

Properties of quadratic equations and their application to power system analysis

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Abstract

A number of facts about quadratic algebraic problems and applied Newton–Raphson like methods are presented. The main results are about solution structure, loading trajectories, load flow feasibility boundaries and Newton–Raphson solutions. Various practically important applications of these properties are discussed. Although motivated by power flow problems, the results are valid for any problem described by an algebraic system of quadratic equations. © 2000 Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

Different problems in power system analysis and control can be described by algebraic sets of quadratic equations of the form

$$f(x) = y + g(x) = 0 \quad (1)$$

where $x \in \mathbb{R}_x^n$ is a vector of unknown variables, $y \in \mathbb{R}_y^n$ is a vector of specified parameters, and g is a quadratic function of x .

A classical example is the load flow problem where x consists of nodal voltages, y represents specified nodal parameters such as nodal powers, and g is the nodal power mismatch function. There are two main forms used for the power mismatch equations: the polar and rectangular forms. Both of them have some advantages. The polar form provides significant reduction of computations. For instance, the method, which uses P–Q decomposition of the load flow problem [1], is widely used in practice. The rectangular form of power flow equations can be effectively used as well—see Refs. [2–8]. The most important features of that form are that the power mismatch function can be exactly expressed using linear and second order terms of the

Taylor series, and its Jacobian matrix $J(x) = \partial f/\partial x$ is a linear function of active and reactive components of nodal voltages. Note that highly nonlinear static characteristics of loads can be always presented as quadratic functions by means of Taylor series approximations and introducing additional variables to reduce their nonlinearity.

The load flow problem is certainly not the only example of quadratic algebraic problems met in power system analysis. In fact, solutions of any optimisation problem with linear and quadratic objective function and constraints can be found by solving a quadratic system whose equations are obtained by differentiating the corresponding Lagrange function with respect to unknown variables and Lagrange multipliers. For instance, the problem of finding the shortest distance from a current operating point to the load flow feasibility boundary, which is often considered as the voltage collapse boundary, can be expressed as a quadratic task [9]. Once again, the problem can be represented as a quadratic equations set with the Jacobian matrix $J(x)$ being a linear function of unknown variables.

In any nonlinear problem, there are some important theoretical issues including solution existence, numerical method behaviour and their convergence to particular solutions, location of multiple solutions, configuration of singular boundaries where $\det J(x) = 0$, mutual arrangement of solutions and singular boundaries and solution behaviour in vicinity of singular boundaries. Such studies have already given valuable practical results.

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Studies of the multiple solutions of the load flow problem play a role in determining proximity to voltage collapse [5,10]. In order to obtain multiple load flow solutions, Tamura et al. used a set of quadratic load flow equations and the Newton–Raphson (NR) optimal multiplier method [7]. Iba et al. used Tamura’s approach and some newly discovered convergence peculiarities of the NR method to find a pair of closest multiple solutions [8]. It is observed from experimental results [8] that if a point x comes close to a line connecting a couple of distinct solutions, a further NR iterative process in rectangular form goes along this line. The next observation in Ref. [8] is that, in the vicinity of a singular point, the NR method with the optimal multiplier gives a trajectory which tends to the straight line connecting a pair of closely located but distinct solutions. These features are effectively used in Ref. [8] to locate multiple load flow solutions. The authors asked for a theoretical background of these experimentally discovered properties.

The present paper is written to further explore some fundamental properties of quadratic algebraic problems and use them in a number of practical applications. The main results establish the following properties:

- A variation of x along a straight line through a pair of distinct solutions of the problem $f(x) = 0$ results in variation of the mismatch vector $f(x)$ along a straight line in \mathbb{R}_y^n .
- There is a singular point in the middle of a straight line connecting a pair of distinct solutions x_1, x_2 in \mathbb{R}_x^n [5,6,11 and others].
- A vector co-linear to a straight line connecting a pair of distinct solutions in \mathbb{R}_x^n nullifies the Jacobian matrix $J(x) = \partial f/\partial x$ at the centre point of the line [5,6 and others].
- If x belongs to a straight line connecting a pair of distinct solutions, the NR iterative process goes along that line.
- The maximal number of solutions on any straight line in \mathbb{R}_x^n is two.
- Along a straight line through two distinct solutions x_1, x_2 , the problem can be reduced to a single scalar quadratic equation which locates these solutions.
- If a loading process $y(\beta)$ in \mathbb{R}_y^n reaches a singular point $\det J(x) = 0$, the corresponding trajectory of $x(\beta)$ in \mathbb{R}_x^n tends to the right eigenvector nullifying $J(x)$ at the singular point (except in some special cases).
- At any singular point, there are two merging solutions (except in some special cases).
- For any two points $x_1 \neq x_2$ and the Jacobian matrix $\det J(x_1) \neq 0$, the number and location of singularities of the quadratic problem $f(x) = 0$ on the straight line through x_1, x_2 is defined by real eigenvalues of the matrix $J^{-1}(x_1)J(x_2)$.

Some of the results are already known—see references above. They are assembled here with new results and clearer explanations. Earlier versions of the results presented here are given in Refs. [2,12,13].

2. Quadratic power flow studies

This section deals with some basic properties of the quadratic power flow problems, their solutions, singularities, and the NR method.

2.1. Property 1

2.1.1. Formulation

For a quadratic problem $f(x) = 0$, there is a point of singularity in the centre of a straight line connecting a pair of distinct solutions in \mathbb{R}_x^n , and a vector co-linear to this line nullifies the Jacobian matrix evaluated in the centre point.

