Antenna Arrays in Wireless Communications

by

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I dedicate this work to the whole wide world.

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Introduction

Wireless communications have been gaining popularity because of better antenna technologies, lower costs, easier deployment of wireless systems, greater flexibility, better reliability and need for mobile communications. In some cases, like in very remote areas, wireless connections may be the only option.

Eventhough the popularity of mobile wireless systems is a more recent phenomenon fixed-wireless systems have a long history. Point-to-point microwave connections have long been used for voice and data communications, generally in backhaul networks operated by phone companies, cable TV companies, utilities, railways, paging companies and government agencies, and will continue to be an important part of the communications infrastructure. Improvements in technology have allowed higher frequencies and thus smaller antennas to be used resulting in lower costs and easier-to-deploy systems.

Another reason for the popularity of wireless systems is that offlate consumers demand for data rates has been insatiable. Wireline models have topped of at a rate of 56Kbps and people have been looking for ISDN and DSL connections. Companies with T1 connections of 1.54Mbps have found the connections inadequate and are turning to T3 fiber connections. But, due to very expensive deployment of fiber connections companies have been turning to fixed wireless links.

This has resulted in wireless communications being found in a host of applications ranging over fixed microwave links, wireless LANs, data over cellular networks, wireless WANs, satellite links, digital dispatch networks, one-way and two-way paging networks, diffuse infrared, laser-based communications, keyless car entry, the Global Positioning System, mobile communications, indoor-radio and more.

There is such a wide variety of research in antennas that in future [27] we can expect "a hand-held terminal the size of a wristwatch capable of steering beams toward a satellite. The system would also consist of many radiating elements fabricated by microstrip technology, each with its own phase-shifting network, power amplifier, and so on with other required processors manufactured by the microwave monolithic integrated circuits technology."

One challenge in wireless systems not present in wireline systems is the issue of fading. Fading arises due to the possibility of multiple paths from the transmitter to the receiver with distructive combination at the receiver output. There are many models describing fading in wireless channels. The classic ones being rayleigh and rician flat fading models. rayleigh and rician models are typically for narrowband signals and don't include the doppler shift induced due to the motion of the transmitter or the receiver. For other emerging models see [18].

In wireless systems, there are three different ways to combat fading; 1) Frequency diversity 2) Time diversity and 3) spatial diversity. Frequency diversity makes use of the fact that multipath structure for different frequencies is different. This fact can be exploited to mitigate the effect of fading. But, the positive effects of frequency diversity are limited due to bandwidth limitations. Wireless communications uses radio spectrum, a finite resource. This limits the number of wireless users and the amount of spectrum available to any user at any moment in time. Time diversity makes use of the fact that fading for different time intervals is different. By using channel coding the effect of bad fading intervals can be mitigated by good fading intervals. However, due to delay constraints time diversity can't be really exploited.

The third way to do it is to exploit spatial diversity using multiple antennas either separated in space or differently polarized [21, 22, 47]. Different antennas have different multipath characteristics or different fading characterisitics and this can be used to generate a stronger signal. Spatial diversity techniques don't have the drawbacks associated with time diversity and frequency diversity techniques though spatial diversity does involve deployment of multiple antennas at the transmitter and the receiver which is not always feasible.

In this thesis, we will concentrate on spatial diversity offered with multiple antennas. Spatial diversity, receive (multiple antennas in the receiver) and transmit (multiple antennas at the transmitter), helps in improving system performance by [27]

1. Improving spectrum efficiency: Using multiple antennas we can accomodate more than one user in a given spectral bandwidth.

2. Extending range coverage: Multiple antennas can be used to direct the energy of a signal in a given direction and hence minimize unnecessary transmission of signal energy.

3. Tracking multiple mobiles: The outputs of antennas can be combined in different ways to isolate signals from each and every mobile.

4. Increasing channel reuse

5. Reducing power usage: By directing the energy in a certain direction and in-

creasing range coverage lesser energy can be used to reach a user at a given distance.

6. Generating multiple access: Appropriately combining the outputs of the antennas to selectively provide access to users.
7. Increasing channel capacity: Improving spectral efficiency allows more than one user to operate in a cell.
8. Reducing co-channel interference
9. Combating fading
10. increasing information channel capacity: Multiple antennas have been used to increase the maximum acheival data rates.

Traditionally, all the gains listed above have been realized by explicitly directing the receive or transmit antenna array to point in specific directions. This process is called beamforming. For receive antennas beamforming can be achieved electronically by appropriately weighting the antenna outputs and combining them to make the antenna response to certain directions more sensitive than others. Most of the research on antenna arrays for beamforming has mostly been on receivers. Transmit beamforming is different and requires different algorithms and hardware [28].

Different kinds of beamforming at the recieve antenna array currently in use are based on array processing algorithms such as signal copy, direction finding and signal separation algorithms [28] that include conventional beamforming, Null steering beamforming, Optimal beamforming, Beam-Space Processing, Blind Beamforming, Optimum Combining and Maximal Ratio combining [28, 41, 57, 2, 58, 53, 71]. Some of the beamformers require a reference signal and use adaptive algorithms like LMS to get to the optimal signal [28, 42, 69, 70].

Adaptive algorithms can also be used for the problem of adaptive beamforming where it is required to track multiple users in motion or to track varying channel conditions. More popular ones [28] are the LMS Algorithm, Constant Modulus Algorithm and the Recursive Least Squares algorithm. The algorithm of interest in this work is the LMS Algorithm.

Another research topic in the field of beamforming that has generated sufficient interest involves investigating the effect of calibration errors in direction finding and signal copy problems [44, 52, 63, 62, 43, 73, 23]. An array with Gaussian calibration errors operating in a non-fading environment has the same model as a rician fading channel. Thus the work done in this thesis can be easily translated to the case of array calibration errors.

Beamforming at the receiver is a way of exploiting receive diversity. Most of the research in the literature concentrates on this kind of diversity. Exploiting transmit diversity usually involved [65] using the channel state information obtained via feedback for reassigning energy at different antennas via waterpouring, linear processing of signals to spread the information across transmit antennas and using channel codes and transmitting the codes using different antennas in an orthogonal manner.

The multiple transmit antennas were probably first used to send multiple copies of a signal over orthogonal time or frequency slices. This of course incurs a bandwidth expansion factor equal to the number of antennas. A transmit diversity technique without bandwidth expansion was first suggested by Wittenben [72]. Wittenben's diversity technique of sending time-delayed copies of a common input signal over the multiple antennas was also independently discovered by Seshadri and Winters [60] and by Weerackody [68]. An information theoretic approach to transmit diversity schemes has been undertaken by Narula [3, 51].

Recently, people have realized that explicit beamforming may not be the most optimal way to increase data rates. Foschini with his BLAST project showed that multipaths are not as harmuful as thought out to be and in fact can be exploited to increase capacity. This has given rise to the concept of space-time codes [1, 65, 66, 35, 34, 39, 38, 8]. Space-time coding is a coding technique that is designed for use with multiple transmit antennas. The codes are designed in such a way to induce spatial and temporal correlations into signals that can be exploited at the receiver. Space-time codes are simply a different way of looking at space-time processing of signals before transmission [1].

Design of space-time codes has taken many forms. Tarokh et. al. [65, 66] have taken the approach of designing space-time codes for both rayleigh and rician fading channels that maximizes a distance criterion. The distance criterion was derived from an upper bound on probability of decoding error. There have been code designs where the receiver has no knowledge about the channel. Hero and Marzetta [34] design space-time codes with a design criterion of maximizing the cut-off rate for multiple-antenna rayleigh fading channel. Hochwald et. al [35, 9] propose a design based on signal structures that asymptotically achieve capacity in the non-coherent case for rayleigh fading channels. Hughes [39, 38] considered the design of space-time based on the concept of Group codes. The codes can be viewed as an extended version of phase shift keying for the case of multiple antenna communications. In [38] the author independently proposed a scheme similar to that proposed by Hochwald and Marzetta in [35].

The research topics this disseration concentrates on are the LMS algorithm and channel capacity of multiple antennas in the presence of rician fading. We will elaborate more on the research topics in the following sections.

1.1 Partial Update LMS Algorithms

The LMS algorithm is a popular algorithm for adaptation of weights in the field of adaptive beamforming using antenna arrays or for channel equalization to combat intersymbol interference. This has application in many areas including interference cancellation, space time modulation and coding, signal copy in surveillance and wireless communications. Although there are algorithms with faster convergence rates like RLS, LMS is really popular because of ease of implementation and low computational costs.

One of the variants of LMS existing in literature is the Partial Update LMS Algorithm. Partial updating of the LMS adaptive filter has been proposed to reduce computational costs [46, 29, 11]. In this era of mobile computing and communications, such implementations are also attractive for reducing power consumption. However, theoretical performance predictions on convergence rate and steady state tracking error are more difficult to derive than for standard full update LMS. Accurate theoretical predictions are important as it has been observed that the standard LMS conditions on the step size parameter fail to ensure convergence of the partial update algorithm.

Two of the partial update algorithms prevalent in the literature have been described in [14]. They are referred to as the "Periodic LMS algorithm" and the "Sequential LMS algorithm". To reduce computation by a factor of P, the Periodic LMS algorithm (P-LMS) updates all the filter coefficients every P^{th} iteration instead of every iteration. The Sequential LMS (S-LMS) algorithm updates only a fraction of coefficients every iteration.

Another variant referred to as "Max Partial Update LMS algorithm" has been proposed in [12, 13] and [5]. In this algorithm, the subset of coefficients to be updated is dependent on the input signal. The subset is so chosen as to maximize the reduction in the mean squared error. The input signals multiplying each coefficient are ordered according to their magnitude and the coefficients corresponding to the largest $\frac{1}{P}$ of input signals are chosen for update in an iteration. Some analysis of this algorithm has been done in [13] for the special case of one coefficient per iteration but, analysis for more general cases still needs to be completed.

1.2 Multiple-Antenna Capacity

The paper by Foschini et. al. [21, 22] showed that a significant gain in capacity can be achieved by using multiple antennas in the presence of rician fading. Foschini and Telatar showed ([64]) that with perfect channel knowledge at the receiver, for high SNR a capacity gain of $\min(M, N)$ bits/second/Hz, where M is the number of antennas at the transmitter and N is the number of antennas at the receiver, can be achieved with every 3 dB increase in SNR. Channel knowledge at the receiver however requires that the time between different fades be sufficiently large to enable the receiver to learn the channel. This might not be true in the case of fast mobile receivers and large numbers of transmit antennas.

Following Foschini [21], there have been many papers written on the subject of calculating capacity for a multiple antenna channel [16, 24, 25, 47, 17, 26, 50, 31, 30, 54, 48]. The most notable of these is the work done by Marzetta and Hochwald [48]. There also have been attempts to evaluate the achievable rate regions for the multiple antenna channel in terms of cut-off rate [34] and error exponents [4].

