Optimal Detection of SAR/IR Targets Lying Across a Clutter Boundary

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OUTLINE

1. Problem Statement and Background
2. Detection Background
3. A Poisson Imaging Example
4. Gaussian Image Model for Target Detection
5. SAR/IR Examples

Problem Statement and Background

Principal Objective: To develop weak target detection/classification algorithms which are robust to target and clutter variability yet maintain highest possible discrimination capability.

Methods: Extraction of invariant statistics followed by CFAR detection and feature classification (*Exact Robustness*).

Key concept: Decision directed image processing

Familiar systems: (STARLOS, MONITOR, SAIP, MSTAR)

- Downsampling/local averaging/non-coherent averaging (FOA):

$$x_1, x_2, x_3, \ldots \rightarrow \boxed{\begin{array}{c}\textsf{Decimate}\\ \textsf{Average}\end{array}} \rightarrow \; x_1, x_5, x_9, \ldots$$

- Binary masking and morphological filtering (Secondary Detection):

$$x_1, x_2, x_3, \ldots \rightarrow \boxed{\textsf{Quantize}} \rightarrow \; b_1, b_2, b_3, \ldots$$

- Template matching (MSE Classification)

$$x_1, x_2, x_3, \ldots \Rightarrow \boxed{\textsf{MF Bank}} \Rightarrow \; \hat{s}_1, \ldots, \hat{s}_P,$$

$\Rightarrow$ Standard processing adequate for bright targets in homogeneous clutter ($0.8 < P_d < 0.95$).

$\Rightarrow$ Inadequate for weak targets in inhomogeneous clutter ($P_d < 0.5$).

Detection Background

Ingredients:

- Statistical model for observations $\mathbf{X}$

$$\mathbf{X} \sim f(X;\ \theta)$$

- Parameter space $\Theta = \Theta_0 \cup \Theta_1$
- two hypotheses:

$$H_0 \ :\ \mathbf{X} \sim f(X;\ \theta_0), \qquad \theta_0 \in \Theta_0$$

$$H_1 \ :\ \mathbf{X} \sim f(X;\ \theta_1), \qquad \theta_1 \in \Theta_1$$

Detection Objective:

Find test function $\phi(\mathbf{X}) \in [0,1]$ to maximize family of *ROC* curves $(P_{FA}, P_D)_\Theta$

$$\begin{aligned} P_{FA} &= E_\theta[\phi], \quad \theta \in \Theta_0 \qquad (\mathit{Level}) \\ P_D &= E_\theta[\phi], \quad \theta \in \Theta_1 \qquad (\mathit{Power}) \end{aligned}$$

Desirable Properties of a Detector

- *Uniformly Most Powerful* (UMP): ROC curves for ϕ dominate those of any other test
- *Unbiasedness*: P_D always at least as great as P_{FA}
- *Min Max Robustness*: worst case ROC curves for ϕ dominate those of any other test
- *Exact Robustness*:
  - *Null Robustness* (CFAR): P_{FA} of ϕ is constant over Θ_0
  - *Optimality Robustness*: ϕ remains optimal under model variations

Likelihood Ratio Test (LRT):

$$\frac{f_{\theta_1}(\mathbf{X})}{f_{\theta_0}(\mathbf{X})} \mathop{\gtrless}_{H_0}^{H_1} \eta$$

η selected such that $P_{FA} = \alpha$, $\alpha \in [0, 1]$

Note: LRT is MP test for $\Theta_0 = \{\theta_0\}$ and $\Theta_1 = \{\theta_1\}$

Difficulties:

- LRT is not usually CFAR
- LRT is not usually unbiased
- LRT is not usually UMP

Generalized Likelihood Ratio Test (GLRT)

$$\frac{\max_{\theta\in\Theta_1} f_\theta(\mathbf{X})}{\max_{\theta\in\Theta_0} f_\theta(\mathbf{X})} \overset{H_1}{\underset{H_0}{\gtrless}} \eta$$

η selected such that $P_{FA} = \alpha$, $\alpha \in [0, 1]$

1. CLRT is a function of MLEs $\hat{\theta}_0$ and $\hat{\theta}_1$
2. GLRT is not asymptotically CFAR or UMP unless MLEs are consistent
3. GLRT optimization may be intractible
4. GLRT performance can be very poor (not even unbiased) in finite sample regime

$\Rightarrow$ GLRT decision region may overly condense data $\mathbf{X}$ eroding H_0 vs. H_1 discrimination ability

Poisson imaging example

Observe: Planar Poisson process $\{\underline{x}_i,\ i=1,\ldots,N\}$

Intensity: $\lambda(\underline{x})$, $\underline{x}=(x_1,x_2)$.

$$H_0 \ :\ \lambda(\underline{x}) = \lambda_0(\underline{x})$$

$$H_1 \ :\ \lambda(\underline{x}) = \lambda_\rho(\underline{x}), \quad \rho \in \mathcal{R}.$$

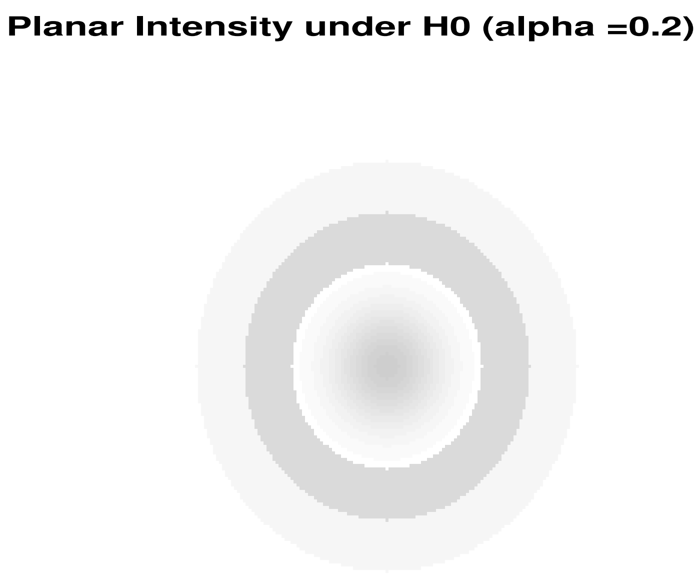
where

$$\tilde{\lambda}_0(\underline{x}) = \begin{cases} \frac{\alpha}{a_3}, & \|\underline{x}\| \in (r_2, r_3] \\ \frac{1-2\alpha}{a_2}, & \|\underline{x}\| \in (r_1, r_2] \\ \frac{\alpha}{a_1}, & \|\underline{x}\| \in [0, r_1] \end{cases}.$$

$$\tilde{\lambda}_\rho(\underline{x}) = \begin{cases} \frac{\rho(\underline{x})}{a_3}, & \|\underline{x}\| \in (r_2, r_3] \\ 0, & \|\underline{x}\| \in (r_1, r_2] \\ \frac{1-1/a_3}{a_1}, & \|\underline{x}\| \in [0, r_1] \end{cases}.$$

and $\rho = \rho(\underline{x}) \in \mathcal{R}$ has known radial marginals

$$\int_0^{2\pi} \rho(r\cos(\psi), r\sin(\psi))d\psi = \frac{2\pi}{a_3}, \quad r_2 < r \le r_3.$$