2.1.2. Proof

Let x_1, x_2 be two distinct solutions of a quadratic problem $f(x) = 0$. A line connecting these solutions can be defined by

$$x = x_1 + \mu(x_2 - x_1) = x_1 + \mu\Delta x_{21} \quad (2)$$

where μ is a parameter, and $\Delta x_{21} = x_2 - x_1$. Due to quadratic nonlinearity, we have

$$f(x_1) = f(x_2) - J(x_2)\Delta x_{21} + 0.5W(-\Delta x_{21}) \quad (3)$$

$$f(x_2) = f(x_1) + J(x_1)\Delta x_{21} + 0.5W(\Delta x_{21}) \quad (4)$$

where $0.5W(\Delta x_{21})$ is the quadratic term of the Taylor series expansion, Eq. (4). It is clear that $W(-\Delta x) = W(\Delta x)$. From Eqs. (3) and (4),

$$[J(x_1) + J(x_2)]\Delta x_{21} = 0 \quad (5)$$

For a quadratic function $f(x)$, the Jacobian matrix contains elements which are linear functions of x . So, it can be represented as

$$J(x) = \sum_{i=1}^n A_i x_i + J(0) \quad (6)$$

where $A_i, J(0)$ are $(n \times n)$ constant matrices of Jacobian coefficients, $x_i \in x$. Using Eq. (6), the equality (5) can be rewritten as

$$2J(x_0)\Delta x_{21} = 0 \quad (7)$$

where $x_0 = (x_1 + x_2)/2$. As $\Delta x_{21} \neq 0$, the vector Δx_{21} is the right eigenvector corresponding to a zero eigenvalue of the Jacobian matrix. Moreover, for all $x \neq 0$, which are co-linear vectors with respect to Δx_{21} , we get $J(x_0)x = 0$.

2.1.3. Comments

Both the first and second parts were proved in Refs. [2,5,11 and others]. The above proof seems to be more simple and compact.

2.1.4. Examples

Let us take an example of the New England Test System [14]. The system consists of 39 buses, 10 generators, and 18

Table 1
Distinct load flow solutions

Bus	Solution x_1		Solution x_2		Solution x_3	
	V, kV	δ , grad	V, kV	δ , grad	V, kV	δ , grad
1	103.1	-24.9	0.0	-191.2	89.3	-42.4
2	101.7	-18.6	86.9	-11.0	64.1	-41.1
3	95.5	-20.4	92.0	-15.1	3.8	-125.2
4	84.2	-19.6	94.6	-15.7	44.5	-39.2
5	77.0	-16.4	96.6	-14.1	64.0	-29.0
6	78.7	-15.0	97.3	-13.1	67.7	-26.4
7	68.5	-19.2	96.0	-16.3	66.3	-32.7
8	64.1	-20.4	95.9	-17.3	66.5	-34.5
9	0.0	104.0	100.8	-23.4	88.9	-40.2
10	88.2	-12.9	98.9	-10.8	73.5	-25.3
11	84.9	-13.6	98.2	-11.6	71.2	-25.7
12	84.6	-13.9	96.8	-11.7	67.6	-26.7
13	87.9	-14.1	98.2	-11.7	68.9	-27.5
14	87.9	-16.9	96.9	-13.6	59.0	-34.3
15	94.2	-18.7	97.5	-14.5	64.3	-45.0
16	98.6	-17.7	99.4	-13.1	69.5	-45.2
17	98.3	-19.1	97.2	-14.3	52.3	-49.7
18	97.1	-20.1	95.1	-15.1	32.0	-56.4
19	103.5	-12.9	103.8	-8.4	93.6	-39.1
20	98.3	-14.2	98.5	-9.7	92.6	-40.8
21	100.0	-15.1	100.6	-10.6	79.5	-41.2
22	103.4	-10.5	103.7	-6.0	92.5	-35.1
23	102.7	-10.7	103.1	-6.2	91.5	-35.3
24	99.5	-17.5	100.3	-13.0	73.0	-45.1
25	103.5	-17.3	93.9	-12.0	75.6	-45.8
26	102.4	-17.9	98.3	-12.9	73.4	-48.1
27	99.9	-19.7	97.2	-14.9	62.5	-51.7
28	103.6	-14.3	101.5	-9.2	88.8	-43.2
29	104.1	-11.5	102.7	-6.3	94.1	-39.9
30	104.8	-16.2	104.8	-8.3	104.8	-37.4
31	98.2	-0.0	98.2	0.0	98.2	0.0
32	98.3	-4.8	98.3	-3.7	98.3	-15.6
33	99.7	-8.3	99.7	-3.8	99.7	-34.4
34	101.2	-9.1	101.2	-4.6	101.2	-35.5
35	104.9	-5.7	104.9	-1.2	104.9	-29.7
36	106.4	-2.7	106.4	1.7	106.4	-26.52
37	102.8	-10.7	102.8	-4.9	102.8	-37.3
38	102.7	-4.7	102.7	0.5	102.7	-32.7
39	103.0	-28.6	103.0	-27.3	103.0	-42.5

loads. Bus number 31 is chosen as a slack bus. All other generators are represented by means of constant active powers and terminal voltages. Loads have fixed active and reactive demands. We consider three distinct load flow solutions x_1 , x_2 and x_3 given in Table 1. These solutions were selected from a solution set obtained by the method for computing multiple solutions of quadratic algebraic problems given in Section 3. All three points are low voltage solutions. As the points x_1 , x_2 and x_3 do not belong to a straight line, they define a plane in \mathbb{R}_x^n which we call the Δ -plane. The Δ -plane is very convenient for viewing the mutual arrangement of solutions and singularities of a load flow problem [15]. Actually, solutions x_1 , x_2 and x_3 can be shown as points in the plane—see Fig. 1. Straight lines connecting these three solutions belong to the plane—they are drawn as dashed lines. The cut-set of the feasibility

domain boundary where $\det J(x) = 0$ by the Δ -plane is given by dotted curves. To get the cut-set, the X-ray theorem and Δ -plane method were used—see Section 3 and Ref. [15]. In full compliance with Property 1, the singular points marked by small circles lie exactly in the middle of the dashed lines connecting the points x_1 , x_2 and x_3 .

2.2. Property 2

2.2.1. Formulation

For quadratic mismatch functions $f(x)$, a variation of x along a straight line through a pair of distinct solutions of the problem $f(x) = 0$ results in variation of the mismatch vector $f(x)$ along a straight line in \mathbb{R}_y^n .