Marzetta and Hochwald [48] considered the case when neither the receiver nor the transmitter has any knowledge of the fading coefficients where the fading coefficients remain constant for T symbol periods and instantaneously change to new independent realizations every T symbol periods. They established that to achieve capacity it is sufficient to use $M = T$ antennas at the transmitter and the capacity achieving signal matrix consists of a product of two independent matrices, a $T \times T$ isotropically random unitary matrix and a $T \times M$ real nonnegative diagonal matrix. Hence, it is sufficient to optimize over the density of a smaller parameter set of size $\min\{M, T\}$ instead of the original one of size $T \cdot M$.

Zheng and Tse [74] derived explicit capacity results for the case of high SNR in the case of no channel knowledge. They showed that the number of degrees of freedom for non-coherent communication is $M^*(1 - M^*/T)$ where $M^* = \min\{M, N, T/2\}$ as opposed to $\min\{M, N\}$ in the case of coherent communications.

The literature cited above has limited its attention to rayleigh fading channel models for computing capacity of multiple-antenna wireless links. However, rayleigh fading models are inadequate in describing the gamut of fading channels one comes across in practice. Another popular model used in the literature to fill this gap is the rician fading channel. rician fading components traditionally have been modeled as independent Gaussian components with a deterministic non-zero mean [56, 65, 49, 15, 19, 59]. Farrokhi et. al. [19] used this model to analyze the capacity of a MIMO channel with a specular component. They assume that the specular component is deterministic and unchanging and unknown to the transmitter but, known to the receiver. They also assume that the receiver has complete knowledge about the fading coefficients (i.e. has knowledge about the rayleigh component as well). They work with the premise that since the transmitter has no knowledge about the specular component the signaling scheme has to be designed to guarantee a given rate irrespective of the value of the deterministic specular component. They conclude that the signal matrix has to be composed of independent circular Gaussian random variables of mean 0 and equal variance to maximize the rate.

1.3 Organization of the Dissertation and Significant Contributions

In this work, we have made significant contributions in the field of LMS algorithm for adaptive arrays and in evaluating shannon capacity for multiple antennas in the presence of rician fading. The contributions in both fields will find a lot of application in multiple-antenna wireless communications.

1. In chapter II we analyze the Sequential PU-LMS for stability and come up with more stringent conditions on stability. We validate the analysis via experimental simulations.

    - **Contributions**: Rigorously proved the stability of the algorithm for stationary signals without restrictive assumptions and analyzed the algorithm for cyclo-stationary signals which led to an understanding of the reason behind the algorithm's instability of sequential LMS algorithm. This understanding led to the design of a more stable algorithm.

2. chapter III contains the description of a new Partial Update Algorithm called the Stochastic Partial Update LMS where the coefficients to be updated in an iteration are chosen randomly. We derive conditions for stability and also analyze the algorithm for performance. We demonstrate the effectiveness via examples.

    - **Contributions**: Designed a new Partial Update algorithm with much better convergence properties than those of existing Partial Update LMS algorithms for the case of non-stationary signals and similar performance for the case of stationary signals. Also, analyzed the algorithm for different scenarios including stationary signals, deterministic signals and generic signals.

3. In chapter IV, we use a non-traditional model where the specular component is also modeled as random but, with an isotropically uniform density [48]. With this model the concept of channel capacity is clearly defined. We also derive a lower bound to capacity.

   - **Contributions**: Proposed a new tractable model for analysis enabling characterization of capacity achieving signals and also derived a very useful lower bound to channel capacity which is also applicable to the case of rayleigh fading.

4. In chapter V, we use a slight variation of the traditional and well-established model where the specular component is modeled as deterministic and non-changing. The variation is that we assume the transmitter has no knowledge about the specular mean. In this case, the concept of channel capacity is not defined and we have to maximize the worst possible rate available for communication over the ensemble of values the unknown specular component can take.

   - **Contributions**: Proposed a tractable formulation of the problem and derived capacity expressions, lower bound to capacity and characterized the properties of capacity achieving signals.

5. In chapter VI, we use the traditional and well-established model where the specular component is modeled as deterministic and non-changing. We assume that both the transmitter and the receiver have complete knowledge about the

specular component. In this case, the concept of channel capacity in terms of Shannon theory is well defined.

- **Contributions**: Derived coherent and non-coherent capacity expressions in the low and high SNR regimes for a popular rician fading model. Also, showed the contrast between rician and rayleigh fading channels based on capacity for training based communication systems.

CHAPTER II

Sequential Partial Update LMS Algorithm

2.1 Introduction

In [14], condition for convergence in mean for the Sequential Partial Update LMS (S-LMS) Algorithm were derived under the assumption of small step-size parameter (μ) which turned out to be the same as those for the standard LMS algorithm. This condition is however unreliable because of the underlying small μ assumption. In this chapter, we prove without the forementioned assumption that for stationary input signals convergence in mean for the regular LMS algorithm guarantees convergence in mean for S-LMS.

We also derive bounds on the step-size parameter μ for Sequential Partial Update LMS (S-LMS) Algorithm which ensures convergence in mean for the special case involving alternate even and odd coefficient updates when the input signal is cyclo-stationary. The bounds are based on extremal properties of the matrix 2-norm. We derive bounds for the case of stationary and cyclo-stationary signals. For simplicity we make the standard independence assumptions used in the analysis of LMS [6].

The organization of the chapter is as follows. First in section 2.2, a brief description of the sequential partial update algorithm is given. The algorithm with arbitrary sequence of updates is analyzed for the case of stationary signals in section 2.3. This is followed by the analysis of algorithm with the special case of alternate even and odd coefficient updates for cyclo-stationary signals in section 2.4. In section 2.5 an example is given to illustrate the usefulness of the bounds on step-size derived in section 2.4. Finally, conclusions and directions for future work are indicated in section 2.6.

2.2 Algorithm Description

The block diagram of S-LMS for a N-tap LMS filter with alternating even and odd coefficient updates is shown in Figure 2.1

It is assumed that the LMS filter is a standard FIR filter of even length, N. For convenience, we start with some definitions. Let $\{x_{i,k}\}$ be the input sequence and let $\{w_{i,k}\}$ denote the coefficients of the adaptive filter. Define

$$\begin{aligned} W_k &= [w_{1,k}\ w_{2,k}\ \ldots\ w_{N,k}]^T \\ X_k &= [x_{1,k}\ x_{2,k}\ x_{3,k}\ \ldots\ x_{N,k}]^T \end{aligned}$$

where the terms defined above are for the instant k. In addition, Let d_k denote the desired response. In typical applications d_k is a known training signal which is transmitted over a noisy channel with unknown FIR transfer function.

In this paper we assume that d_k itself obeys an FIR model given by $d_k = W_{opt}^{\dagger} X_k + n_k$ where W_{opt} are the coefficients of an FIR model given by $W_{opt} = [w_{1,opt}\ \ldots\ w_{N,opt}]^T$. Here $\{n_k\}$ is assumed to be a zero mean i.i.d sequence that is independent of the input sequence X_k.

For description purposes we will assume that the filter coefficients can be divided into P mutually exclusive subsets of equal size, i.e. the filter length N is a multiple of P. For convenience, define the index set $S = \{1, 2, \ldots, N\}$. Partition S into P mutually exclusive subsets of equal size, $S_1,\ S_2, \ldots, S_P$. Define I_i by zeroing out the j^{th} row of the identity matrix I if $j \notin S_i$. In that case, $I_i X_k$ will have precisely $\frac{N}{P}$ non-zero entries. Let the sentence "choosing S_i at iteration k" stand to mean "choosing the weights with their indices in S_i for update at iteration k".

The S-LMS algorithm is described as follows. At a given iteration, k, one of the sets S_i, $i = 1, \ldots, P$, is chosen in a pre-determined fashion and the update is performed.

$$w_{k+1,j} = \begin{cases} w_{k,j} + \mu e_k^* x_{k,j} & \text{if } j \in S_i \\ w_{k,j} & \text{otherwise} \end{cases} \tag{2.1}$$

where $e_k = d_k - W_k^\dagger X_k$. The above update equation can be written in a more compact form in the following manner

$$W_{k+1} = W_k + \mu e_k^* I_i X_k \tag{2.2}$$

In the special case of even and odd updates, $P = 2$ and S_1 consists of all even indices and S_2 of all odd indices as shown in Figure 2.1.

We also define the coefficient error vector as

$$V_k \;=\; W_k - W_{opt}$$

which leads to the following coefficient error vector update for S-LMS when k is odd

$$\begin{aligned} V_{k+2} &= (I - \mu I_2 X_{k+1} X_{k+1}^\dagger)(I - \mu I_1 X_k X_k^\dagger) V_k + \\ &\quad \mu (I - \mu I_2 X_{k+1} X_{k+1}^\dagger) n_k I_1 X_k + \mu n_{k+1} I_2 X_{k+1} \end{aligned} \tag{2.3}$$

and the following when k is even

$$\begin{aligned} V_{k+2} &= (I - \mu I_1 X_{k+1} X_{k+1}^\dagger)(I - \mu I_2 X_k X_k^\dagger) V_k + \\ &\quad \mu (I - \mu I_1 X_{k+1} X_{k+1}^\dagger) n_k I_2 X_k + \mu n_{k+1} I_1 X_{k+1} \end{aligned} \tag{2.4}$$

2.3 Analysis: Stationary Signals

Assuming that X_k is a WSS random sequence, we analyze the convergence of the mean coefficient error vector $E[V_k]$. We make the standard assumptions that V_k and X_k are mutually uncorrelated and that X_k is independent of X_{k-1} [6] which is not an unreasonable assumption for the case of antenna arrays. For regular full update LMS algorithm the recursion for $E[V_k]$ is given by

$$E[V_{k+1}] = (I - \mu R)E[V_k] \tag{2.5}$$

where I is the N-dimensional identity matrix and $R = E\left[X_k X_k^\dagger\right]$ is the input signal correlation matrix. The necessary and sufficient condition for stability of the recursion is given by

$$0 < \mu < 2/\lambda_{max} \tag{2.6}$$

where λ_{max} is the maximum eigen-value of the input signal correlation matrix R.

Taking expectations under the same assumptions as above, using the independence assumption on the sequences X_k, n_k, the mutual independence assumption on X_k and V_k, and simplifying we obtain for odd k when S-LMS is operating under the special case of alternate even and odd updates

$$E[V_{k+2}] = (I - \mu I_2 R)(I - \mu I_1 R)E[V_k] \tag{2.7}$$

and for even k

$$E[V_{k+2}] = (I - \mu I_1 R)(I - \mu I_2 R)E[V_k] \tag{2.8}$$

It can be shown that under the above assumptions on X_k, V_k and d_k, the convergence conditions for even and odd update equations are identical. We therefore focus on (2.7). Now to ensure stability of (2.7), the eigenvalues of $(I - \mu I_2 R)(I - \mu I_1 R)$ should lie inside the unit circle. We will show that if the eigenvalues of $I - \mu R$ lie inside the unit circle then so do the eigenvalues of $(I - \mu I_2 R)(I - \mu I_1 R)$.