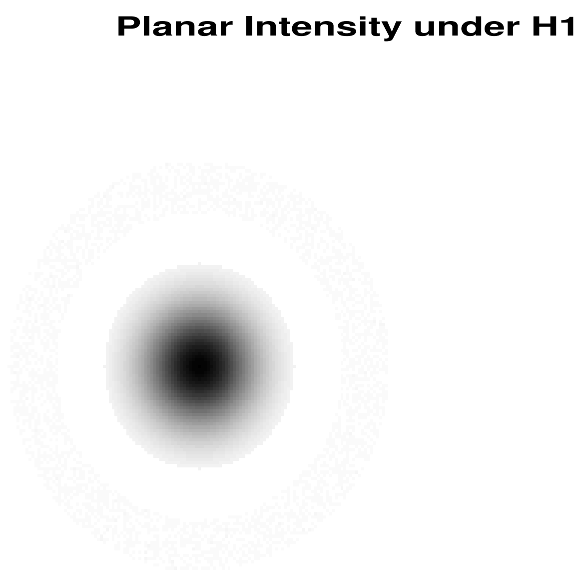


Figure 1: Planar intensities $\lambda(\underline{x})$ supported on nested rings at known radii. Under H_0 the intensity, shown for parameter $\alpha = 0.2$, is completely known. Under H_1 the outer ring has an unknown intensity whose integral over any fixed radius contour is equal to a known constant.

GLRT:

$$\max_{\rho \in \mathcal{R}} \frac{f_\rho(\underline{x})}{f_0(\underline{x})} = \left[\frac{1 - 1/a_3}{\alpha}\right]^{N_1} \max_{\rho \in \mathcal{R}} \prod_{i=1}^{N_3} \frac{\rho(\underline{x}_i)}{\alpha} \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \gamma$$

GLR critical region: decide H_1 if $N_3 > 0$.

$$P_{FA} = 1 - P(N_3 = 0|H_0) = 1 - \exp\left(-\alpha\Lambda\right)$$

$$P_D = 1 - P(N_3 = 0|H_1) = 1 - \exp\left(-\frac{\Lambda}{a_3}\right).$$

Alternative: notice that model is *invariant* with respect to rotation of coordinate system about O:

$$\boxed{\underline{x} = re^{j\psi}}$$

$\Rightarrow$ Data reduction: throw away phase angle ψ of $\underline{x}$

$$H_0 \ : \ \lambda(r) = \lambda_0(r)$$

$$H_1 \ : \ \lambda(r) = \lambda_1(r)$$

$$\hat{\lambda}_0(r) = \begin{cases} \alpha \, \frac{2\pi r}{a_3}, & r \in (r_2, r_3] \\ 1 - 2\alpha \, \frac{2\pi r}{a_2}, & r \in (r_1, r_2] \\ \alpha \, \frac{2\pi r}{a_1} & r \in [0, r_1] \end{cases}$$

$$\hat{\lambda}_1(r) = \begin{cases} \frac{1}{a_3} \, \frac{2\pi r}{a_3}, & r \in (r_2, r_3] \\ 0, & r \in (r_1, r_2] \\ 1 - 1/a_3 \, \frac{2\pi r}{a_1}, & r \in [0, r_1] \end{cases}$$

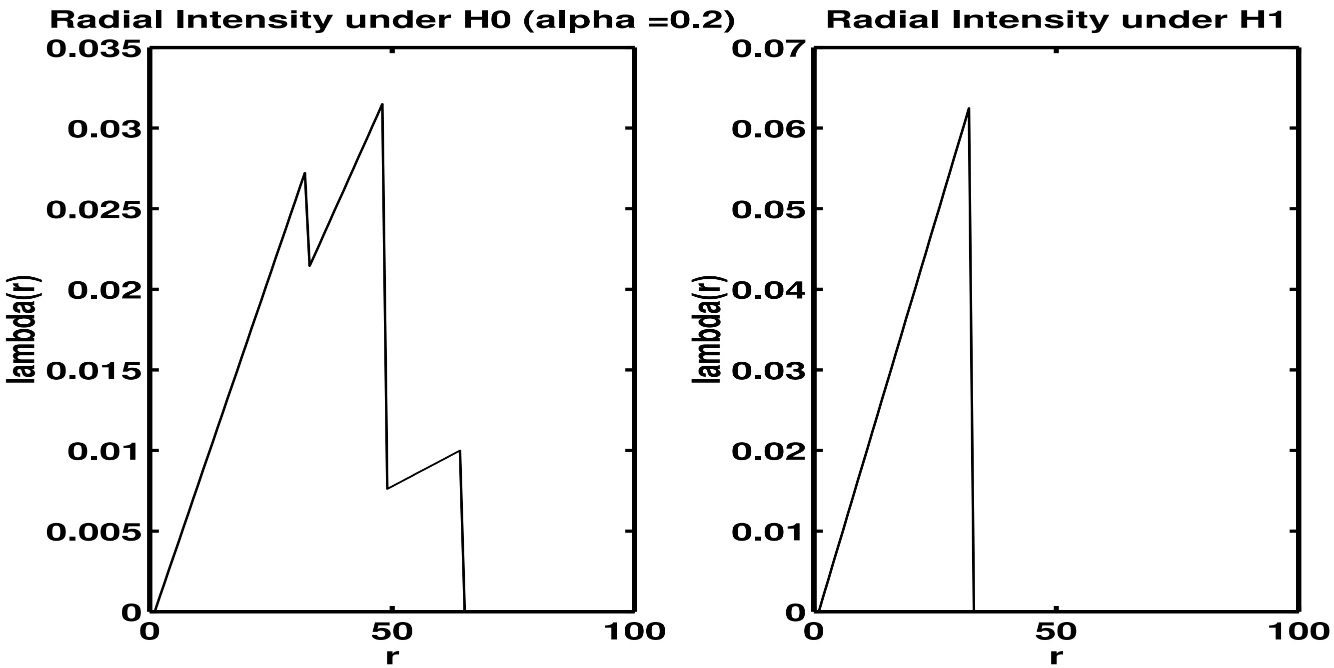


Figure 2: Marginal radial intensities $\lambda(r)$ under H_0 and H_1 derived from the planar intensities in Fig. 1.