2.2.2. Proof

Let x be a point on the straight line connecting two distinct solutions x_1 , x_2 described by Eq. (2). For quadratic mismatch functions,

$$f(x) = f(x_1) + \mu J(x_1)\Delta x_{21} + 0.5\mu^2 W(\Delta x_{21}) \quad (8)$$

$$f(x_2) = f(x_1) + J(x_1)\Delta x_{21} + 0.5W(\Delta x_{21}) \quad (9)$$

where $0.5W(\Delta x_{21})$ is the quadratic term of the Taylor series expansion, Eq. (9). At points x_1 , x_2 , we have $f(x_1) = f(x_2) = 0$. So, from Eq. (9),

$$0.5W(\Delta x_{21}) = -J(x_1)\Delta x_{21} \quad (10)$$

Using Eq. (10), Eq. (8) transforms to

$$f(x_1 + \mu\Delta x_{21}) = \mu(1 - \mu)J(x_1)\Delta x_{21} = \beta\Phi \quad (11)$$

where $\beta = \mu(1 - \mu)$, $\Phi = J(x_1)\Delta x_{21}$. Thus the mismatch function $f(x_1 + \mu\Delta x_{21})$ varies along the straight line $\beta\Phi$ in \mathbb{R}_y^n .

2.2.3. Comments

This fact was mentioned in Ref. [5]. An interesting conclusion follows from Properties 1 and 2. Variations of x along a straight line connecting a couple of distinct solutions are actually motions of x along the right eigenvector nullifying $J(x)$ in the middle of the line.

2.2.4. Examples

In the same New England test system and for the same solution set x_1 , x_2 and x_3 , let us consider a map $y = -y_0 - g(x)$ of the Δ -plane in Fig. 1 into the space \mathbb{R}_y^n —see Fig. 2. Point 0 represents the condition $y_0 + g(x_i) = 0$, $x_i = x_1, x_2, x_3$. As the straight lines defined by $x_1(\mu_1) = x_1 + \mu_1\Delta x_{21}$ and $x_2(\mu_2) = x_1 + \mu_2\Delta x_{31}$ are mapped into the distinct straight lines $y_1(\mu_1) = \mu_1(1 - \mu_1)J(x_1)\Delta x_{21}$ and $y_2(\mu_2) = \mu_2(1 - \mu_2)J(x_1)\Delta x_{31}$ which intersect at point 0, the last two lines define a plane in \mathbb{R}_y^n . This plane is referred to as the Δ -plane in \mathbb{R}_y^n . All points which belong to the straight lines $x_1(\mu_1)$ and $x_2(\mu_2)$ are given accurately as the corresponding points on the lines $y_1(\mu_1)$ and $y_2(\mu_2)$ including all singular points with $\det J(x) = 0$. Singular

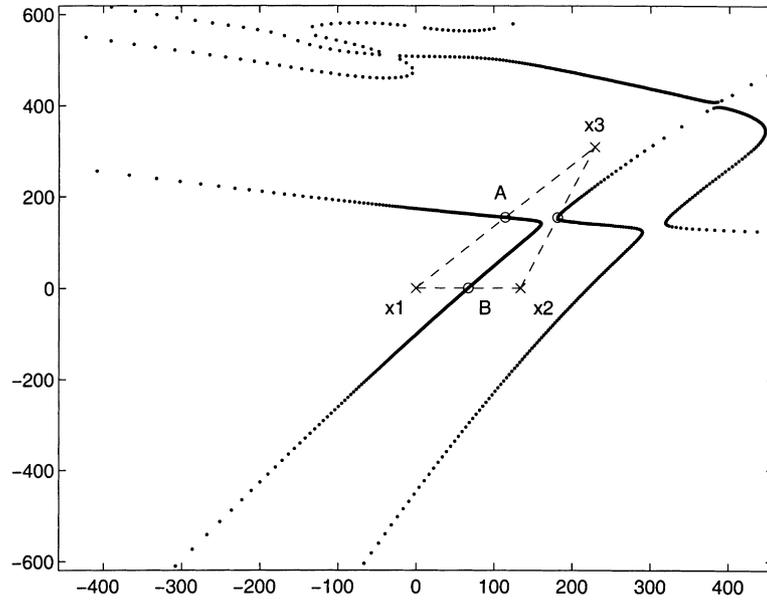


Fig. 1. Δ -plane in \mathbb{R}_x^2 (New England Test System).

points A and B correspond to the points A and B in Fig. 1. They have a specific nature. For example, a loading process $x_1(\mu_1)$ results in a straight motion from point 0 to point A , $\mu_1 < 1/2$. At point A , the maximum loadability point with $\mu_1 = 1/2$ is reached. A further loading along $x_1(\mu_1)$, $\mu_1 > 1/2$, results in backward straight motion from A to 0. All other singular points (dotted curves beyond the straight lines 0– A and 0– B) are obtained as projections of multidimensional singular curves on the Δ -plane in \mathbb{R}_y^2 .

2.3. Property 3

2.3.1. Formulation

If any point x is on a straight line connecting two distinct solutions of the quadratic problem $f(x) = 0$, the Newton–Raphson iterative process with the initial point from x follows this line.