Now, if instead of just two partitions of even and odd coefficients ($P = 2$) we have any number of arbitrary partitions ($P \geq 2$) then the update equations can be similarly written as above with $P > 2$. Namely,

$$E[V_{k+P}] = \prod_{i=1}^{P}(I - \mu I_i R)E[V_k] \tag{2.9}$$

We will show that for any arbitrary partition of any size ($P \geq 2$); S-LMS converges in the mean if LMS converges in the mean(Theorem II.2). The case $P = 2$ follows as a special case.

We will show that if R is a positive definite matrix of dimension $N \times N$ with eigenvalues lying in the open interval $(0, 2)$ then $\prod_{i=1}^{P}(I - I_i R)$ has eigenvalues inside the unit circle. I_i, $i = 1, \ldots, P$ is obtained from I, the identity matrix of dimension $N \times N$, by zeroing out some rows in I such that $\sum_{i=1}^{M} I_i$ is positive definite.

The following theorem is used in proving the main result in Theorem II.2.

Theorem II.1. *[36, Prob. 16, page 410] Let B be an arbitrary $N \times N$ matrix. Then $\rho(B) < 1$ if and only if there exists some positive definite $N \times N$ matrix A such that $A - B^{\dagger}AB$ is positive definite. $\rho(B)$ denotes the spectral radius of B ($\rho(B) = \max_{1,\ldots,N} |\lambda_i(B)|$).*

Theorem II.2. *Let R be a positive definite matrix of dimension $N \times N$ with $\rho(R) = \lambda_{max}(R) < 2$ then $\rho(\prod_{i=1}^{P}(I - I_i R)) < 1$ where I_i, $i = 1, \ldots, P$ are obtained by zeroing out some rows in the identity matrix I such that $\sum_{i=1}^{P} I_i$ is positive definite. Thus S-LMS converges in the mean if LMS converges in the mean.*

Proof: Let $\mathbf{x}_0 \in \mathbb{C}^N$ be an arbitrary non-zero vector of length N. Let $\mathbf{x}_i = (I - I_i R)\mathbf{x}_{i-1}$. Also, let $\mathbf{P} = \prod_{i=1}^{P}(I - I_i R)$.

First we will show that $\mathbf{x}_i^\dagger R\mathbf{x}_i \le \mathbf{x}_{i-1}^\dagger R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1}$, where $\alpha = \frac{1}{2}(2 - \lambda_{max}(R)) > 0$.

$$\begin{aligned}\mathbf{x}_i^\dagger R\mathbf{x}_i &= \mathbf{x}_{i-1}^\dagger (I - RI_i)R(I - I_iR)\mathbf{x}_{i-1} \\ &= \mathbf{x}_{i-1}^\dagger R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1} - \\ &\quad \beta\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1} + \mathbf{x}_{i-1}^\dagger RI_iRI_iR\mathbf{x}_{i-1}\end{aligned}$$

where $\beta = 2 - \alpha$. If we can show $\beta RI_iR - RI_iRI_iR$ is positive semi-definite then we are done. Now

$$\beta RI_iR - RI_iRI_iR = \beta RI_i(I - \frac{1}{\beta}R)I_iR$$

Since $\beta = (1+\lambda_{max}(R)/2) > \lambda_{max}(R)$ it is easy to see that $I - \frac{1}{\beta}R$ is positive definite. Therefore, $\beta RI_1R - RI_1RI_1R$ is positive semi-definite and

$$\mathbf{x}_i^\dagger R\mathbf{x}_i \le \mathbf{x}_{i-1}^\dagger R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1}$$

Combining the above inequality for $i = 1, \ldots, P$, we note that $\mathbf{x}_P^\dagger R\mathbf{x}_P < \mathbf{x}_0^\dagger R\mathbf{x}_0$ if $\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1} > 0$ for at least one i, $i = 1, \ldots, P$. We will show by contradiction that is indeed the case.

Suppose not, then $\mathbf{x}_{i-1}^\dagger RI_iR\mathbf{x}_{i-1} = 0$ for all i, $i = 1, \ldots, P$. Since, $\mathbf{x}_0^\dagger RI_1R\mathbf{x}_0 = 0$ this implies $I_1R\mathbf{x}_0 = \mathbf{0}$. Therefore, $\mathbf{x}_1 = (I - I_1R)\mathbf{x}_0 = \mathbf{x}_0$. Similarly, $\mathbf{x}_i = \mathbf{x}_0$ for all i, $i = 1, \ldots, P$. This in turn implies that $\mathbf{x}_0^\dagger RI_iR\mathbf{x}_0 = 0$ for all i, $i = 1, \ldots, P$ which is a contradiction since $R(\sum_{i=1}^P I_i)R$ is a positive-definite matrix and $0 = \sum_{i=1}^P \mathbf{x}_0^\dagger RI_iR\mathbf{x}_0 = \mathbf{x}_0^\dagger R(\sum_{i=1}^P I_i)R\mathbf{x}_0 \neq 0$.

Finally, we conclude that

$$\begin{aligned}\mathbf{x}_0^\dagger \mathbf{P}^\dagger R\mathbf{P}\mathbf{x}_0 &= \mathbf{x}_P^\dagger R\mathbf{x}_P \\ &< \mathbf{x}_0^\dagger R\mathbf{x}_0\end{aligned}$$

Since $\mathbf{x}_0$ is arbitrary we have $R - \mathbf{P}^\dagger R\mathbf{P}$ to be positive definite so that applying Theorem II.1 we conclude that $\rho(\mathbf{P}) < 1$.

Finally, if LMS converges in the mean we have $\rho(I - \mu R) < 1$ or $\lambda_{max}(\mu R) < 2$. Which from the above proof is sufficient for concluding that $\rho(\prod_{i=1}^{P}(I - \mu I_i R)) < 1$. Therefore, S-LMS also converges in the mean. ∎

2.4 Analysis: Cyclo-stationary Signals

Next, we consider the case when X_k is cyclo-stationary. We limit our attention to S-LMS with alternate even and odd updates as shown in Figure 2.1. Let X_k be a cyclo-stationary signal with period L. i.e, $R_{i+L} = R_i$. For simplicity, we will assume L is even. For the regular LMS algorithm we have the following L update equations

$$E\left[V_{k+L}\right] = \prod_{i=0}^{L-1}(I - \mu R_{i+d})E\left[V_k\right] \tag{2.10}$$

for $d = 1,\ 2,\ \ldots, L$, in which case we would obtain the following sufficient condition for convergence

$$0 < \mu < \min_i\{2/\lambda_{i,max}\} \tag{2.11}$$

where $\lambda_{i,max}$ is the largest eigenvalue of the matrix R_i.

Define $A_k = (I - \mu I_1 R_k)$ and $B_k = (I - \mu I_2 R_k)$ then for the partial update algorithm the $2L$ valid update equations are

$$E\left[V_{k+L}\right] = \left(\prod_{i=0}^{\frac{L-1}{2}} B_{2*i+1+d}A_{2*i+d}\right) E\left[V_k\right] \tag{2.12}$$

for $d = 1,\ 2, \ldots, L$ and

$$E\left[V_{k+L}\right] = \left(\prod_{i=0}^{\frac{L-1}{2}} A_{2*i+1+d}B_{2*i+d}\right) E\left[V_k\right] \tag{2.13}$$

for $d = 1,\ 2, \ldots, L$.

Let $\|A\|$ denote the spectral norm $\lambda_{max}(AA^{\dagger})$ of the matrix A. Then for ensuring the convergence of the iteration (2.12) and (2.13) a sufficient condition is

$$\|B_{i+1}A_i\| < 1 \quad \text{and} \quad \|A_{i+1}B_i\| < 1 \quad \text{for } i = 1,\ 2,\ \ldots, L \tag{2.14}$$

Since we can write $B_{i+1}A_i$ as

$$B_{i+1}A_i = (I - \mu R_i) + \mu I_2(R_i - R_{i+1}) + \mu^2 I_2 R_{i+1} I_1 R_i \tag{2.15}$$

and $A_{i+1}B_i$ as

$$A_{i+1}B_i = (I - \mu R_i) + \mu I_1(R_i - R_{i+1}) + \mu^2 I_1 R_{i+1} I_2 R_i \tag{2.16}$$

we have the the following expression which upper bounds both $\|B_{i+1}A_i\|$ and $\|A_{i+1}B_i\|$

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\| \tag{2.17}$$

This tells us that the sufficient condition to ensure convergence of both (2.12) and (2.13) is

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\| < 1 \tag{2.18}$$

for $i = 1, \ldots, L$.

If we make the assumption that

$$\mu < \min_i\{\frac{2}{\lambda_{i,max} + \lambda_{i,min}}\}$$

and

$$\delta_i = \|R_{i+1} - R_i\| < \max\{\lambda_{i,min}, \lambda_{i+1,min}\} = \eta_i$$

for $i = 1,\ 2,\ \ldots,\ L$ then (2.18) translates to

$$1 - \mu\eta_i + \mu\delta_i + \mu^2\lambda_{i,max}\lambda_{i+1,max} < 1 \tag{2.19}$$

which gives

$$0 < \mu < \min_{i=1}^{L}\{\frac{\eta_i - \delta_i}{\lambda_{i,max}\lambda_{i+1,max}}\} \tag{2.20}$$

(2.20) is the sufficient condition for the convergence of S-LMS.

2.5 Example

The usefulness of the bound on step-size for the cyclo-stationary case can be gauged from the following example. Consider a 2-tap filter and a cyclo-stationary $\{x_{i,k} = x_{k-i+1}\}$ with period 2 having the following auto-correlation matrices

$$R_1 = \begin{bmatrix} 5.1354 & -0.5733 - 0.6381i \\ -0.5733 + 0.6381i & 3.8022 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 3.8022 & 1.3533 + 0.3280i \\ 1.3533 - 0.3280i & 5.1354 \end{bmatrix}$$

For this choice of R_1 and R_2 η_1 and η_2 turn out to be 3.38 and we have $\|R_1 - R_2\| = 2.5343 < 3.38$. Therefore, R_1 and R_2 satisfy the assumption made for analysis. Now, $\mu = 0.33$ satisfies the condition for the regular LMS algorithm but, the eigenvalues of B_2A_1 for this value of μ have magnitudes 1.0481 and 0.4605. Since one of the eigenvalues lies outside the unit circle (2.12) is unstable for this choice of μ. Where as (2.20) gives $\mu = 0.0254$. For this choice of μ the eigenvalues of B_2A_1 turn out to have magnitudes 0.8620 and 0.8773. Hence (2.12) is stable.

We have plotted the evolution trajectory of the 2-tap filter with input signal satisfying the above properties. We chose $W_{opt} = [0.4\ 0.5]$ in Figures 2.2 and 2.3. For Figure 2.2 μ was chosen according to be 0.33 and for Figure 2.3 μ was chosen to be 0.0254. For simulation purposes we set $d_k = W_{opt}^{\dagger}S_k + n_k$ where $S_k = [s_k\ s_{k-1}]^{\tau}$ is a vector composed of the cyclo-stationary process $\{s_k\}$ with correlation matrices given as above, and $\{n_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$. We set $\{x_k\} = \{s_k\} + \{v_k\}$ where $\{v_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$.