Invariant LRT:

$$\frac{f_1(r)}{f_0(r)} = \left[\frac{1-1/a_3}{\alpha}\right]^{N_1} \left[\frac{1/a_3}{\alpha}\right]^{N_3} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \gamma,$$

or equivalently (assuming $a_3 \geq 1/\alpha$ and $\alpha \in [0, \frac{1}{2}]$)

$$N_1 + N_3\, c \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \eta$$

where $c = \log\left[\frac{1}{a_3}\,\frac{1}{\alpha}\right] / \log\left[(1-1/a_3)\,\frac{1}{\alpha}\right]$.

$\boxed{\text{No dependence on } \rho \Rightarrow \text{UMPI test}}$

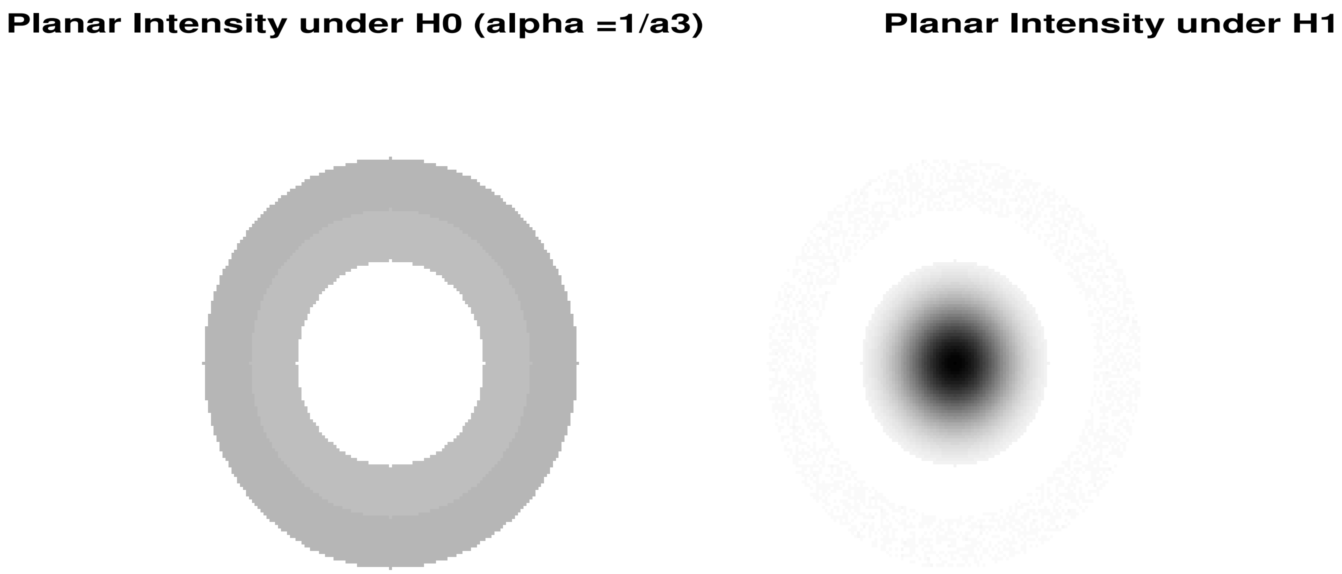


Figure 3: Same H_0 and H_1 planar intensities as in Fig. 1 except that $\alpha = 1/a_3$ where a_3 is the area of the outer ring.

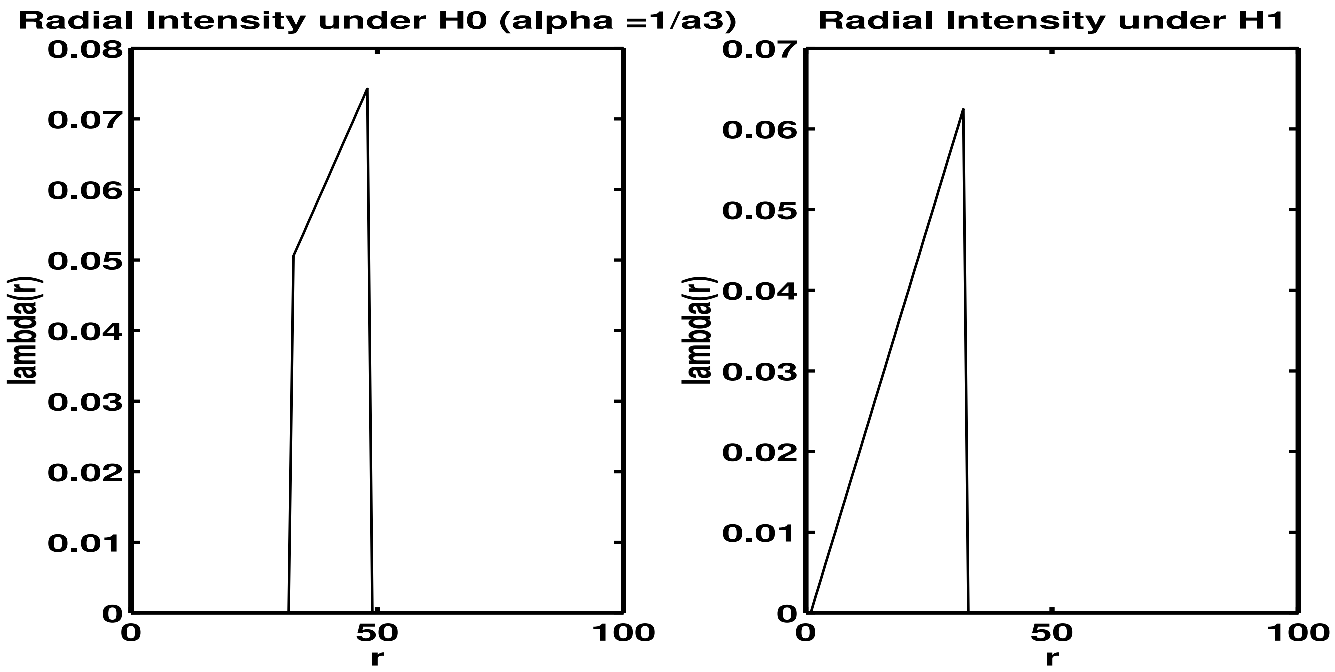


Figure 4: Marginal radial intensities $\lambda(r)$ under H_0 and H_1 derived from the planar intensities in Fig. 2.