2.3.2. Proof

If any point x^j is on the line connecting two distinct

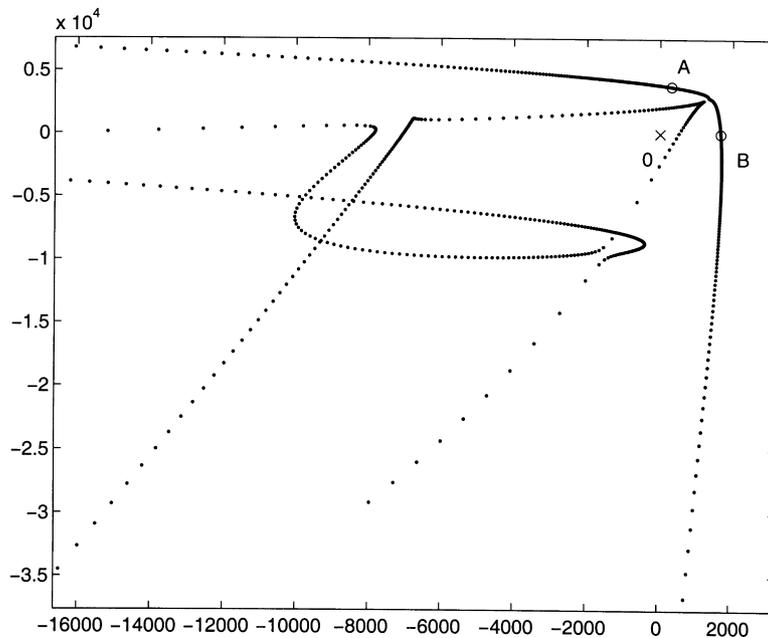


Fig. 2. Δ -plane in \mathbb{R}_y^2 (New England Test System).

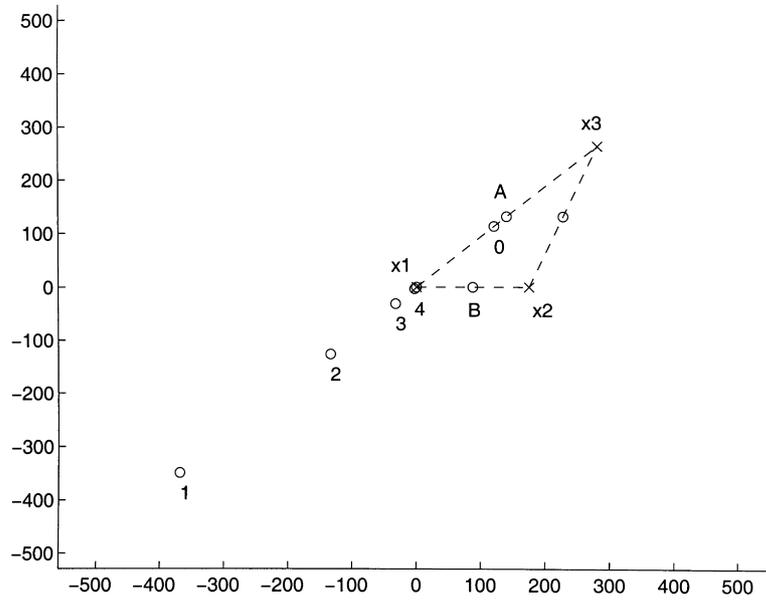


Fig. 3. Newton–Raphson process along the line connecting two distinct solutions (New England Test System).

solutions it can be described in the form (2). The middle point of Δx_{21} is $x_0 = 0.5(x_1 + x_2)$. The quadratic mismatch function can be expressed as

$$f(x^i) = f(x_0) + J(x_0)(x^i - x_0) + 0.5W(x^i - x_0) \quad (12)$$

To express the last term of Eq. (12) in terms of Jacobian matrices, we write

$$f(\Delta x) = f(0) + J(0)\Delta x + 0.5W(\Delta x) \quad (13)$$

$$f(0) = f(\Delta x) - J(\Delta x)\Delta x + 0.5W(\Delta x) \quad (14)$$

By summing of the last two equalities,

$$W(\Delta x) = [J(\Delta x) - J(0)]\Delta x,$$

and so

$$0.5W(x^i - x_0) = 0.5[J(x^i - x_0) - J(0)](x^i - x_0)$$

On the contrary, taking into account the quadratic nonlinearity of $f(x)$ and Eq. (6),

$$J(x^i - x_0) = J(x^i) - J(x_0) + J(0)$$

Therefore, in Eq. (12), we have

$$0.5W(x^i - x_0) = 0.5[J(x^i) - J(x_0)](x^i - x_0)$$

Noting Eq. (7), then $J(x_0)(x^i - x_0) = 0$. From Eq. (12),

$$f(x^i) = f(x_0) + 0.5J(x^i)(x^i - x_0) \quad (15)$$

For the NR method with initial point x^i , we have the following expression for the correction vector Δx^i

$$f(x^i) + J(x^i)\Delta x^i = 0 \quad (16)$$

From Eq. (11),

$$f(x^i) = \mu(1 - \mu)J(x_1)\Delta x_{21} \quad (17)$$

and so

$$J(x_1)\Delta x_{21} = \mu^{-1}(1 - \mu)^{-1}f(x^i), \quad \mu \neq 0, 1 \quad (18)$$

At the point $\mu = 0.5$, we have $x^i = x_0$, and it follows from Eqs. (17) and (18) that

$$f(x_0) = 0.25\mu^{-1}(1 - \mu)^{-1}f(x^i) \quad (19)$$

By the substitution of Eq. (19) into Eq. (15), it follows

$$f(x^i) = 0.25\mu^{-1}(1 - \mu)^{-1}f(x^i) + 0.5J(x^i)(x^i - x_0) \quad (20)$$

Multiplying Eq. (20) by $J^{-1}(x^i)$ and taking into account Eq. (16),

$$\Delta x^i = -2\mu(1 - \mu)[4\mu(1 - \mu) - 1]^{-1}(x^i - x_0) \quad (21)$$

Eq. (21) shows that the NR correcting vector Δx^i belongs to the straight line directed by the vector $(x^i - x_0)$, i.e. the iterative process goes along the line connecting x_1, x_2 .

2.3.3. Comments

This fact was experimentally discovered in Ref. [8], where again the authors asked for some theoretical proof of the phenomena.

2.3.4. Examples

In the same New England test system, let us consider Newton–Raphson iterations starting from the point $x = x_1 + 0.43(x_3 - x_1)$ which lies exactly on the line $(x_1 - x_3)$ —see point 0 in Fig. 3. A numerical simulation has been done, and it has been discovered that the entire NR process goes along the same line on the Δ -plane—see points 1, 2, 3 and 4 in Fig. 3. The initial leap from point 0 to point 1 is explained by the fact that the point 0 is taken rather close to the singular point A.

