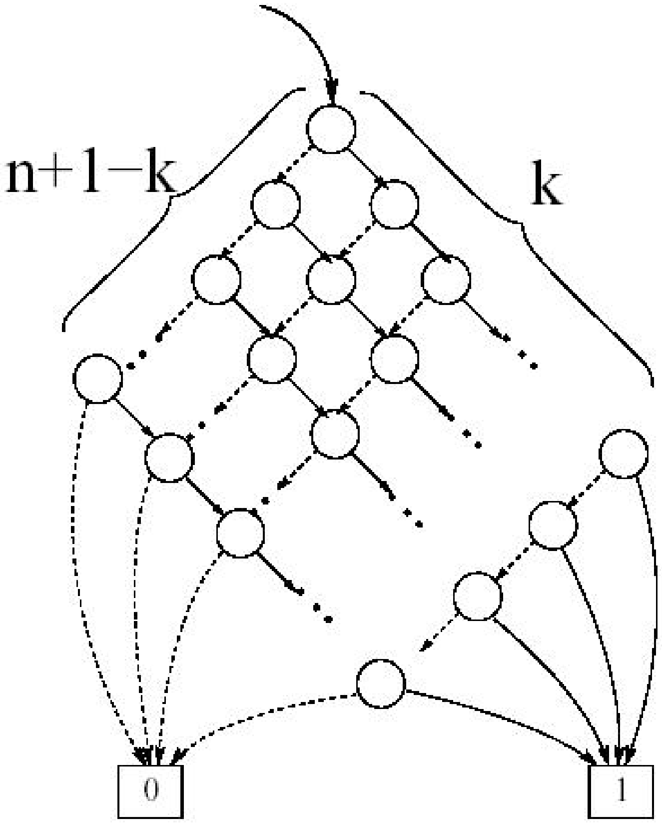


Bounds Between H_k

- **Lemma**. Let $k \in \{1, 2, \ldots, n\}$. After assigning values to variables $x_1, x_2, \ldots, x_{k(n+1)}$, we may satisfy at most k of the n+1 pigeon clauses.
  - ☐ Valid partial truth assignment to the first k(n+1) variables
  - ⇒Must set only one variable in H_i true, for each i<k.
- For CBFS
  - ☐ Remove subsumed sets
  - ⇒ front contains all sets of (n+1-k) pigeon clauses
  - ☐ How many nodes does this take?