Special Case $\alpha = 1/a_3$

MPI critical region: decide H_1 if $N_1 > 0$

The false alarm probability and probability of detection of the MPRI test are

$$P_{FA} = 1 - P(N_1 = 0|H_0) = 1 - \exp(-\alpha\Lambda)$$

$$P_D = 1 - P(N_1 = 0|H_1) = 1 - \exp(-(1 - 1/a_3)\Lambda).$$

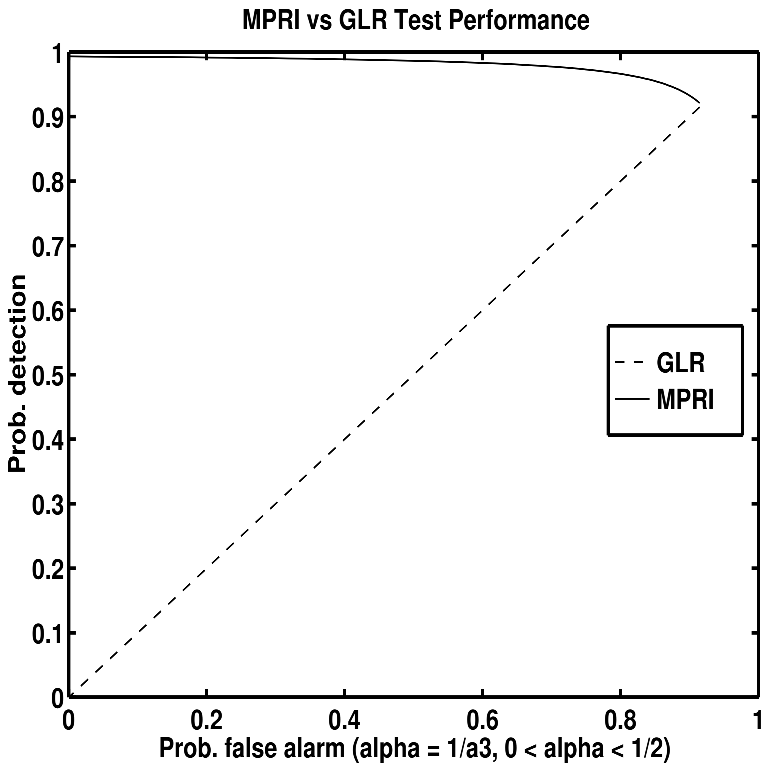


Figure 5: Probability of detection as a function of $P_F = 1 - e^{-\alpha\Lambda}$ associated with GLR and MPRI tests of H_0 vs. H_1 for the case of $a_3 = 1/\alpha$ shown Fig. 2.

STATISTICAL REDUCTION VIA SUFFICIENCY

Sufficient Statistic: $\mathbf{T} = T(\mathbf{X})$ is a sufficient statistic for testing between H_0 and H_1 if for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$ the likelihood ratio depends on $\mathbf{X}$ only through $T(\mathbf{X})$:

$$\Lambda(\mathbf{X}, \theta_0, \theta_1) \stackrel{\text{def}}{=} \frac{f(\mathbf{X}; \theta_1)}{f(\mathbf{X}; \theta_0)} = g(T(\mathbf{X}), \theta_1, \theta_0)$$

Note:

- The sets $\{\mathbf{X} : T(\mathbf{X}) = t\}_t$ are **orbits** of $\mathbf{X}$ induced by likelihood ratio and specify constant contours of $\Lambda(\mathbf{X})$.
- The values t of the sufficient statistic index the orbits and preserve information needed to discriminate between H_0 vs H_1.

STATISTICAL REDUCTION VIA INVARIANCE

Main Idea: find *maximal invariant* statistic $\mathbf{Z} = Z(\mathbf{X})$ which indexes the orbits of the set function

$$\tilde{\Lambda}(X) \stackrel{\text{def}}{=} \left\{ \frac{f(\mathbf{X};\theta_1)}{f(\mathbf{X};\theta_0)} : \theta_0 \in \Theta_0, \theta_1 \in \Theta_1 \right\}$$

This is possible when Θ_0 and Θ_1 have topological group structure.

Let $\mathcal{G}$ be a group of transformations $g : \mathcal{X} \to \mathcal{X}$ acting on $\mathbf{X}$.

Assume that for each $\theta \in \Theta$ there exists a unique $\overline{\theta} = \overline{g}(\theta)$ such that

$$f_\theta(g(\mathbf{X})) = f_{\overline{\theta}}(\mathbf{X})$$

$\overline{g}$ is called the induced group action on Θ.

Note:

- P.d.f. of $\mathbf{Z}$ only depends on θ through reduced set of parameters $\delta = \delta(\theta)$ (induced maximal invariant)

- MP Test constructed on $\mathbf{Z}$ is MP Invariant Test (MPI)

- use of $\mathbf{Z}$ to perform detection gives **CFAR** test if $\delta(\Theta_0) =$ constant.

General Gaussian Image Model

$$\mathbf{X} = [\underline{X}_1, \ldots, \underline{X}_n]$$

$\underline{X}_k = m \times 1$ lexicographic ordered k-th sub-image

$$\mathbf{X} = \mathbf{S}\, \mathrm{a}\, \mathbf{B} + \mathbf{N}$$

- $\mathbf{S} = [\underline{s}_1, \ldots, \underline{s}_p] =$ an $m \times p$ matrix of target signatures
- $\mathrm{a} = \mathrm{diag}(\underline{a}) =$ a $p \times p$ diagonal matrix
- $\mathbf{B}$ a $p \times n$ matrix
- $\mathbf{N}$ – matrix of i.i.d. multivariate Elliptical random variables: $\mathrm{vec}\{\mathbf{N}\} \sim \mathcal{N}_{mn}\left(\underline{0}, \mathbf{R} \otimes \mathbf{I}_n\right)$

Example: Known target, unknown homogenous region:

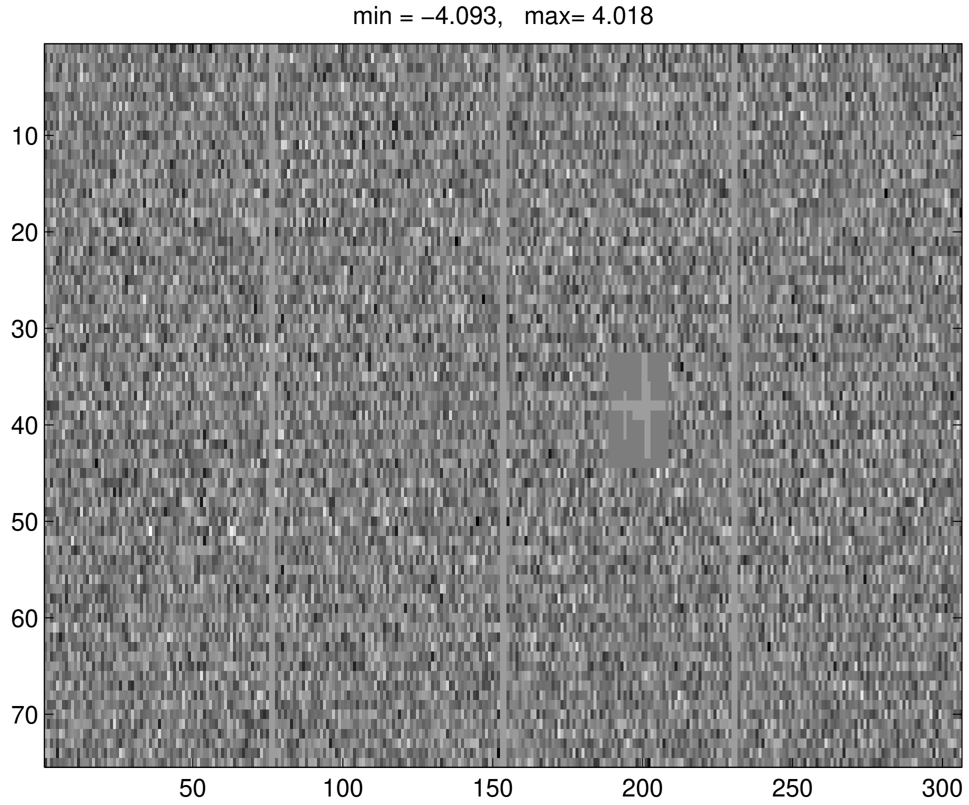


Figure 6: Single target in unknown clutter

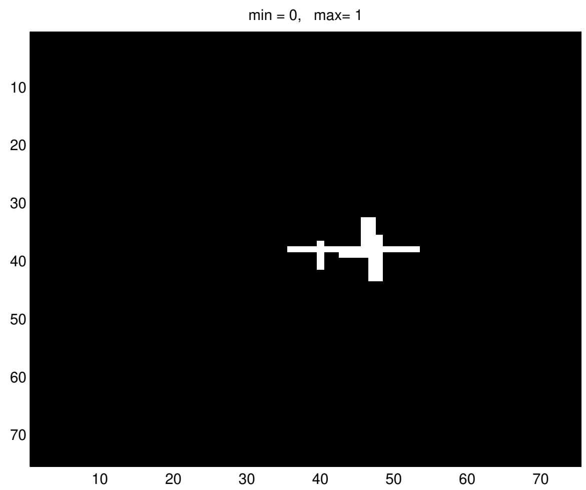


Figure 7: Single target alone

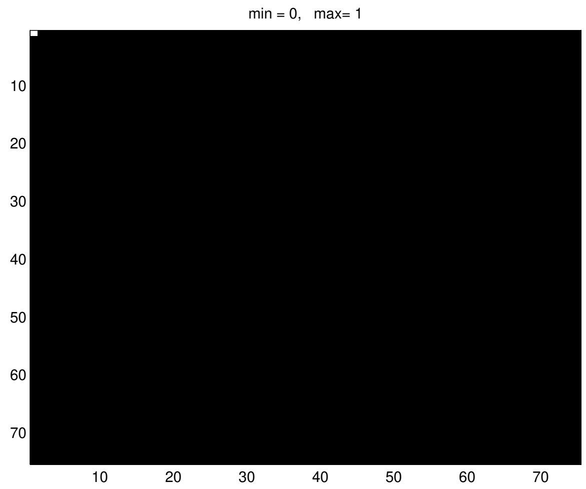


Figure 8: Transformed Single target

$\Rightarrow$ canonical form for $\mathbf{X} = a\underline{s} \cdot \underline{b}^T + \mathbf{N}$:

$$\tilde{\mathbf{X}} = a \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot [1, 0 \ldots, 0] + \mathbf{N}$$

$$\Theta_0 = \{a = 0, \mathbf{R} > 0\}, \qquad \Theta_1 = \{a \neq 0, \mathbf{R} > 0\}$$

Least restricted group of transformations leaving decision problem invariant: $\mathcal{G}$ has group action: $g(\tilde{\mathbf{X}}) = \mathbf{F}\tilde{\mathbf{X}}\mathbf{H}$,

- $\mathbf{F}$ invertible $m \times m$ of form:

$$\mathbf{F} = \begin{bmatrix} \beta_1 & \underline{\beta}^T \\ \underline{0} & \mathbf{M} \end{bmatrix}$$

- $\mathbf{H}$ unitary $n \times n$ of form:

$$\mathbf{H} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbf{U} \end{bmatrix}$$

Note:

1. $g(\tilde{\mathbf{X}})$ remains multivariate Normal
2. $\overline{g}(a, \mathbf{R}) = \{\beta_1 a, \mathbf{F}\mathbf{R}\mathbf{F}^T\} \Rightarrow \overline{g}(\Theta_0) = \Theta_0$ and $\overline{g}(\Theta_1) = \Theta_1$.

Let

$$\tilde{\mathbf{X}} = \left[\underline{\tilde{x}}_1, \tilde{\mathbf{X}}_2\right] = \begin{bmatrix} \tilde{x}_{11} & \underline{\tilde{x}}_{12} \\ \underline{\tilde{x}}_{21} & \tilde{\mathbf{X}}_{22} \end{bmatrix}$$

<u>**Maximal Invariant**</u> is two dimensional:

$$z_1(\tilde{\mathbf{X}}) = \underline{\tilde{x}}_1^T [\tilde{\mathbf{X}}_2 \tilde{\mathbf{X}}_2^T]^{-1} \underline{x}_1$$

$$z_2(\tilde{\mathbf{X}}) = \underline{\tilde{x}}_{21}^T [\tilde{\mathbf{X}}_{22} \tilde{\mathbf{X}}_{22}^T]^{-1} \underline{\tilde{x}}_{21} \quad \text{multiple correlation factor}$$

Spatial prediction interpretation of maximal invariant

Using partioned matrix inverse identities, can show that z_1, z_2 equivalent to:

$$\begin{aligned}
z_1^{'}(\mathbf{X}) &= \frac{\left|\tilde{x}_{11} - \underline{\tilde{x}}_{12}\tilde{\mathbf{X}}_{22}^T[\tilde{\mathbf{X}}_{22}\tilde{\mathbf{X}}_{22}^T]^{-1}\underline{\tilde{x}}_{21}\right|^2}{\underline{\tilde{x}}_{12}^T\left[\mathbf{I} - \tilde{\mathbf{X}}_{22}^T[\tilde{\mathbf{X}}_{22}\tilde{\mathbf{X}}_{22}^T]^{-1}\tilde{\mathbf{X}}_{22}\right]\underline{\tilde{x}}_{12}} \\
&= \frac{\left|\tilde{x}_{11} - \hat{\tilde{x}}_{11|21}\right|^2}{\|\underline{\tilde{x}}_{12} - \underline{\hat{\tilde{x}}}_{12|22}\|^2} \qquad \text{(predicted TCR)} \\
z_2(\tilde{\mathbf{X}}) &= \underline{\tilde{x}}_{21}^T[\tilde{\mathbf{X}}_{22}\tilde{\mathbf{X}}_{22}^T]^{-1}\underline{\tilde{x}}_{21}
\end{aligned}$$

sample (clutter) correlation between primary and secondaries

Induced Maximal Invariant: $a^2\ \underline{s}^T \mathbf{R}^{-1} \underline{s}$

Given invariant data $z(\tilde{\mathbf{X}})$ we have equivalent hypotheses:

$$\Theta_0 = \{\delta = 0\}, \qquad \Theta_1 = \{\delta > 0\}$$

$\Rightarrow$ Any test based on $z(\tilde{\mathbf{X}})$ will be CFAR

Density function of $z_1(\tilde{\mathbf{X}}), z_2(\tilde{\mathbf{X}})$ is bivariate Fisher-F and no UMP exists. However:

- LBI test (UMP for small $\|\underline{s}\|$) exists [Kariya&Sinha].
- LBI test is superior to GLRT for low SNR.