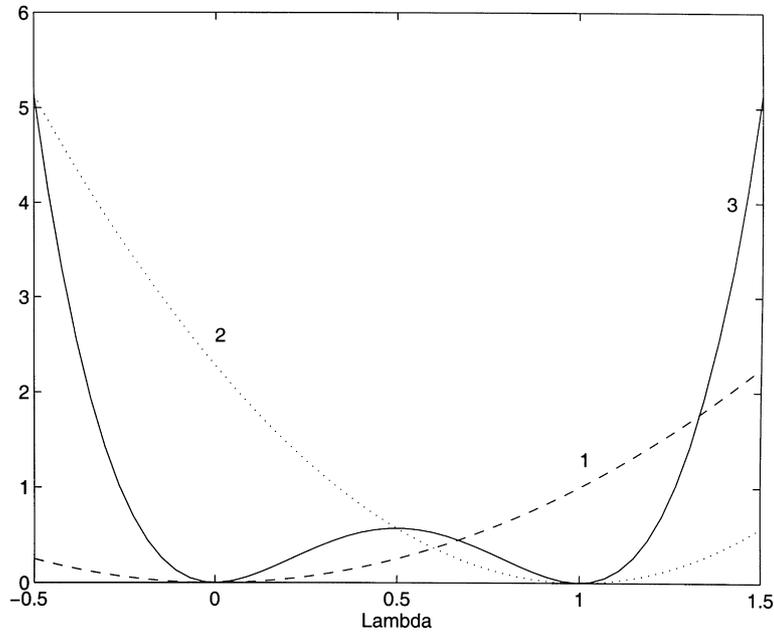


Fig. 4. Two distinct solutions along the connecting line (New England Test System): 1-Curve λ^2 ; 2-Curve $a\lambda^2 + b\lambda + c$; 3-Curve $\phi(x_*, \lambda) = (a\lambda^2 + b\lambda + c)\lambda^2$.

2.4. Property 4

2.4.1. Formulation

The maximum number of solutions of a quadratic equation $f(x) = 0$ on each straight line in the state space \mathbb{R}_∞^n is two.

2.4.2. Proof

Let us take the function

$$\phi(\lambda, x) = f^t(x + \lambda\Delta x)f(x + \lambda\Delta x) \quad (22)$$

For a quadratic mismatch function $f(x)$,

$$f(x + \lambda\Delta x) = f(x) + \lambda J(x)\Delta x + 0.5\lambda^2 W(\Delta x)$$

So,

$$\begin{aligned} \phi(\lambda, x) &= \|f(x) + \lambda J(x)\Delta x + 0.5\lambda^2 W(\Delta x)\|^2 \\ &= \|f(x)\|^2 + \|\lambda J(x)\Delta x\|^2 + \|0.5\lambda^2 W(\Delta x)\|^2 \\ &\quad + 2\lambda f^t(x)J(x)\Delta x + \lambda^3 W^t(x)J(x)\Delta x + \lambda^2 f^t(x)W(\Delta x) \end{aligned}$$

The function $\phi(\lambda, x)$ equals zero if and only if $f(x + \lambda\Delta x) = 0$. At a solution point $x = x_*$, $f(x_*) = 0$, and the function (22) is

$$\begin{aligned} \phi(\lambda, x_*) &= 0.25\lambda^4 \|W(\Delta x)\|^2 + \lambda^3 W^t(\Delta x)J(x_*)\Delta x \\ &\quad + \lambda^2 \|J(x_*)\Delta x\|^2 \\ &= (a\lambda^2 + b\lambda + c)\lambda^2 \end{aligned}$$

where a, b, c are the obvious functions of Δx . For any fixed

direction $\Delta x \neq 0$, $\phi(\lambda, x_*)$ equals zero in the two following cases

- (a) $\lambda = 0$;
- (b) $a\lambda^2 + b\lambda + c = 0$.

The first case gives us the original solution point $x = x_*$. The second case corresponds to solutions $x \neq x_*$ on the straight line directed by Δx . However, as it is clear from Eq. (22), that function (22) cannot be negative. Thus $a\lambda^2 + b\lambda + c \geq 0$, and in case (b) it is possible to have only one additional solution except x_* , but not two or more. So, on the line we get one root $x = x_*$, and we can have only one additional root corresponding to condition (b).

2.4.3. Examples

Fig. 4 shows a typical shape of the scaled functions λ^2 (curve 1), $a\lambda^2 + b\lambda + c$ (curve 2) and $\phi(\lambda, x_*) = (a\lambda^2 + b\lambda + c)\lambda^2$ (curve 3) along a straight line $x_* + \lambda\Delta x$ connecting two distinct solutions in the New England test system. Two zero minima of $\phi(\lambda, x_*)$ at $\lambda = 0$ and $\lambda = 1$ correspond to the first solution x_* and additional solution $x \neq x_*$.

2.5. Property 5

2.5.1. Formulation

For a straight line, connecting two solutions in \mathbb{R}_x^n , the system of quadratic equations can be reduced to a single scalar quadratic equation, which locates these solutions.

2.5.2. Proof

Let x_1, x_2 be unknown distinct solutions of a quadratic

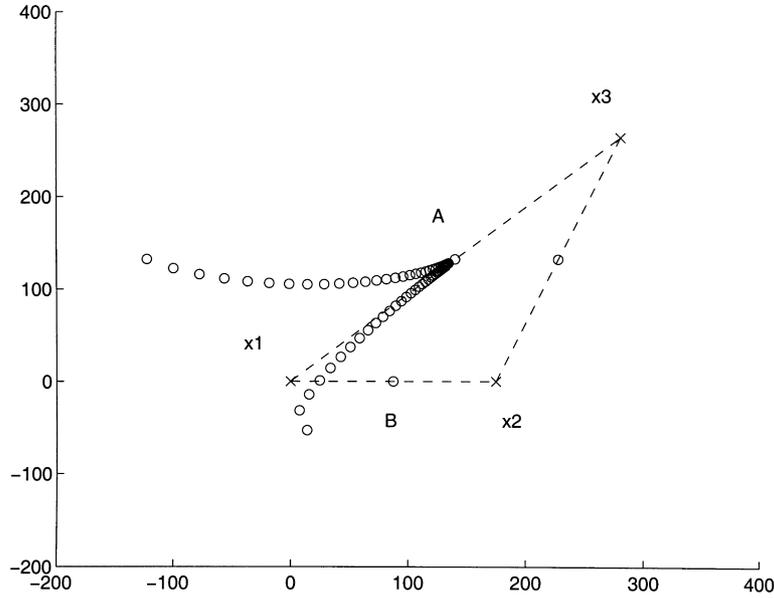


Fig. 5. Two loading trajectories $x(\beta)$ ending at singular point A (New England Test System).