2.6 Conclusion

We have analyzed the alternating odd/even partial update LMS algorithm and we have derived stability bounds on step-size parameter μ for wide sense stationary and cyclo-stationary signals based on extremal properties of the matrix 2-norm. For the case of wide sense stationary signals we have shown that if the regular LMS algorithm converges in mean then so does the sequential LMS algorithm for the general case of arbitrary but fixed ordering of the sequence of partial coefficient updates. For cyclo-stationary signals the bounds derived may not be the weakest possible bounds but they do provide the user with a useful sufficient condition on μ which ensures convergence in the mean. We believe the analysis undertaken in this paper is the first step towards deriving concrete bounds on step-size without making small μ assumptions. The analysis also leads directly to an estimate of mean convergence rate.

In the future, it would be useful to analyze partial update algorithm, without the assumption of independent snapshots and also, if possible, perform a second order analysis (mean square convergence). Furthermore, as S-LMS exhibits poor convergence in non-stationary signal scenarios (illustrative example given in the following chapter) it is of interest to develop new partial update algorithms with better convergence properties. One such algorithm based on randomized partial updating of filter coefficients is described in the following chapter (chapter III).

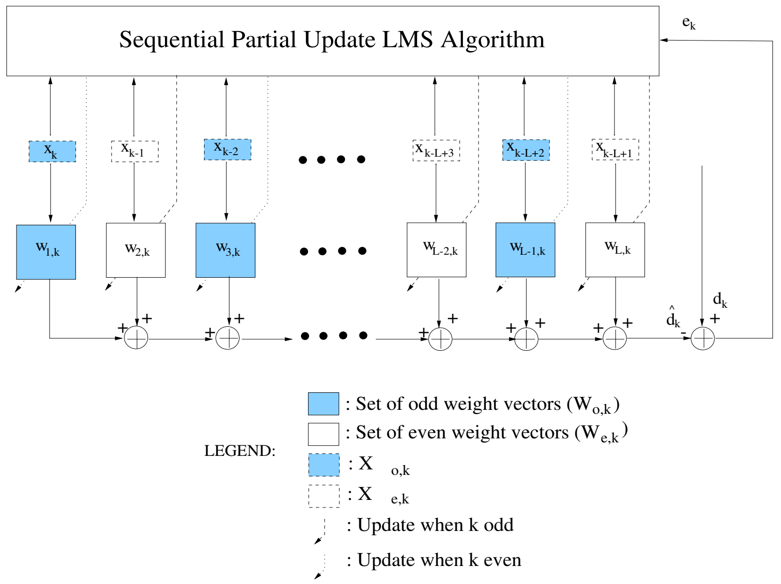


Figure 2.1: Block diagram of S-LMS for the special case of alternating even/odd coefficient update

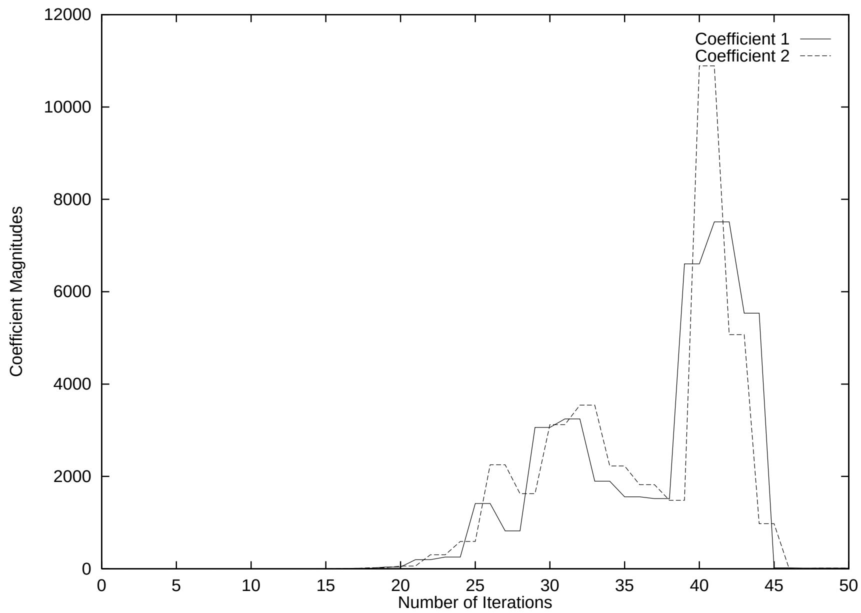


Figure 2.2: Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.33$

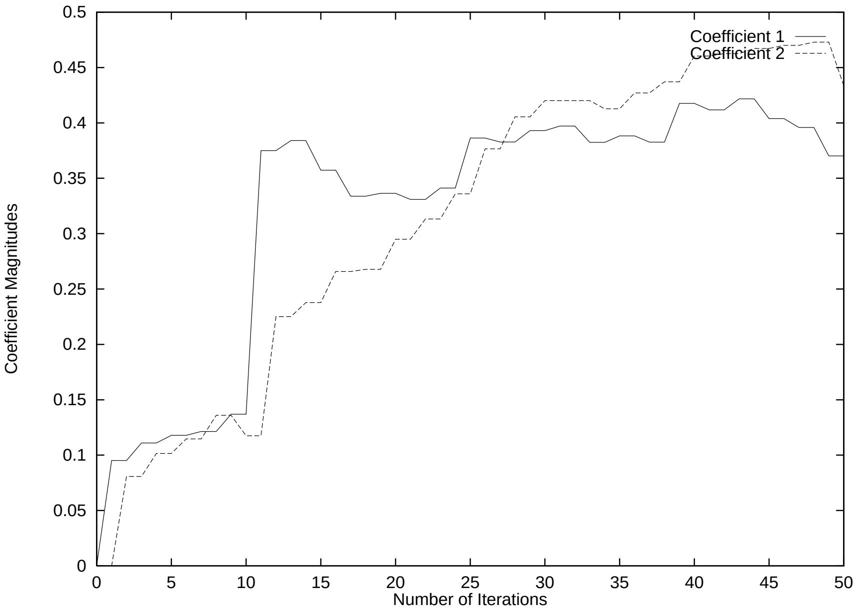


Figure 2.3: Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.0254$

CHAPTER III

Stochastic Partial Update LMS Algorithm

3.1 Introduction

The important characteristic of the partial update algorithms described in section 1.1 is that the coefficients to be updated at an iteration are pre-determined. It is this characteristic which renders P-LMS (see 1.1) and S-LMS unstable for certain signals and which makes random coefficient updating attractive. The algorithm proposed in this chapter is similar to S-LMS except that the subset of the filter coefficients that are updated each iteration is selected at random. The algorithm, referred to as Stochastic Partial Update LMS algorithm (SPU-LMS), involves selection of a subset of size $\frac{N}{P}$ coefficients out of P possible subsets from a fixed partition of the N coefficients in the weight vector. For example, filter coefficients can be partitioned into even and odd subsets and either even or odd coefficients are randomly selected to be updated in each iteration. In this chapter we derive conditions on the step-size parameter which ensures convergence in the mean and the mean square sense for stationary signals, generic signals and deterministic signals.

The organization of the chapter is as follows. First, a brief description of the algorithm is given in section 3.2 followed by analysis of the stochastic partial update algorithm for the stationary stochastic signals in section 3.3, deterministic signals in section 3.4 and for generic signals in 3.5. section 3.6 gives a description of of the existing Partial Update LMS algorithms. This is followed by section 3.7 consisting of examples where in section 3.7.1 verification of theoretical analysis of the new algorithm is carried out via simulations and examples are given to illustrate the usefulness of SPU-LMS. In sections 3.7.2 and 3.7.3 techniques developed in section 3.5 are used to show that the performance of SPU-LMS is very close to that of LMS in terms of final misconvergence. Finally conclusions and directions for future work are indicated in section 3.8.

3.2 Algorithm Description

Unlike in the standard LMS algorithm where all the filters taps are updated every iteration the algorithm proposed in this chapter updates only a subset of coefficients at each iteration. The subset to be updated is chosen in a random manner so that eventually every weight is updated.

The description of SPU-LMS is very similar to that of S-LMS (section 2.2). The only difference is as as follows. At a given iteration, k, for S-LMS one of the sets S_i, $i = 1, \ldots, P$ is chosen in a *pre-determined fashion* whereas for SPU-LMS, one of the sets S_i is chosen at *random* from $\{S_1,\ S_2, \ldots,\ S_P\}$ with probability $\frac{1}{P}$ and the update is performed. i.e.

$$w_{k+1,j} = \begin{cases} w_{k,j} + \mu e_k^* x_{k,j} & \text{if } j \in S_i \\ w_{k,j} & \text{otherwise} \end{cases} \tag{3.1}$$

where $e_k = d_k - W_k^\dagger X_k$. The above update equation can be written in a more compact form in the following manner

$$W_{k+1} = W_k + \mu e_k^* I_i X_k \tag{3.2}$$

where I_i now is a randomly chosen matrix.

3.3 Analysis SPU-LMS: Stationary Stochastic Signals

In the stationary signal setting the offline problem is to choose an optimal W such that

$$\begin{aligned}\xi^{(W)} &= E\left[(d_k - y_k)(d_k - y_k)^*\right] \\ &= E\left[(d_k - W^\dagger X_k)(d_k - W^\dagger X_k)^*\right]\end{aligned}$$

is minimized, where a^* denotes the complex conjugate of a. The solution to this problem is given by

$$W_{opt} = R^{-1} r \tag{3.3}$$

where $R = E[X_k X_k^\dagger]$ and $r = E[d_k^* X_k]$. The minimum attainable $\xi^{(W)}$ is given by

$$\xi_{min} = E[d_k d_k^*] - r^\dagger R^{-1} r$$

For the following analysis, we assume that the desired signal, d_k satisfies the following relation

$$d_k = W_{opt}^\dagger X_k + n_k \tag{3.4}$$

where X_k is a zero mean circular Gaussian random vector and n_k is a zero mean circular complex Gaussian (not necessarily white) noise, with variance ξ_{min}, uncorrelated with X_k. It can be easily verified that the model assumed for d_k is same as assuming d_k and X_k are jointly zero mean complex circular Gaussian sequences.

We also make the independence assumption used in the analysis of standard LMS [6] which is reasonable for the present application of adaptive beamforming. We assume that X_k is a Gaussian random vector and that X_k is independent of X_j for $j < k$. We also assume that I_i and X_k are mutually independent.

For convergence-in-mean analysis we obtain the following update equation conditioned on a choice of S_i.

$$E[V_{k+1}|S_i] = (I - \mu I_i R)E[V_k|S_i]$$

which after averaging over all choices of S_i gives

$$E[V_{k+1}] = (I - \frac{\mu}{P}R)E[V_k]$$

To obtain the above equation we have made use of the fact that the choice of S_i is independent of V_k and X_k. Therefore, μ has to satisfy $0 < \mu < \frac{2P}{\lambda_{max}}$ to guarantee convergence in mean.