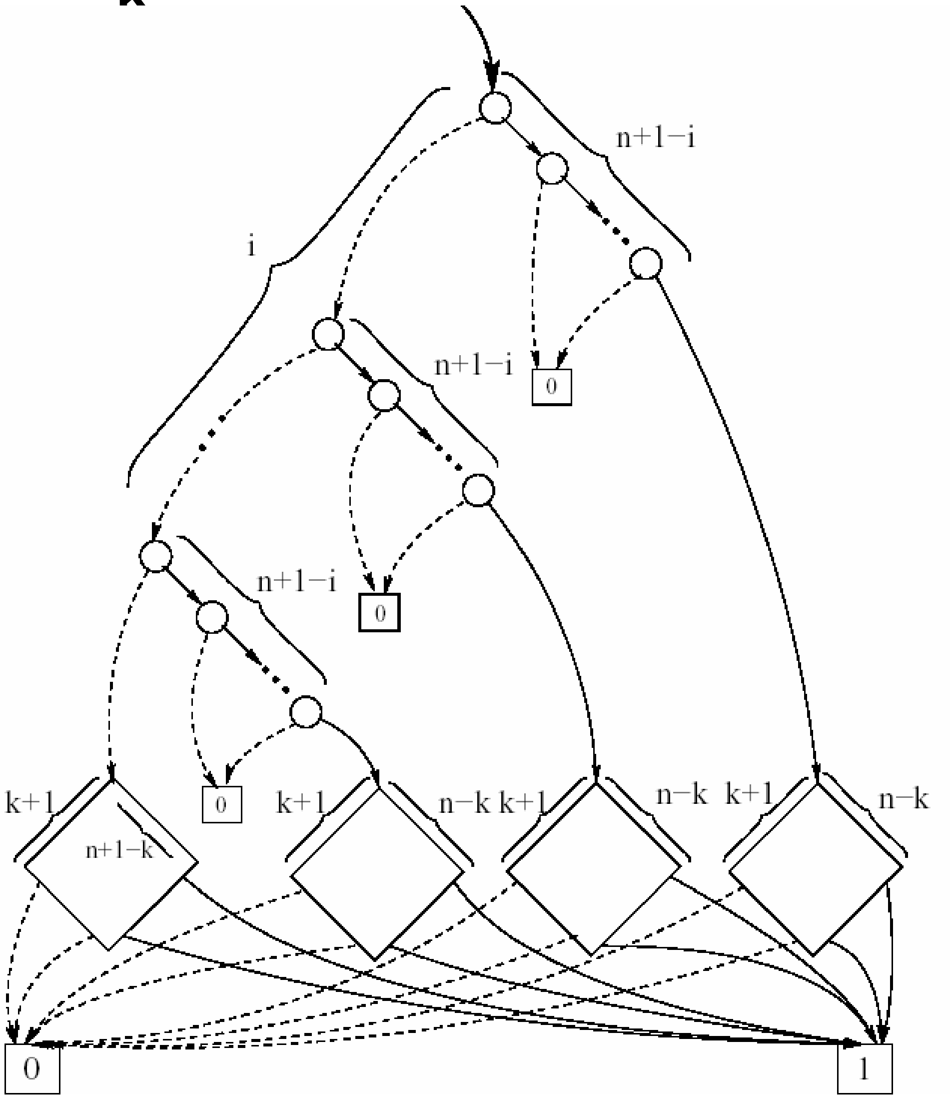
ZDD of all k-Element Subsets



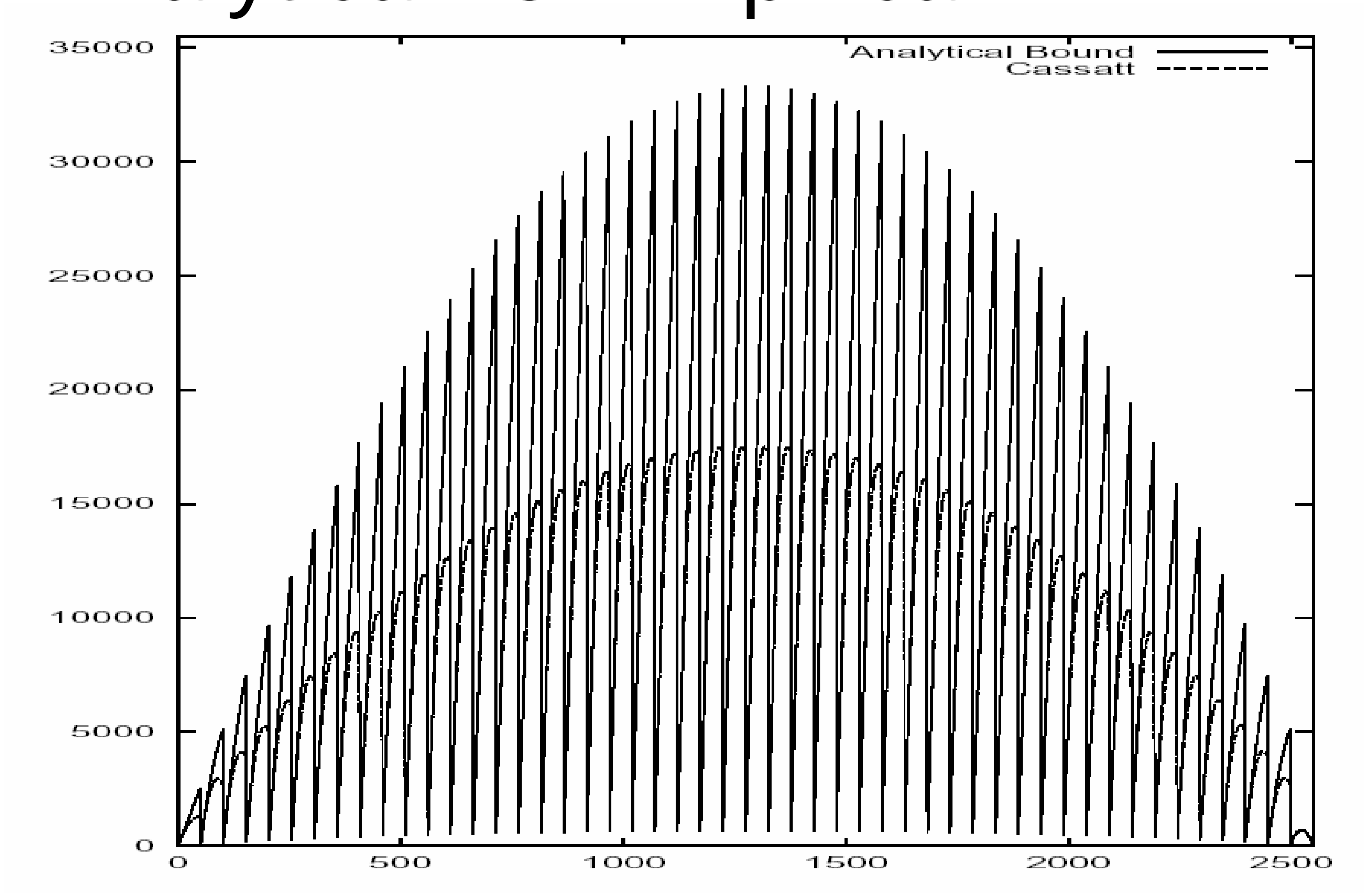
- To reach **1** ⇒ function must select the T-Child on exactly k indices
  - ☐ Less than k ⇒ Traverse to 0
  - ☐ More than k ⇒ Zero-Supression Rule
- Contains (n+1-k)k nodes
- ZDDs are a canonical representation
  - ☐ When this is encountered in CBFS, we are assured of this structure
  - ⇒ CBFS uses (n+1-k)(k+1) nodes after variable $x_{k(n+1)}$

The *front* within H_k

- After variable $x_{k(n+1)+i}$ the ZDD contains (i+1) “branches”
- Main branch corresponds to all $x_{k(n+1) + 1}, \ldots, x_{k(n+1) + i}$ false
- i+1 other branches correspond to one of $x_{k(n+1) + 1}, \ldots, x_{k(n+1) + i}$ true
- Squares correspond to ZDDs of all subsets of a given size
- Can show this structure is correct by induction
- Bound comes from counting nodes in this structure