problem $f(x) = 0$. Suppose we have a point x_* and direction Δx_* which define a line connecting the pair of solutions. The mismatch function calculated along this line is

$$F(\alpha) = f(x_* + \alpha \Delta x_*) \quad (23)$$

where α is a scalar parameter. Let us take any fixed value $\alpha = \alpha_*$ and define a constant

$$\Phi_* = F(\alpha_*) \neq 0 \quad (24)$$

Having Eqs. (23) and (24), consider the equation

$$\Phi_*^t F(\alpha) = 0 \quad (25)$$

It follows from Property 1 that Φ_* and $F(\alpha)$ are co-linear vectors, and Eq. (25) is true only if $F(\alpha) = 0$. Using Eq. (24) and the Taylor series expansion

$$F(\alpha) = f(x_*) + \alpha J(x_*) \Delta x_* + 0.5 \alpha^2 W(\Delta x_*)$$

we get the scalar quadratic equation

$$a \alpha^2 + b \alpha + c = 0 \quad (26)$$

where $a = 0.5 \Phi_*^t W(\Delta x_*)$, $b = \Phi_*^t J(x_*) \Delta x_*$, $c = \Phi_*^t f(x_*)$. Eq. (26) has a pair of distinct real roots α_1, α_2 corresponding to x_1, x_2 , and we can define these solutions as $x_1 = x_* + \alpha_1 \Delta x_*$, $x_2 = x_* + \alpha_2 \Delta x_*$.

2.6. Property 6

2.6.1. Formulation

For almost all cases, if any loading process $y(\beta)$ ends at a singular point x_0 of the problem $y(\beta) + g(x) = 0$, where $g(x)$ is a quadratic function of x and β is a scalar loading parameter, there are two distinct solutions x_1, x_2 merging at the singular point, and the trajectories $x_1(\beta), x_2(\beta)$ tend to the

right eigenvector r corresponding to a zero eigenvalue of the Jacobian matrix $J(x_0)$.

2.6.2. Proof

Let a loading process with β variable and

$$y(\beta) + g(x) = 0 \quad (27)$$

end at a singular point x_0 corresponding to $\beta = \beta_0$. At the singular point, $dy = y'_{\beta}(\beta_0) d\beta = Y_0 d\beta$. The implicit function theorem gives

$$J(x_0) dx + dy = J(x_0) dx + Y_0 d\beta = 0 \quad (28)$$

By multiplying Eq. (28) by s^t , where s is the left eigenvector of $J(x_0)$ corresponding to a zero eigenvalue, we get

$$s^t J(x_0) dx + s^t Y_0 d\beta = s^t Y_0 d\beta = 0$$

For the general case of loading, $s^t Y_0 \neq 0$, and, therefore, $d\beta = 0$. This, means that the loading parameter β reaches its extremal value β_0 at the singular point x_0 . Alternatively, when $s^t Y_0 = 0$, the loading trajectory $y(\beta)$ tends to the tangent hyper-plane to the singular margin ($\det J(x) = 0$) of the problem $f(x) = 0$ at the point x_0 . It follows from the fact that s is an orthogonal vector with respect to the singular margin in \mathbb{R}_y^n .

Using Eq. (28), we have

$$J(x_0) dx = 0 \quad (29)$$

So, the increment dx has the same direction as the right eigenvector r of the Jacobian matrix corresponding to its zero eigenvalue, and the last part of Property 6 has been proved.

On the contrary, having Eq. (29),

$$-Y_0 \Delta \beta = J(x_0) \Delta x + \frac{1}{2} W(\Delta x) \xrightarrow{\Delta \beta \rightarrow 0} \frac{1}{2} W(\Delta x)$$

and $W(\Delta x) = W(-\Delta x)$. So, for a small increment $\Delta\beta$ at the point x_0 , we have two increments of x of opposite signs directed along r . Therefore, the first part of Property 6 has been proved.

2.6.3. Comments

The property follows from [16, p. 217]. A similar fact was mentioned in Ref. [17]. The proof presented here was first given in Ref. [9]. It explains the experimental fact obtained in Ref. [8]. It was observed that in the vicinity of a singular point the NR method with the optimal multiplier gives a trajectory which tends to the straight line connecting a pair of closely located load flow solutions x_1, x_2 . Actually, the loading trajectory $y(\beta)$ in Eq. (27) can be represented as the convergence trajectory of the NR method. If it comes close to the singular margin, it tends to the right eigenvector r , which nullifies the Jacobian matrix in the middle point between closely located solutions. Property 3 says that the further iterative process goes along the line directed by the vector (x_1, x_2) , and that the line is co-linear to r . So, Properties 3 and 6 explain these phenomena.

2.6.4. Examples

Fig. 5 shows two curves $x(\beta)$ (circled curves) corresponding to two different loading trajectories $y(\beta)$ ending at singular point A . It is seen that both the trajectories $x(\beta)$ tend to the right eigenvector $r = x_2 - x_1$ which nullifies the Jacobian matrix at point A .

2.7. Property 7 (X-ray theorem)

2.7.1. Formulation

For any two points $x_1 \neq x_2$ and $\det J(x_1) \neq 0$, the number and location of singularities of the quadratic problem $f(x) = 0$ on the straight line through x_1, x_2 is defined by real eigenvalues of the matrix $J^{-1}(x_1)J(x_2)$.