For convergence-in-mean square analysis we are interested in the convergence of $E[e_k e_k^*]$. Under the assumptions we obtain $E[e_k e_k^*] = \xi_{min} + \mathrm{tr}\{RE[V_k V_k^\dagger]\}$ where ξ_{min} is as defined earlier.

We have followed the procedure of [37] for our mean-square analysis. First, conditioned on a choice of S_i, the evolution equation of interest for $\mathrm{tr}\{RE[V_k V_k^\dagger]\}$ is given by

$$\begin{aligned} RE[V_{k+1}V_{k+1}^\dagger|S_i] &= RE[V_k V_k^\dagger|S_i] - 2\mu R I_i RE[V_k V_k^\dagger|S_i] + \\ &\quad \mu^2 I_i R I_i E[X_k X_k^\dagger A_k X_k X_k^\dagger|S_i] + \mu^2 \xi_{min} R I_i R I_i \end{aligned}$$

where $A_k = E[V_k V_k^\dagger]$. For simplicity, consider the case of block diagonal R satisfying $\sum_{i=1}^{P} I_i R I_i = R$. Then, we obtain the final equation of interest for convergence-in-mean square to be

$$G_{k+1} = (I - \frac{2\mu}{P}\Lambda + \frac{\mu^2}{P}\Lambda^2 + \frac{\mu^2}{P}\Lambda^2 \mathbf{1}\mathbf{1}^\tau)G_k + \frac{\mu^2}{P}\xi_{min}\Lambda^2\mathbf{1} \qquad (3.5)$$

where G_k is a vector of diagonal elements of $\Lambda E[U_k U_k^\dagger]$ where $U_k = QV_k$ with Q such that $QRQ^\dagger = \Lambda$. It is easy to obtain the following necessary and sufficient conditions (see Appendix .1) for convergence of the SPU-LMS algorithm

$$0 < \mu < \tfrac{2}{\lambda_{max}} \qquad (3.6)$$
$$\eta(\mu) \stackrel{\text{def}}{=} \textstyle\sum_{i=1}^{N} \tfrac{\mu\lambda_i}{2-\mu\lambda_i} < 1$$

which is independent of P and identical to that of LMS.

We used the integrated MSE difference $J = \sum_{k=0}^{\infty}[\xi_k - \xi_\infty]$ introduced in [20] as a measure of the convergence rate and $M(\mu) = \frac{\xi_\infty - \xi_{min}}{\xi_{min}}$ as a measure of misadjustment. The misadjustment factor is simply (see Appendix .3)

$$M(\mu) = \frac{\eta(\mu)}{1-\eta(\mu)} \qquad (3.7)$$

which is same as that of the standard LMS. Thus, we conclude that random update of subsets has no effect on the final excess mean-squared error.

Finally, it is straightforward to show (see Appendix .2) the integrated MSE difference is

$$J = P\ \mathrm{tr}\{[2\mu\Lambda - \mu^2\Lambda^2 - \mu^2\Lambda^2\mathbf{1}\mathbf{1}^{\tau}]^{-1}(G_0 - G_\infty)\} \qquad (3.8)$$

which is P times the quantity obtained for standard LMS algorithm. Therefore, we conclude that for block diagonal R, random updating slows down convergence by a factor of P without affecting the misadjustment. Furthermore, it can be easily verified that $0 < \mu < \frac{1}{\mathrm{tr}\{R\}}$ is a sufficient region for convergence of SPU-LMS and the standard LMS algorithm.

3.4 Analysis SPU-LMS: Deterministic Signals

Here we followed the analysis given in [61, pp. 140–143] which can be extended to SPU-LMS with complex signals in a straightforward manner. We assume that the input signal X_k is bounded, that is $\sup_k(X_k^\dagger X_k) \leq B < \infty$ and that the desired signal d_k follows the model

$$d_k = W_{opt}^{\dagger} X_k$$

Define $V_k = W_k - W_{opt}$ and $e_k = d_k - W_k^{\dagger} X_k$. Then we can show that if $\mu < 2/B$ then $\overline{e_k^2} \to 0$ as $k \to \infty$, and if in addition the signal satisfies the following persistence of excitation condition:

for all k, there exist $K < \infty$, $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1 I < \sum_{i=k}^{k+K} X_i X_i^{\dagger} < \alpha_2 I \tag{3.9}$$

then $\overline{V_k}^{\dagger} \overline{V_k} \to 0$ exponentially fast and $\overline{V_k^{\dagger} V_k} \to 0$ at a rate $o(\frac{1}{k})$. Here, $\overline{\{\cdot\}}$ indicates statistical expectation over all possible choices of S_i, where each S_i is chosen uniformly from $\{S_1, \ldots, S_P\}$.

Condition (3.9) is identical to the persistence of excitation condition for standard LMS. Therefore, the sufficient condition for exponential stability of LMS is enough to guarantee asymptotic stability of SPU-LMS.

3.5 Analysis SPU-LMS: Generic Signals

In this section, we analytically compare the performance of LMS and SPU-LMS in terms of stability and misconvergence when the independent snapshots assumption is invalid. For this we employ the theory developed in [45] and [55]. Eventhough the theory developed is for the case of real random variables it can easily be adapted to the case of complex circular random variables.

In this section, results for stability and performance for the case of SPU-LMS are developed for describing the performance hit taken when going from LMS to SPU-LMS. One of the important results obtained is that for stability we establish that

LMS and SPU-LMS have the same necessary and sufficient conditions. The theory used for stability analysis and performance analysis is from [45] and [55], respectively.

3.5.1 Stability Analysis

Notations and Definitions

Notations are the same as those used in [45]. $\|X\|_p$ is used to denote the L_p-norm of a random matrix X given as $\|X\|_p \stackrel{\text{def}}{=} \{E\|X\|^p\|^{1/p}$ for $p \geq 1$ where $\|X\| \stackrel{\text{def}}{=} \{\sum_{i,j} |x|_{ij}^2\}^{1/2}$ is the Euclidean norm of the matrix X. Note that in [45], $\|X\| \stackrel{\text{def}}{=} \{\lambda_{max}(XX^\dagger)\}^{1/2}$. Since the two norms are related by a constant the results in [45] could as well have been stated with the definition used here. We use this definition since this is the one used in [55].

A process X_k is said to be ϕ-mixing if there is a function $\phi(m)$ such that $\phi(m) \to 0$ as $m \to \infty$ and

$$\sup_{A \in \mathcal{M}_{-\infty}^{k}(X), B \in \mathcal{M}_{k+m}^{\infty}} (X)|P(B|A) - P(B)| \leq \phi(m), \forall m \geq 0, k \in (-\infty, \infty)$$

where $\mathcal{M}_i^j(X)$, $-\infty \leq i \leq j \leq \infty$ is the σ-algebra generated by $\{X_k\}$, $i \leq k \leq j$

For any random matrix sequence $F = \{F_k\}$, define $\mathcal{S}_p(\alpha, \mu^*)$ for $\mu* > 0$ and $0 < \alpha < 1/\mu^*$ by

$$\begin{aligned}\mathcal{S}_p(\alpha, \mu^*) &= \Bigg\{F : \left\| \prod_{j=i+1}^{k} (I - \mu F_j) \right\|_p \leq K_{\alpha,\mu^*}(F)(1-\mu\alpha)^{k-i} \\ &\quad \forall \mu \in (0, \mu^*], \forall k \geq i \geq 0\}\end{aligned}$$

Basically, $\mathcal{S}_p(\alpha, \mu^*)$ is the family of L_p-stable random matrices.

Similarly, the averaged exponentially stable family is defined as $\mathcal{S}(\alpha, \mu^*)$ for $\mu* > 0$ and $0 < \alpha < 1/\mu^*$ by

$$\begin{aligned}\mathcal{S}(\alpha, \mu^*) &= \Bigg\{F : \left\| \prod_{j=i+1}^{k} (I - \mu E[F_j]) \right\|_p \leq K_{\alpha,\mu^*}(E[F])(1-\mu\alpha)^{k-i} \qquad (3.10) \\ &\quad \forall \mu \in (0, \mu^*], \forall k \geq i \geq 0\}\end{aligned}$$

We also define $\mathcal{S}_p$ and $\mathcal{S}$ as $\mathcal{S}_p \stackrel{\text{def}}{=} \cup_{\mu^* \in (0,1)} \cup_{\alpha \in (0,1/\mu^*)} \mathcal{S}_p(\alpha, \mu^*)$ and $\mathcal{S} \stackrel{\text{def}}{=} \cup_{\mu^* \in (0,1)} \cup_{\alpha \in (0,1/\mu^*)} \mathcal{S}(\alpha, \mu^*)$.

Results

Let X_k be the input signal vector generated from the following process

$$X_k = \sum_{j=-\infty}^{\infty} A(k,j)\epsilon_{k-j} + \psi_k \tag{3.11}$$

with $\sum_{j=-\infty}^{\infty} \sup_k \|A(k,j)\| < \infty$. $\{\psi_k\}$ is a d-dimensional deterministic process, and $\{\epsilon_k\}$ is a general m-dimensional ϕ-mixing sequence. The weighting matrices $A(k,j) \in \mathcal{R}^{d \times m}$ are assumed to be deterministic.

We prove the following theorem which is similar to Theorem 2 in [45].

Theorem III.1. *Let X_k be as defined above with $\{\epsilon_k\}$ a ϕ-mixing sequence such that it satisfies for any $n \geq 1$ and any integer sequence $j_1 < j_2 \ldots j_n$*

$$E\left[\exp\left(\alpha \sum_{i=1}^{n} \|\epsilon_{j_i}\|^2\right)\right] \leq M \exp(Kn) \tag{3.12}$$

where α, M, and K are positive constants. Then for any $p \geq 1$, there exist constants $\mu* > 0$, $M > 0$, and $\alpha \in (0,1)$ such that for all $\mu \in (0, \mu*]$ and for all $t \geq k \geq 0$

$$\left[E\left\|\prod_{j=k+1}^{t}(I - \mu I_j X_j X_j^\dagger)\right\|^p\right]^{1/p} \leq M(1-\mu\alpha)^{t-k}$$

where I_j is a sequence of i.i.d $d \times d$ masking matrices, if and only if there exists an integer $h > 0$ and a constant $\delta > 0$ such that for all $k \geq 0$

$$\sum_{i=k+1}^{k+h} E[X_i X_i^\dagger] \geq \delta I \tag{3.13}$$

Proof: For proof see Appendix .4. ■

Note that the LMS algorithm has the same necessary and sufficient condition for convergence. Therefore, SPU-LMS behaves exactly like LMS in this respect.

3.5.2 Performance Analysis

For perforance analysis, we assume that

$$d_k = X_k^\dagger W_{opt,k} + n_k$$

$W_{opt,k}$ varies as follows $W_{opt,k+1} - W_{opt,k} = w_{k+1}$, where w_{k+1} is the lag noise. Then for LMS we can write the evolution equation for the tracking error $V_k \stackrel{\text{def}}{=} W_k - W_{opt,k}$ as

$$V_{k+1} = (I - \mu X_k X_k^\dagger)V_k + \mu X_k n_k - w_{k+1}$$

and for SPU-LMS the corresponding equation can be written as

$$V_{k+1} = (I - \mu I_k X_k X_k^\dagger)V_k + \mu X_k n_k - w_{k+1}$$

In the example used in this report it is assumed that $w_k = 0$ for all k.