Analytical vs. Empirical



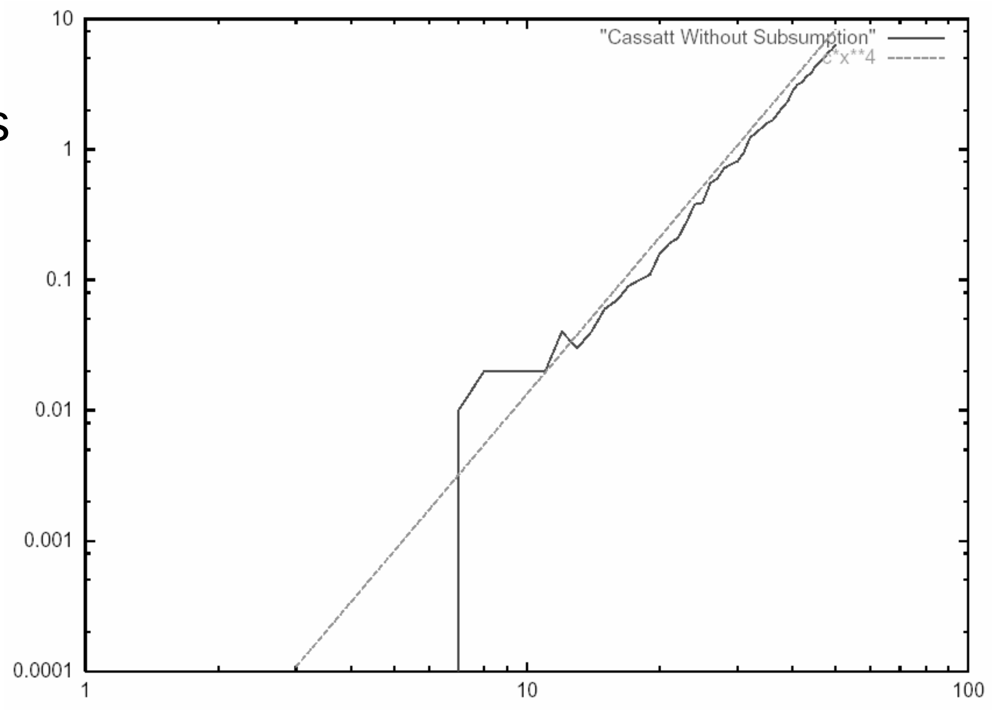
Conclusions and Ongoing Work

- Understanding why CBFS can quickly solve pigeonhole instances depends on recognizing structural invariants within the ZDD
- We hope to understand exactly what proof system is behind CBFS
- We hope to improve the performance of CBFS
  - DLL solvers have been augmented with many ideas (BCP, clause subsumption, etc)
  - These ideas may have an analogue with CBFS giving a performance increase

Thank you!!!

The Utility of Subsumption

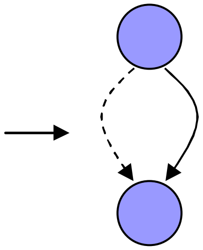
- Cassatt empirically solves pigeonhole instances in O(n^4) without removing subsumptions
- Without subsumption removal
    - ☐ Instead of ZDD’s for all k-element subsets
    - ☐ ZDDs for all (k or greater)-element subsets
        - Still O(n^2)
- To find a bound, need to factor in the additional nodes due to keeping all (k or greater) element subsets



Opportunistic Subsumption Finding

- ‘Subsume’-able sets can occur as the result of Existential Abstraction or Union
  - In pigeonhole instances, this only occurs when we satisfy 1 pigeon clause

  ⇒Smaller sets will have only one less element than larger sets they subsume
- Can detect some subsumptions by recursively searching for nodes of the form



  - Captures subsumptions which occur in CBFS’s solution of pigeonhole instances

Thanks again!!!

Processing a Single Variable

- Given:
  - Assignment of 0 or 1 to a single variable x
- It violates some clauses: $V_{x\leftarrow\{0,1\}}$
  - $V_{x\leftarrow\{0,1\}}$: Clauses which are unit, and this assignment makes the remaining literal false
    - If any clause in $V_{x\leftarrow\{0,1\}}$ is open then the partial truth assignment for that set of open clauses cannot yield satisfiability
  - Remove all such sets of open clauses

    ⇒ Can use ZDD Intersection

Processing a Single Variable

- Given:
  - Assignment of 0 or 1 to a single variable x
- It satisfies some clauses: $S_{x \leftarrow \{0,1\}}$
  - $S_{x \leftarrow \{0,1\}}$: Clauses in which x appears, and the assignment makes the corresponding literal true
    - If any clause in $S_{x \leftarrow \{0,1\}}$ is open, it should no longer be
  - Remove all such clauses $S_{x \leftarrow \{0,1\}}$ from any set

    $\Rightarrow$ ZDD $\exists$Abstraction

Processing a Single Variable

- Given:
  - Assignment of 0 or 1 to a single variable x
- It activates some clauses, $A_{x\leftarrow\{0,1\}}$
  - $A_{x\leftarrow\{0,1\}}$: Clauses in which x is the first literal encountered, and x does not satisfy
    - These clauses are open in any branch of the search now
  - Add these clauses $A_{x\leftarrow\{0,1\}}$ to each set

    ⇒ ZDD Cartesian Product