2.7.2. Proof

Let us define the line through x_1, x_2 as Eq. (2). Using Eq. (6), it is easy to show that

$$J[x_1 + \mu(x_2 - x_1)] = (1 - \mu)J(x_1) + \mu J(x_2) \quad (30)$$

As x_1 is a nonsingular point, for $\mu \neq 0$, expression (30) can be written as

$$J(x) = \mu J(x_1)[J^{-1}(x_1)J(x_2) - (\mu - 1)\mu^{-1}I]$$

where I is the identity matrix. Therefore, all singular points on the line, Eq. (2) can be computed as real eigenvalues of the matrix $J^{-1}(x_1)J(x_2)$.

2.7.3. Comments

The X-ray theorem can be effectively used to locate all singularities of the load flow Jacobian matrix along a given ray in the space of dependent variables x . In particular, it forms a basis of the Δ -plane method given in Section 3.

3. Power system applications

This section illustrates how the properties analysed in Section 2 can be used in power system analysis.

3.1. Finding multiple solutions of a quadratic problem

Property 1 allows to find multiple solutions of a quadratic problem $f(x) = 0$, for example, the load flow problem. Note that Eq. (11) can be rewritten as

$$f(x_1 + \Delta x) + (\mu - 1)J(x_1)\Delta x = 0 \quad (31)$$

where x_1 is a known solution, Δx an unknown increment of state variables and μ an unknown scalar parameter. Except the trivial case $\Delta x = 0$, the last equation corresponds to a different solution

$$x_2 = x_1 + \mu^{-1}\Delta x, \quad |\mu| < \infty, \quad \mu \neq 0$$

System (31) has n equations and $n + 1$ unknown variables, so it is necessary to add an additional equation in Eq. (31), for instance,

$$r^t \Delta x - 1 = 0 \quad (32)$$

where r is a nonzero vector. By varying r and substitution of newly discovered solutions instead of x_1 in Eqs. (31) and (32), it is possible to get all solutions of a quadratic problem. Tables 1 contains some of the distinct solutions in the New England test system. The total number of solutions obtained exceeds 70.

3.2. Δ -plane method

The X-ray theorem allows us to develop a new robust method for finding the power system load flow feasibility boundary and, more generally speaking, the singular boundaries of the power flow problem on the Δ -plane defined by any three vectors of dependent variables (nodal voltages). An advantage of the method is that it does not require an iterative solution of nonlinear equations (except the eigenvalue problem). Besides benefits for visualisation and fully flexible observation, the method is a useful tool for topological studies of power system multiple solution structures and stability domains. For instance, any three equilibria (load flow solutions) can be chosen to form the Δ -plane, and the relative positions of these points and bifurcation surfaces will be reflected on the Δ -plane in all details.

3.2.1. Obtaining bifurcation curves on the Δ -plane

A plane in \mathbb{R}_x^n can be defined by any three distinct points x_1, x_2 and x_3 provided the condition

$$x_3 \neq x_1 + \mu(x_2 - x_1) \quad (33)$$

is satisfied for any scalar parameter μ . Condition (33) means that the points x_1, x_2 and x_3 do not belong to a straight line in \mathbb{R}_x^n . Once Eq. (33) is true, the points x_1, x_2 and x_3 form a triangular Δ , which defines a plane in \mathbb{R}_x^n and gives the name for the proposed method. Any point on the Δ -plane

can be expressed by means of scalar parameters γ_1 and γ_2 :

$$x(\gamma_1, \gamma_2) = x_1 + \gamma_1(x_2 - x_1) + \gamma_2(x_3 - x_1) \quad (34)$$

It is clear that $x(0, 0) = x_1$, $x(1, 0) = x_2$ and $x(0, 1) = x_3$. Suppose that at the point x_1 , $\det J(x_1) \neq 0$. Then the following procedure can be used to find out all singularities of $J(x)$ in the Δ -plane. The idea consists in rotation of a vector $x(\gamma_1, \gamma_2) - x_1$ in the Δ -plane, and subsequent computations of all singularities of $J(x)$ on each line defined by this vector. Let θ be the angle, which defines the current orientation of the vector $x(\gamma_1, \gamma_2) - x_1$ with respect to the vector $x_2 - x_1$ in the Δ -plane. The following steps are used:

1. Choose the angle $\theta = 0$.
2. Compute $\gamma_1 = l \cos \theta$ and $\gamma_2 = l \sin \theta$, $l \gg 0$.
3. Define a point $x = x(\gamma_1, \gamma_2)$ as in Eq. (34).
4. Find eigenvalues of the matrix $J^{-1}(x_1)J(x)$.
5. Compute $\mu_j = (1 - \lambda_j)^{-1}$ for all real eigenvalues $\lambda_j \neq 1$ from the previous step.
6. For each value of μ_j , define the corresponding point in the Δ -plane as

$$x_j(\theta) = x_1 + \mu_j[\gamma_1(x_2 - x_1) + \gamma_2(x_3 - x_1)] \quad (35)$$

7. Change $\theta = \theta + \Delta\theta$, where $\Delta\theta$ is an increment, and go to step 2 unless $\theta \geq \pi$.

The set of points $x_j(\theta)$ computed for different θ forms a cut-set of the feasibility domain boundary by the Δ -plane. The procedure does not require an iterative solution except as needed for the eigenvalue problem. The reliable QR technique is recommended to be used in step 4.