Now, V_{k+1} can be decomposed [55] as $V_{k+1} = {}^uV_k + \mu {}^nV_k + {}^wV_k$ where

$$\begin{aligned}
{}^uV_{k+1} &= (I - \mu P_k X_k X_k^\dagger){}^uV_k \quad {}^uV_0 = V_0 = -W_{opt,0} \\
{}^nV_{k+1} &= (I - \mu P_k X_k X_k^\dagger){}^nV_k + P_k X_k n_k \quad {}^nV_0 = 0 \\
{}^wV_{k+1} &= (I - \mu P_k X_k X_k^\dagger){}^wV_k - w_{k+1} \quad {}^nV_0 = 0
\end{aligned}$$

where $P_k = I$ for LMS and $P_k = I_k$ for SPU-LMS. $\{{}^uV_k\}$ denotes the transient term, reflecting the way the successive estimates of the regression coefficients forget the initial conditions. $\{{}^nV_k\}$ accounts for the errors introduced by the measurement noise, n_k and $\{{}^vV_k\}$ accounts for the errors associated with the lag-noise $\{w_k\}$.

So, in general ${}^nV_k$ and ${}^wV_k$ obey the following inhomogenous equation

$$\delta_{k+1} = (I - \mu F_k)\delta_k + \xi_k, \quad \delta_0 = 0$$

δ_k can be represent by a set of recursive equations as follows

$$\delta_k = J_k^{(0)} + J_k^{(1)} + \ldots + J_k^{(n)} + H_k^{(n)}$$

where the processes $J_k^{(r)}, 0 \leq r < n$ and $H_k^{(n)}$ are described by

$$\begin{aligned}
J_{k+1}^{(0)} &= (I - \mu \bar{F}_k) J_k^{(0)} + \xi_k; \, J_0^{(0)} = 0 \\
J_{k+1}^{(r)} &= (I - \mu \bar{F}_k) J_k^{(r)} + \mu Z_k J_k^{(r-1)}; \quad J_k^{(r)} = 0, 0 \leq k < r \\
H_{k+1}^{(n)} &= (I - \mu F_k) H_k^{(n)} + \mu Z_k J_k^{(n)}; \quad H_k^{(n)} = 0, 0 \leq k < n
\end{aligned}$$

where $\bar{F}_k$ is an appropriate deterministic process. It usually is chosen as $\bar{F}_k = E[F_k]$. In [55] under appropriate conditions it was shown that there exists some constant $C < \infty$ and $\mu_0 > 0$ such that for all $0 < \mu \leq \mu_0$, we have

$$\sup_{k \geq 0} \|H_k^{(n)}\|_p \leq C \mu^{n/2}$$

Notations and Definitions

Now, we modify the definition of weak dependence as given in [55] for circular complex random variables. The theory developed in [55] can be easily adapted for circular random variables using this definition. Let $q \geq 1$ and $X = \{X_n\}_{n \geq 0}$ be a $(l \times 1)$ matrix valued process. Let $\beta = (\beta(r))_{r \in N}$ be a sequence of positive numbers decreasing to zero at infinity. The complex process $X = \{X_n\}_{n \geq 0}$ is said to be (δ, q)-weak dependent if there exist finite constants $C = \{C_1, \ldots, C_q\}$, such that for any $1 \leq m < s \leq q$ and m-tuple $k_1, \ldots, k_m$ and any $(s-m)$-tuple $k_{m+1}, \ldots, k_s$, with $k_1 \leq \ldots \leq k_m < k_m + r \leq k_{m+1} \leq \ldots \leq k_s$, it holds that

$$\sup_{1 \leq i_1, \ldots, i_s \leq l, f_{k_1,i_1}, f_{k_2,i_2} \cdots f_{k_m,i_m}} \Big| \mathrm{cov}\Big(f_{k_1,i_1}(\tilde{X}_{k_1,i_1}) \cdot \ldots \cdot f_{k_m,i_m}(\tilde{X}_{k_m,i_m}), \\ f_{k_{m+1},i_{m+1}}(\tilde{X}_{k_{m+1},i_{m+1}}) \cdot \ldots \cdot f_{k_s,i_s}(\tilde{X}_{k_s,i_s}) \Big) \Big| \leq C_s \beta(r)$$

where $\tilde{X}_{n,i}$ denotes the i-th component of $X_n - E(X_n)$ and the set of functions $f_{n,i}()$ that the sup is being taken over are given by $f_{n,i}(\tilde{X}_{n,i}) = \tilde{X}_{n,i}$ and $f_{n,i}(\tilde{X}_{n,i}) = \tilde{X}^*_{n,i}$.

Define $\mathcal{N}(p)$ from [55] as follows

$$\begin{aligned}\mathcal{N}(p) = \quad & \left\{\epsilon : \left\|\sum_{k=s}^{t} D_k \epsilon_k\right\|_p \leq \rho_p(\epsilon)\left(\sum_{k=s}^{t} |D_k|^2\right)^{1/2} \ \forall 0 \leq s \leq t \right. \\ & \left. \text{and } \forall D = \{D_k\}_{k \in N}(q \times l) \text{ deterministic matrices } \right\}\end{aligned}$$

F_k can be written as $F_k = P_k X_k X_k^\dagger$ where $P_k = I$ for LMS and $P_k = I_k$ for SPU-LMS. It is assumed that the following hold true for F_k. For some $r, q \in N$, $\mu_0 > 0$ and $0 < \alpha < 1/\mu_0$

- **F1**(r, α, μ_0) $\{F_k\}_{k \geq 0}$ is in $\mathcal{S}(r, \alpha, \mu_0)$ that is $\{F_k\}$ is L_r-exponentially stable.
- **F2**(α, μ_0) $\{E[F_k]\}_{k \geq 0}$ is in $\mathcal{S}(\alpha, \mu_0)$, that is $\{E[F_k]\}_{k \geq 0}$ is averaged exponentially stable.

Conditions **F3** and **F4** stated below are trivially satisfied for $P_k = I$ and $P_k = I_k$.

- **F3**(q, μ_0) $\sup_{k \in N} \sup_{\mu \in (0, \mu_0]} \|P_k\|_q < \infty$ and $\sup_{k \in N} \sup_{\mu \in (0, \mu_0]} |E[P_k]| < \infty$
- **F4**(q, μ_0) $\sup_{k \in N} \sup_{\mu \in (0, \mu_0]} \mu^{-1/2} \|P_k - E[P_k]\|_q < \infty$

The excitation sequence $\xi = \{\xi_k\|_{k \geq 0}$ [55] is assumed to be decomposed as $\xi_k = M_k \epsilon_k$ where the processes $M = \{M_k\}_{k \geq 0}$ is a $d \times l$ matrix valued process and $\epsilon = \{\epsilon_k\}_{k \geq 0}$ is a $(l \times 1)$ vector-valued process that verifies the following assumptions

- **EXC1** $\{M_k\}_{k \in Z}$ is $\mathcal{M}_0^k(X)$-adapted and $\mathcal{M}_0^k(\epsilon)$ and $\mathcal{M}_0^k(X)$ are independent.
- **EXC2**(r, μ_0), $(r > 0, \mu_0 > 0)$ $\sup_{\mu \in (0, \mu_0]} \sup_{k \geq 0} \|M_k\|_r < \infty$
- **EXC3**(p, μ_0), $(p > 0, \mu_0 > 0)$ $\epsilon = \{\epsilon_k\}_{k \in N}$ belongs to $\mathcal{N}(p)$.

Results

The following theorems from [55] are relevant.

Theorem III.2 (Theorem 1 in [55]). *Let $n \in N$ and let $q \geq p \geq 2$. Assume* **EXC1**, **EXC2**$(pq/(q-p), \mu_0)$ *and* **EXC3**(p, μ_0). *For $a, b, \alpha > 0$, $a^{-1} + b^{-1} = 1$, and some $\mu_0 > 0$, assume in addition* **F2**(α, μ_0), **F4**(aqn, μ_0) *and*

- $\{G_k\}_{k \geq 0}$ *is $(\beta, (q+2)n)$ weakly dependent and* $\sum(r+1)^{((q+2)n/2)-1}\beta(r) < \infty$
- $\sup_{k \geq 0} \|G_k\|_{bqn} < \infty$

Then, there exists a constant $K < \infty$ (depending on $\beta(k)$, $k \geq 0$ and on the numerical constants $p, q, n, q, b, \mu_0, \alpha$ but not otherwise on $\{X_k\}$, $\{\epsilon_k\}$ or on μ), such that for all $0 < \mu \leq \mu_0$, for all $0 \leq r \leq n$

$$\sup_{s \geq 1} \|J_s^{(r)}\|_p \leq K \rho_p(\epsilon) \sup_{k \geq 0} \|M_k\|_{pq/(q-p)} \mu^{(r-1)/2}$$

Theorem III.3 (Theorem 2 in [55]). *Let $p \geq 2$ and let $a, b, c > 0$ such that $1/a + 1/b + 1/c = 1/p$. Let $n \in N$. Assume* **F1**(a, α, μ_0) *and*

- $\sup_{s \geq 0} \|Z_s\|_b < \infty$
- $\sup_{s \geq 0} \|J_s^{(n+1)}\|_c < \infty$

Then there exists a constant $K' < \infty$ (depending on the numerical constants $a, b, c, \alpha, \mu_0, n$ but not on the process $\{\epsilon_k\}$ or on the stepsize parameter μ), such that for all $0 < \mu \leq \mu_0$,

$$\sup_{s \geq 0} \|H_s^{(n)}\|_p \leq K' \sup_{s \geq 0} \|J_s^{(n+1)}\|_c$$

It is shown that if LMS satisfies the assumptions above (assumptions in section 3.2 in [55]) then so does SPU-LMS. Conditions **F1** and **F2** follow directly from Theorem III.1. It is easy to see that **F3** and **F4** hold easily for LMS and SPU-LMS.