Single Target Unknown Spatial Position

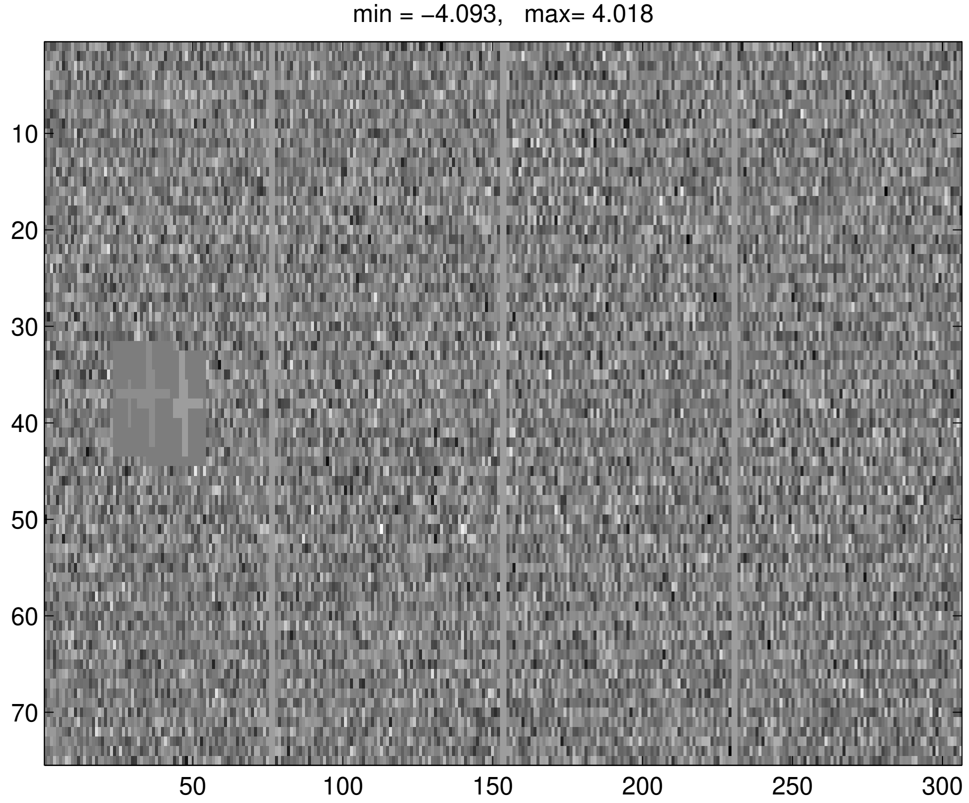


Figure 9: Target at two possible locations in unknown clutter

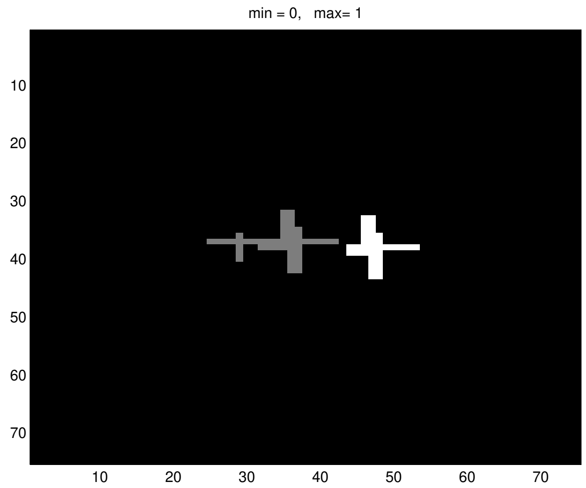


Figure 10: Target alone

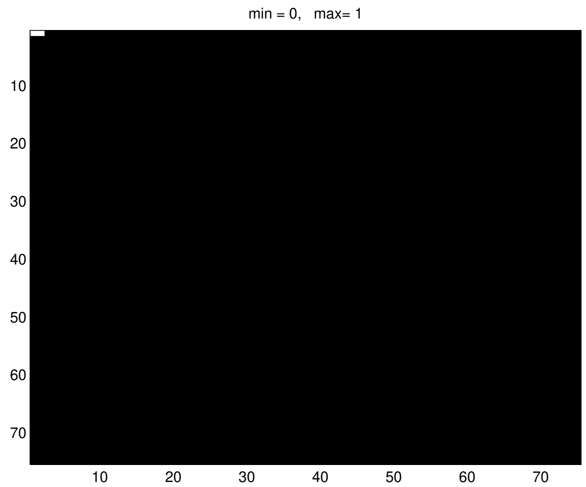


Figure 11: Transformed Target Class

Transformed Model

$$\tilde{\mathbf{X}} = a\, d_k\, \underline{\epsilon}_k\, \underline{e}_1^T + \mathbf{N}$$

Maximal Invariant: Rank ordered spatial correlations

$$z_k(\tilde{\mathbf{X}}) = [\tilde{x}_{1k}, \underline{\tilde{x}}_{21}^T] \begin{bmatrix} \underline{\varepsilon}_k^T \tilde{\mathbf{X}}_{12} \tilde{\mathbf{X}}_{12}^T \underline{\varepsilon}_k & \underline{\varepsilon}_k^T \tilde{\mathbf{X}}_{12} \tilde{\mathbf{X}}_{22}^T \\ \tilde{\mathbf{X}}_{22} \tilde{\mathbf{X}}_{12}^T \underline{\varepsilon}_k & \tilde{\mathbf{X}}_{22} \tilde{\mathbf{X}}_{22}^T \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_{1k} \\ \underline{\tilde{x}}_{21} \end{bmatrix}$$

and

$$z(\tilde{\mathbf{X}}) = \underline{\tilde{x}}_{21}^T [\tilde{\mathbf{X}}_{22} \tilde{\mathbf{X}}_{22}^T]^{-1} \underline{\tilde{x}}_{21}$$

Target at unknown rotation in clutter:

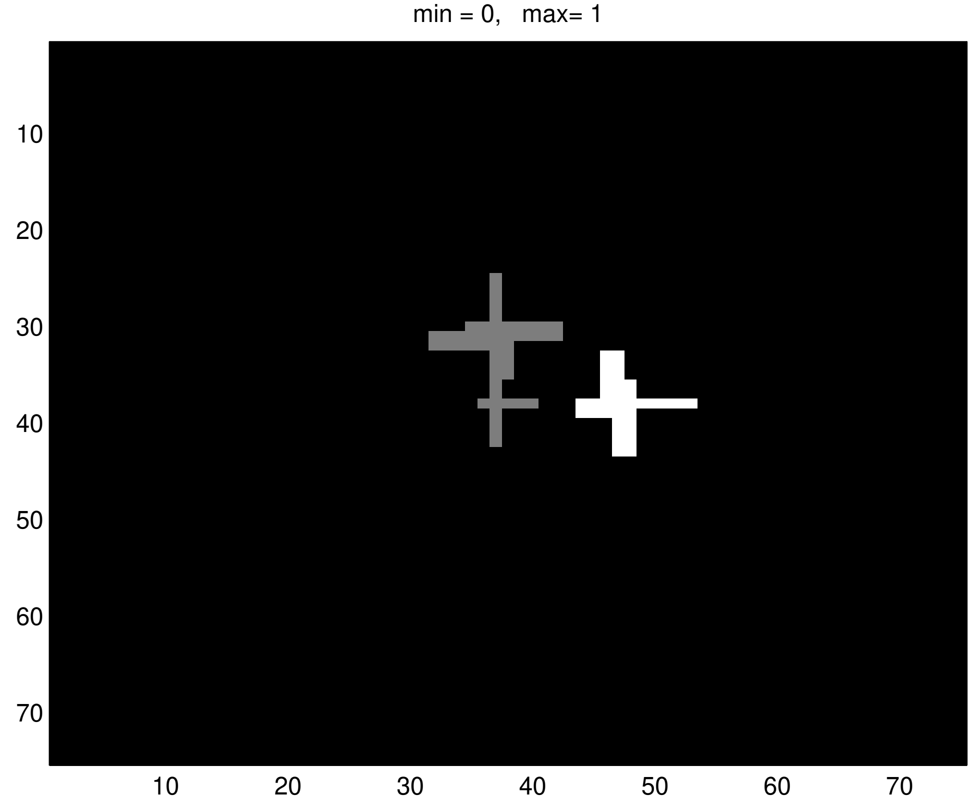


Figure 12: Target at two possible rotations in unknown clutter

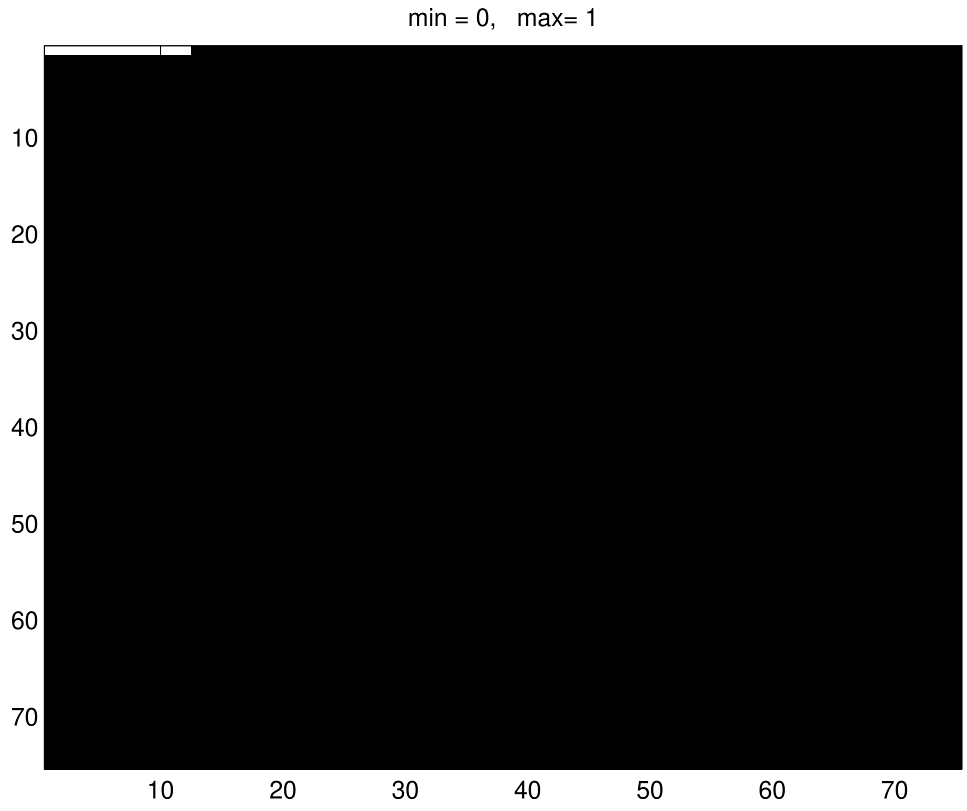


Figure 13: Transformed target class

Two unknown inhomogenous clutter regions:

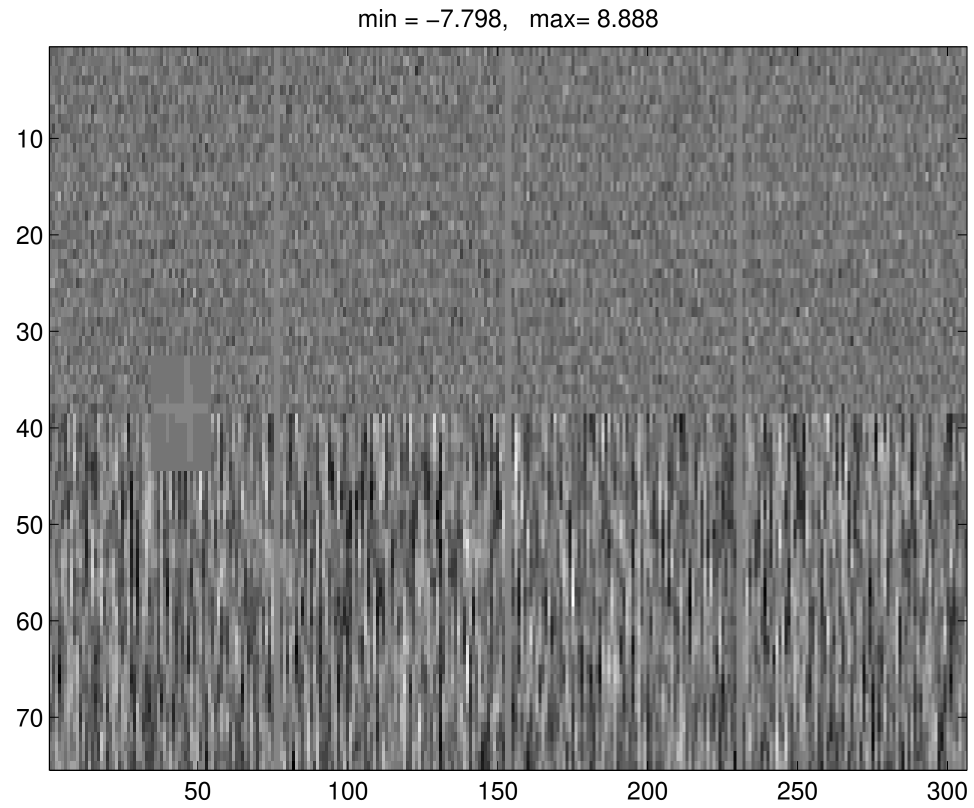


Figure 14: Known deep hide target on clutter boundary

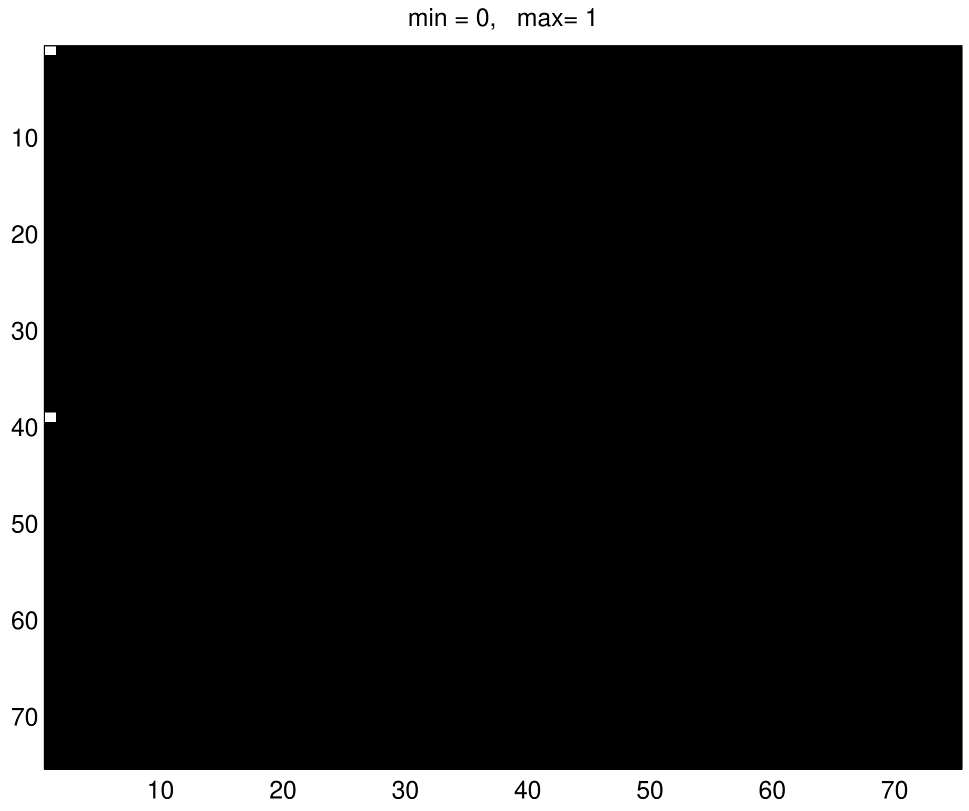


Figure 15: Transformed Target Class

Two unknown inhomogenous clutter regions:

$\Rightarrow$ canonical form for $\mathbf{X} = a\underline{s} \cdot \underline{b}^T + \mathbf{N}$:

$$\tilde{\mathbf{X}} = a \begin{bmatrix} \|s_A\| \\ \underline{0} \\ \|s_B\| \\ \underline{0} \end{bmatrix} \cdot [1, 0 \ldots, 0] + \mathbf{N}$$

$$\Theta_0 = \{a = 0, \mathbf{R}_A > 0, \mathbf{R}_B > 0\}, \qquad \Theta_1 = \{a \neq 0, \mathbf{R}_A > 0, \mathbf{R}_B > 0\}$$

Least restricted group of transformations leaving decision problem invariant:

$\mathcal{G}$ has group action: $g(\tilde{\mathbf{X}}) = \mathbf{F}\tilde{\mathbf{X}}\mathbf{H}$,

- $\mathbf{F}$ invertible $m \times m$ of form:

$$\mathbf{F} = \gamma \begin{bmatrix} \begin{matrix} 1 & \underline{b}_A^T \\ \underline{0} & \mathbf{\Gamma}_A \end{matrix} & \mathrm{O} \\ \mathrm{O} & \begin{matrix} 1 & \underline{b}_B^T \\ \underline{0} & \mathbf{\Gamma}_B \end{matrix} \end{bmatrix}$$

- $\mathbf{H}$ unitary $n \times n$ of form:

$$\mathbf{H} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbf{U} \end{bmatrix}$$

Let

$$\tilde{\mathbf{X}} = \begin{bmatrix} \tilde{\underline{x}}_{A1} & \tilde{\mathbf{X}}_{A2} \\ \tilde{\underline{x}}_{B1} & \tilde{\mathbf{X}}_{B2} \end{bmatrix} \qquad = \begin{bmatrix} \tilde{x}_{A11} & \tilde{\underline{x}}_{A12} \\ \tilde{\underline{x}}_{A21} & \tilde{\mathbf{X}}_{A22} \\ x_{B11} & \tilde{\underline{x}}_{B12} \\ \tilde{\underline{x}}_{B21} & \tilde{\mathbf{X}}_{B22} \end{bmatrix}$$

<u>**Maximal Invariant**</u> is five dimensional:

$$z_1(\tilde{\mathbf{X}}) = \frac{\left|\tilde{x}_{A11} - \hat{\tilde{x}}_{A11|A21}\right|^2}{\|\underline{\tilde{x}}_{A12} - \hat{\underline{\tilde{x}}}_{A12|A22}\|^2}$$

$$z_2(\tilde{\mathbf{X}}) = \underline{\tilde{x}}_{A21}^T[\tilde{\mathbf{X}}_{A22}\tilde{\mathbf{X}}_{A22}^T]^{-1}\underline{\tilde{x}}_{A21}$$

$$z_3(\tilde{\mathbf{X}}) = \frac{\left|\tilde{x}_{B11} - \hat{\tilde{x}}_{B11|B21}\right|^2}{\|\underline{\tilde{x}}_{B12} - \hat{\underline{\tilde{x}}}_{B12|B22}\|^2}$$

$$z_4(\tilde{\mathbf{X}}) = \underline{\tilde{x}}_{B21}^T[\tilde{\mathbf{X}}_{B22}\tilde{\mathbf{X}}_{B22}^T]^{-1}\underline{\tilde{x}}_{B21}$$

$$z_5(\tilde{\mathbf{X}}) = \frac{\tilde{x}_{A11} - \hat{\tilde{x}}_{A11|A21}}{\tilde{x}_{B11} - \hat{\tilde{x}}_{B11|B21}}$$

Future Work

1. Evaluate robustness of CFAR test to boundary specification errors
2. Prior information on one or more clutter types
3. Prior information on spatial articulation of target
4. Random field target models
5. Sensitivity to elliptical clutter distribution assumptions
6. Invariant moment matrices for target classification