3.2.2. Visualisation of the Δ -plane in \mathbb{R}_x^n

Although the bifurcation points $x_j(\theta)$ in Eq. (35) belong to the Δ -plane, they are vectors in the multi-dimensional space \mathbb{R}_x^n . To get a visual representation for them, it is convenient to use a new two-dimensional coordinate system associated with the Δ -plane itself. For this purpose, we use the following oblique-angled coordinate system:

$$\tilde{x}_1 = [0 \ 0]^t \quad (36)$$

$$\tilde{x}_2 = \|x_2 - x_1\| [1 \ 0]^t$$

$$\tilde{x}_3 = \|x_3 - x_1\| [\cos \alpha_x \ \sin \alpha_x]^t$$

where

$$\alpha_x = \arccos \frac{(x_3 - x_1)^t(x_2 - x_1)}{\|x_3 - x_1\| \|x_2 - x_1\|}$$

In the new two-dimensional coordinate system, the expression for computing the power flow singular points

becomes the following:

$$\tilde{x}_j(\theta) = \mu_j(\gamma_1 \tilde{x}_2 + \gamma_2 \tilde{x}_3) \quad (37)$$

3.2.3. Δ -plane in \mathbb{R}_y^n

Consider a particular case when x_1 , x_2 and x_3 are distinct solutions of the load flow problem. It follows from Property 2 in Section 2 that these points cannot lie on a straight line in \mathbb{R}_y^n . Property 3 says that, from $y = -g(x)$, the straight lines

$$x(\gamma_1, 0) = x_1 + \gamma_1(x_2 - x_1) \quad (38)$$

$$x(0, \gamma_2) = x_1 + \gamma_2(x_3 - x_1) \quad (39)$$

in \mathbb{R}_x^n are mapped into the straight lines

$$y_1 = y(\gamma_1, 0) = \gamma_1(\gamma_1 - 1)J(x_1)(x_2 - x_1) \quad (40)$$

$$y_2 = y(0, \gamma_2) = \gamma_2(\gamma_2 - 1)J(x_1)(x_3 - x_1) \quad (41)$$

in \mathbb{R}_y^n , respectively. The last two lines pass through a common point $-g(x_1)$, and they define a plane which we call the Δ -plane in \mathbb{R}_y^n .

3.2.4. Visualisation of the Δ -plane in \mathbb{R}_y^n

From $y = -g(x)$, all singular points computed along the lines (38) and (39) are mapped into points of the lines (40) and (41), respectively. So, they lie on the Δ -plane in the space \mathbb{R}_y^n . The rest of the points $y_j(\theta)$ mapped from the Δ -plane in \mathbb{R}_x^n do not normally belong to the Δ -plane in \mathbb{R}_y^n ; the Δ -plane in \mathbb{R}_x^n is mapped by $y = -g(x)$ into a surface which is not a plane. The only thing, which we can do here, is to find out their projections $\tilde{y}_j(\theta)$ on the Δ -plane in the space \mathbb{R}_y^n . The projection $\tilde{y}_j(\theta)$ can be found by the following way. Let

$$\tilde{y}_j(\theta) = \beta_1 \tilde{y}_1 + \beta_2 \tilde{y}_2 \quad (42)$$

where β_1 and β_2 are parameters, and \tilde{y}_1 , \tilde{y}_2 are defined as

$$\tilde{y}_1 = \|y_1\| [1 \ 0]^t \quad (43)$$

$$\tilde{y}_2 = \|y_2\| [\cos \alpha_y \ \sin \alpha_y]^t \quad (44)$$

where $\alpha_y = \arccos \|y_2\|^{-1} \|y_1\|^{-1} (y_2)^t (y_1)$. The coefficients β_1 and β_2 can be found by solving the linear equation

$$\begin{pmatrix} \|y_1\|^2 & y_1^t y_2 \\ y_1^t y_2 & \|y_2\|^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} y_1^t y_j(\theta) \\ y_2^t y_j(\theta) \end{pmatrix} \quad (45)$$

They are then used in Eq. (42) to get singular points in the two-dimensional oblique-angled coordinate system $(\tilde{y}_1, \tilde{y}_2)$.

As the singular points belong to a nonlinear surface, and we use their projections on the Δ -plane, the resulting plot reflects a qualitative shape of the singular boundary in \mathbb{R}_y^n . Nevertheless, all singular points along the lines AB and AC are defined accurately.

3.2.5. Examples

The proposed method has been tested for the above New England Test System. We consider three distinct load flow solutions x_1 , x_2 and x_3 given in Table 1. Solutions x_1 , x_2 and x_3 define a Δ -plane in \mathbb{R}_x^n shown in Fig. 1. The plane is oriented in such a way that x_1 corresponds to the zero point (0,0), and the vector $x_2 - x_1$ is directed along the horizontal axis. Solution x_1 is taken as a fixed point for the plot: all straight lines, along which the singularities are determined, belong to the Δ -plane and pass through x_1 . The cut-set of the feasibility domain boundary by the Δ -plane is shown by dotted curves (solid parts of the curves are formed by covering of close dots).

A detailed consideration of the feasibility boundary topology is not the main purpose of this paper. Nevertheless, let us make some observations. Firstly, note that the singular points marked by small circles lie exactly in the middle of the dashed lines connecting the points x_1 , x_2 and x_3 . This fact is explained in a number of works—see Refs. [13,18], for example. Secondly, the singular curves in Fig. 1 have an open shape. This appears to say that, by certain unrestricted variations of independent parameters y in \mathbb{R}_y^n , voltages x in \mathbb{R}_x^n may be indefinitely increased without reaching the feasibility boundary. In more realistic situations the limitations for y , for example, equality constraints for generator terminal voltages and zero power injections in empty buses, must be taken into account.

Fig. 2 shows the corresponding Δ -plane in \mathbb{R}_y^n . Point 0 represents the operating condition $y_0 + g(x) = 0$ of the system. The singular boundary is plotted by using the map $y = -y_0 - g(x)$. Points A and B correspond to the points A and B in Fig. 1. All singular points which belong to the straight lines defined by $0-A$ and $0-B$ are given accurately. All other points are obtained as projections of multidimensional singular curves on the Δ -plane in \mathbb{R}_y^n .

4. Conclusions

In this paper, we have carefully studied the properties of quadratic equations with respect to solution structure, loading trajectories, load flow feasibility boundaries and Newton–Raphson solutions. These results were shown to clarify techniques used in power system analysis. This is explored for finding multiple solutions and bifurcation curves.

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