Lemma III.1. *The constants in Theorem III.2 calculated for LMS can also be used for SPU-LMS.*

Proof: Here all that is needed to be shown is that if LMS satisfies the conditions **(EXC1)**, **(EXC2)** and **(EXC3)** then so does SPU-LMS. Moreover, the upper bounds on the norms for LMS are also upper bounds for SPU-LMS. That easily follows because $M_k^{LMS} = X_k$ whereas $M_k^{SPU-LMS} = I_k X_k$ and $\|I_k\| \leq 1$ for any norm $\|\cdot\|$. ■

Lemma III.2. *The constants in Theorem III.3 calculated for LMS can also be used for SPU-LMS.*

Proof: First we show that if for LMS $\sup_{s\geq 0}\|Z_s\|_b < \infty$ then so it is for SPU-LMS. First, note that for LMS we can write $Z_s^{LMS} = X_s X_s^\dagger - E[X_s X_s^\dagger]$ whereas for SPU-LMS

$$Z_s^{SPU-LMS} = I_s X_s X_s^\dagger - \frac{1}{P}E[X_s X_s^\dagger] = I_s X_s X_s^\dagger - I_s E[X_s X_s^\dagger] + (I_s - \frac{1}{P}I)E[X_s X_s^\dagger]$$

That means $\|Z_s^{SPU-LMS}\|_b \leq \|I_s\|_b \|Z_s^{LMS}\|_b + \|I_s - \frac{1}{P}I\|_b \|E[X_s X_s^\dagger]\|_b$. Therefore, since $\sup_{s\geq 0}\|_b E[X_s X_s^\dagger]\|_b < \infty$ and $\sup_{s\geq 0}\|Z_s^{LMS}\|_b < \infty$ we have $\sup_s \|Z_s^{SPU-LMS}\|_b < \infty$. Since all conditions for Theorem 2 have been satisfied by SPU-LMS in a similar manner the constants obtained are also the same. ■

The two lemmas states that the error terms are bounded above by same constants.

3.6 Periodic and Sequential LMS Algorithms

For P-LMS, the update equation can be written as follows

$$W_{k+P} = W_k + \mu e_k X_k$$

For the Sequential LMS algorithm the update equation is same as (3.2) except that the choice of I_i is no longer random. The sequence of I_i as k progresses is pre-determined and fixed.

For the P-LMS algorithm, using the method of analysis described in [37] we conclude that the conditions for convergence are identical to standard LMS. That is (3.6) holds also for P-LMS. Also, the misadjustment factor remains the same. The only difference between LMS and P-LMS is that the measure J for P-LMS is P times that of LMS. Therefore, we see that the behavior of SPU-LMS and P-LMS algorithms is very similar for stationary signals.

The difference between P-LMS and SPU-LMS becomes evident for deterministic signals. From the persistence of excitation condition shown in [14] for P-LMS we conclude that the condition is stricter for P-LMS than for SPU-LMS. In fact, in the next section we construct signals for which P-LMS is guaranteed not to converge whereas SPU-LMS will converge.

The convergence of Sequential LMS algorithm has been analyzed using the small μ assumption in [14]. Theoretical results for this algorithm are not presented here. It is only shown through examples that this algorithm diverges for certain kind of signals and therefore should be employed with caution.

3.7 Examples

3.7.1 Illustration of Utility of SPU-LMS

We simulated an m-element uniform linear antenna array operating in a multiple signal environment. Let A_i denote the response of the array to the i^{th} plane wave signal: $A_i = [e^{-\mathrm{j}(\frac{m}{2}-\tilde{m})\omega_i} \; e^{-\mathrm{j}(\frac{m}{2}-1-\tilde{m})\omega_i} \; \ldots \; e^{\mathrm{j}(\frac{m}{2}-1-\tilde{m})\omega_i} \; e^{\mathrm{j}(\frac{m}{2}-\tilde{m})\omega_i}]^{\tau}$ where $\tilde{m} = (m+1)/2$ and $\omega_i = \frac{2\pi D \sin\theta_i}{\lambda}$, $i = 1, \ldots, M$. θ_i is the broadside angle of the i^{th} signal, D is the inter-element spacing between the antenna elements and λ is the common wavelength of the narrowband signals in the same units as D and $\frac{2\pi D}{\lambda} = 2$. The array output at the k^{th} snapshot is given by $X_k = \sum_{i=1}^{M} A_i s_{k,i} + n_k$ where M denotes the number of signals, the sequence $\{s_{k,i}\}$ the amplitude of the i^{th} signal and n_k the noise present at the array output at the k^{th} snapshot. The objective, in both the examples, is to maximize the SNR at the output of the beamformer. Since the signal amplitudes are random the objective translates to obtaining the best estimate of $s_{k,1}$, the amplitude of the desired signal, in the MMSE sense. Therefore, the desired signal is chosen as $d_k = s_{k,1}$.

In the first example, the array has 4 elements and a single planar waveform with amplitude, $s_{k,1}$ propagates across the array from direction angle, $\theta_1 = \frac{\pi}{2}$. The amplitude sequence $\{s_{k,1}\}$ is a BPSK signal with period four taking values on $\{-1, 1\}$ with equal probability. The additive noise n_k is circular Gaussian with variance 0.25 and mean 0. In all the simulations for SPU-LMS, P-LMS, and S-LMS the number of subsets for partial updating, P was chosen to be 4. It can be easily determined from (3.6) that for Gaussian and independent signals the necessary and sufficient condition for convergence of LMS and SPU-LMS is $\mu < 0.67$. Figure 3.1 shows representative trajectories of the empirical mean-squared error for LMS, SPU-LMS, P-LMS and S-LMS algorithms averaged over 100 trials for $\mu = 0.6$ and $\mu = 1.0$. All algorithms were found to be stable for the BPSK signals even for μ values greater than 0.67. It was only as μ approached 1 that divergent behavior was observed. As expected, LMS and SPU-LMS were observed to have similar μ regions of convergence. It is also clear from Figure 3.1, that as, expected SPU-LMS, P-LMS, and S-LMS take roughly 4 times longer to converge than LMS.

In the second example, we consider an 8-element uniform linear antenna array with one signal of interest propagating at angle θ_1 and 3 interferers propagating at angles θ_i, $i = 2, 3, 4$. The array noise n_k is again mean 0 circular Gaussian but with variance 0.001.

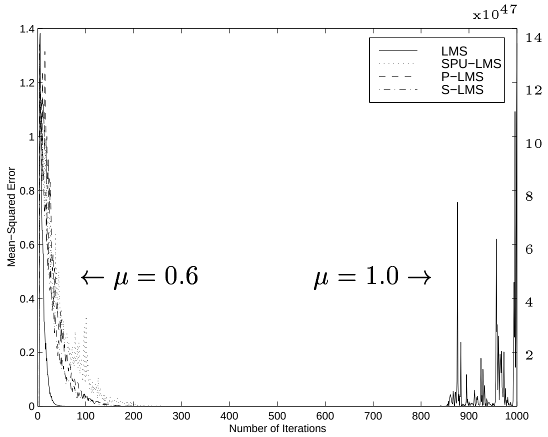


Figure 3.1: Trajectories of MSE for Example 1

We generated signals, such that $s_{k,1}$ is stationary and $s_{k,i}$, $i = 2, 3, 4$ are cyclostationary with period four, which make both S-LMS and P-LMS non-convergent. All the signals were chosen to be independent from time instant to time instant. First, we found signals for which S-LMS doesn't converge by the following procedure. Make the small μ approximation $I - \mu \sum_{i=1}^{P} I_i E[X_{k+i} X_{k+i}^{\dagger}]$ to the transition matrix $\prod_{i=1}^{P}(I - \mu I_i E[X_{k+i} X_{k+i}])$ and generate sequences $s_{k,i}$, $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{P} I_i E[X_{k+i} X_{k+i}^{\dagger}]$ has roots in the negative left half plane. This ensures that $I - \mu \sum_{i=1}^{P} I_i E[X_{k+i} X_{k+i}^{\dagger}]$ has roots outside the unit circle. The sequences found in this manner were then verified to cause the roots to lie outside the unit circle for all μ. One such set of signals found was: $s_{k,1}$ is equal to a BPSK signal with period one taking values in $\{-1, 1\}$ with equal probability. The interferers, $s_{k,i}$, $i = 2, 3, 4$ are cyclostationary BPSK type signals taking values in $\{-1, 1\}$ with the restriction that $s_{k,2} = 0$ if $k \% 4 \neq 1$, $s_{k,3} = 0$ if $k \% 4 \neq 2$ and $s_{k,4} = 0$ if $k \% 4 \neq 3$. Here $a \% b$ stands for a modulo b. θ_i, $i = 1, 2, 3, 4$ are chosen such that $\theta_1 = 1.0388$, $\theta_2 = 0.0737$, $\theta_3 = 1.0750$ and $\theta_4 = 1.1410$. These signals render the S-LMS algorithm unstable for all μ.

The P-LMS algorithm also fails to converge for the signal set described above

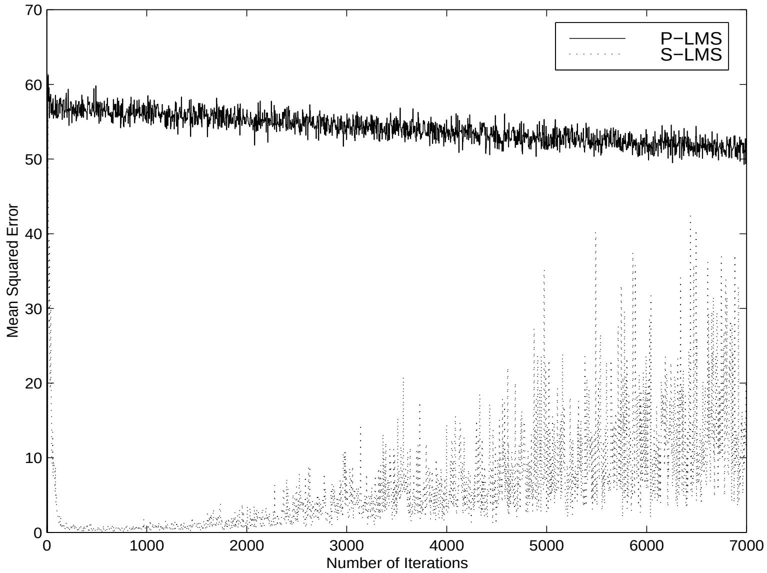


Figure 3.2: Trajectories of MSE for Example 2

irrespective of μ and the choice of θ_1, θ_2, θ_3, and θ_4. Since P-LMS updates the coefficients every 4^{th} iteration it sees at most one of the three interfering signals throughout all its updates and hence can place a null at atmost one signal incidence angle θ_i. Figure 3.2 shows the envelopes of the e_k^2 trajectories of S-LMS and P-LMS for the signals given above with the representative value $\mu = 0.03$. As can be seen P-LMS fails to converge whereas S-LMS shows divergent behavior. SPU-LMS and LMS were observed to converge for the signal set described above when $\mu = 0.03$.

3.7.2 Analysis for an i.i.d Gaussian Input Sequence

In this section, we assume that $X_k = [x_k \; x_{k-1} \; \dots \; x_{k-N+1}]^\tau$ where N is the length of the vector X_k. $\{x_k\}$ is a sequence of zero mean i.i.d Gaussian random variables. We assume that $w_k = 0$ for all $k \geq 0$. In that case

$$V_{k+1} = (I - \mu P_k X_k X_k^\dagger) V_k + X_k n_k \quad V_0 = -W_{opt,0} = W_{opt}$$

where for LMS we have $P_k = I$ and $P_k = I_k$ in case of SPU-LMS. We assume n_k is a white i.i.d. Gaussian noise with variance σ_v^2. We see that since the conditions are satisfied for theorem III.1 both LMS and SPU-LMS are exponentially stable. In fact both have the same α of decay. Therefore, conditions **F1** and **F2** are satisfied.

We rewrite $V_k = J_k^{(0)} + J_k^{(1)} + J_k^{(2)} + H_k^{(2)}$. Since, we have chosen $\bar{F}_k = E[F_k]$ we have $E[P_k X_k X_k^\dagger] = \sigma^2 I$ in the case of LMS and $\frac{1}{P}\sigma^2 I$ in the case of SPU-LMS. By [55] and Lemmas 1 and 2 we can upperbound both $J_k^{(2)}$ and $H_k^{(2)}$ by exactly the same constants for LMS and SPU-LMS. From [55] and Lemmas 1 and 2 we have that there exists some constant $C < \infty$ such that for all $\mu \in (0, \mu_0]$, we have

$$\begin{aligned}
\sup_{t\geq 0} \left| E[J_t^{(1)}(J_t^{(2)} + H_t^{(2)})^\dagger] \right| &\leq C\|X_0\|_{r(r+\delta)/\delta}\rho_r^2(v)\mu^{1/2} \\
\sup_{t\geq 0} \left| E[J_t^{(0)} H_t^{(2)}] \right| &\leq C\rho_r(v)\|X_0\|_{r(r+\delta)/\delta}\mu^{1/2}
\end{aligned}$$

Therefore, for LMS we concentrate on

$$\begin{aligned}
J_{k+1}^{(0)} &= (1-\mu\sigma^2)J_k^{(0)} + X_k n_k \\
J_{k+1}^{(1)} &= (1-\mu\sigma^2)J_k^{(1)} + \mu(\sigma^2 I - X_k X_k^\dagger)J_k^{(0)}
\end{aligned}$$

and for SPU-LMS we concentrate on

$$\begin{aligned}
J_{k+1}^{(0)} &= (1-\frac{\mu}{P}\sigma^2)J_k^{(0)} + I_k X_k n_k \\
J_{k+1}^{(1)} &= (1-\frac{\mu}{P}\sigma^2)J_k^{(1)} + \mu(\frac{\sigma^2}{P} I - I_k X_k X_k^\dagger)J_k^{(0)}
\end{aligned}$$

Solving (see Appendix .5), we obtain for LMS

$$\begin{aligned}
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(0)})^\dagger] &= \frac{\sigma_v^2}{\mu(2-\mu\sigma^2)} I \\
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(1)})^\dagger] &= 0 \\
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(2)})^\dagger] &= 0 \\
\lim_{k\to\infty} E[J_k^{(1)}(J_k^{(1)})^\dagger] &= \frac{N\sigma^2\sigma_v^2}{(2-\mu\sigma^2)^2} I \\
&= \frac{N\sigma^2\sigma_v^2}{4} I + O(\mu) I
\end{aligned}$$

which yields $\lim_{k\to\infty} E[V_k V_k^\dagger] = \frac{\sigma_v^2}{2\mu} I + \frac{N\sigma^2\sigma_v^2}{4} I + O(\mu)I$ and for SPU-LMS we obtain

$$\begin{aligned}
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(0)})^\dagger] &= \frac{\sigma_v^2}{\mu(2-\frac{\mu}{P}\sigma^2)} I \\
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(1)})^\dagger] &= 0 \\
\lim_{k\to\infty} E[J_k^{(0)}(J_k^{(2)})^\dagger] &= 0 \\
\lim_{k\to\infty} E[J_k^{(1)}(J_k^{(1)})^\dagger] &= \frac{\frac{(N+1)P-1}{P}\sigma^2\sigma_v^2}{(2-\frac{\mu}{P}\sigma^2)^2} I \\
&= \frac{\frac{(N+1)P-1}{P}\sigma^2\sigma_v^2}{4} I + O(\mu)I
\end{aligned}$$

which yields $\lim_{k\to\infty} E[V_k V_k^\dagger] = \frac{\sigma_v^2}{2\mu} I + \frac{\frac{(N+1)P-1}{P}\sigma^2\sigma_v^2}{4} I + O(\mu)I$. Therefore, we see that SPU-LMS is marginally worse than LMS.

3.7.3 Temporally Correlated Spatially Uncorrelated Array Output

In this section we consider X_k given by

$$X_k = \kappa X_{k-1} + \sqrt{1-\kappa^2}\, U_k$$

where U_k is a vector of circular Gaussian random variables with unit variance. Similar to section 3.7.2, we rewrite $V_k = J_k^{(0)} + J_k^{(1)} + J_k^{(2)} + H_k^{(2)}$. Since, we have chosen $\bar{F}_k = E[F_k]$ we have $E[P_k X_k X_k^\dagger] = I$ in the case of LMS and $\frac{1}{P}I$ in the case of SPU-LMS. Again, conditions **F1** and **F2** are satisfied because of Theorem III.1. By [55] and Lemmas 1 and 2 we can upperbound both $J_k^{(2)}$ and $H_k^{(2)}$ by exactly the same constants for LMS and SPU-LMS. From [55] and Lemmas 1 and 2 we have that there exists some constant $C < \infty$ such that for all $\mu \in (0, \mu_0]$, we have

$$\begin{aligned}
\sup_{t\geq 0} \left| E[J_t^{(1)}(J_t^{(2)} + H_t^{(2)})^\dagger] \right| &\leq C \|X_0\|_{r(r+\delta)/\delta}\, \rho_r^2(v)\, \mu^{1/2} \\
\sup_{t\geq 0} \left| E[J_t^{(0)} H_t^{(2)}] \right| &\leq C \rho_r(v) \|X_0\|_{r(r+\delta)/\delta}\, \mu^{1/2}
\end{aligned}$$

Therefore, for LMS we concentrate on

$$\begin{aligned}
J^{(0)}_{k+1} &= (1-\mu)J^{(0)}_k + X_k n_k \\
J^{(1)}_{k+1} &= (1-\mu)J^{(1)}_k + \mu(I - X_k X_k^\dagger)J^{(0)}_k
\end{aligned}$$

and for SPU-LMS we concentrate on

$$\begin{aligned}
J^{(0)}_{k+1} &= (1-\frac{\mu}{P})J^{(0)}_k + I_k X_k n_k \\
J^{(1)}_{k+1} &= (1-\frac{\mu}{P})J^{(1)}_k + \mu(\frac{1}{P}I - I_k X_k X_k^\dagger)J^{(0)}_k
\end{aligned}$$

Solving (see Appendix .6), we obtain for LMS

$$\begin{aligned}
\lim_{k\to\infty} E[J^{(0)}_k(J^{(0)}_k)^\dagger] &= \frac{\sigma_v^2}{\mu(2-\mu)}I \\
\lim_{k\to\infty} E[J^{(0)}_k(J^{(1)}_k)^\dagger] &= -\frac{\kappa^2\sigma_v^2 N}{2(1-\kappa^2)}I + O(\mu)I \\
\lim_{k\to\infty} E[J^{(0)}_k(J^{(2)}_k)^\dagger] &= \frac{\kappa^2\sigma_v^2 N}{4(1-\kappa^2)}I + O(\mu)I \\
\lim_{k\to\infty} E[J^{(1)}_k(J^{(1)}_k)^\dagger] &= \frac{(1+\kappa^2)\sigma_v^2 N}{4(1-\kappa^2)}I + O(\mu)I
\end{aligned}$$

which leads to $\lim_{k\to\infty} E[V_k V_k^\dagger] = \frac{\sigma_v^2}{2\mu}I + \frac{N\sigma_v^2}{4}I + O(\mu)I$ and for SPU-LMS we obtain

$$\begin{aligned}
\lim_{k\to\infty} E[J^{(0)}_k(J^{(0)}_k)^\dagger] &= \frac{\sigma_v^2}{\mu(2-\frac{\mu}{P})}I \\
\lim_{k\to\infty} E[J^{(0)}_k(J^{(1)}_k)^\dagger] &= -\frac{\kappa^2\sigma_v^2 N}{2(1-\kappa^2)P}I + O(\mu)I \\
\lim_{k\to\infty} E[J^{(0)}_k(J^{(2)}_k)^\dagger] &= \frac{\kappa^2\sigma_v^2 N}{4(1-\kappa^2)P}I + O(\mu)I \\
\lim_{k\to\infty} E[J^{(1)}_k(J^{(1)}_k)^\dagger] &= \frac{\sigma_v^2}{4}[\frac{N}{P}\frac{1+\kappa^2}{1-\kappa^2} + (N+1)\frac{P-1}{P}]I + O(\mu)I
\end{aligned}$$

which leads to $\lim_{k\to\infty} E[V_k V_k^\dagger] = \frac{\sigma_v^2}{2\mu}I + \frac{\sigma^2}{4}[N+1-\frac{1}{P}]I + O(\mu)I$. Again, SPU-LMS is marginally worse than LMS.

3.8 Conclusion and Future Work

We have proposed a new algorithm based on randomization of filter coefficient subsets for partial updating of filter coefficients. The conditions on step-size for convergence-in-mean and mean-square were shown to be equivalent to those of standard LMS. It was verified by theory and by simulation that LMS and SPU-LMS have similar regions of convergence. We also have shown that the Stochastic Partial Update LMS algorithm has the same performance as the Periodic LMS algorithm for stationary signals but, can have superior performance for some cyclo-stationary and deterministic signals.

The idea of random choice of subsets proposed in the chapter can be extended to include arbitrary subsets of size $\frac{N}{P}$ and not just subsets from a particular partition. No special advantage is immediately evident from this extension though.

CHAPTER IV

Capacity: Isotropically Random Rician Fading

4.1 Introduction

In this chapter, we analyze a mobile wireless link with a line of sight component (specular component) and a diffuse component (rayleigh component) both changing over time. We model the specular component as isotropically random independent of the rayleigh component. Traditionally, in a rician model the fading coefficients are modeled as Gaussian with non-zero mean. We depart from the traditional model in the sense that we model the mean (specular component) as time-varying and stochastic. The specular component is modeled as an isotropic rank one matrix with the specular component staying constant for T symbol durations and taking independent values every T^{th} instant. We establish that it is sufficient to optimize over a smaller parameter set of size $\min\{T, M\}$ of real valued magnitudes of the transmitted signals instead of $T \cdot M$ complex valued symbols. The capacity achieving signal matrix is shown to be the product of two independent matrices, a $T \times T$ isotropically random unitary matrix and a $T \times M$ real nonnegative matrix. This model is described in detail in section 4.2. In section 4.4, we derive a new lower bound on capacity. The lower bound also holds for the case of a purely rayleigh fading channel. In section 4.5 we show the utility of this bound by computing capacity regions for both rayleigh and rician fading channels.

4.2 Signal Model

The fading channel is assumed to stay constant for T symbol periods and then take on a completely independent realization, and so on. Let there be M transmit antennas and N receive antennas. We transmit a $T \times M$ signal matrix S and receive a $T \times N$ signal matrix X which are related as follows

$$X = \sqrt{\frac{\rho}{M}} SH + W \tag{4.1}$$

where the elements, w_{tn} of W are independent circularly symmetric complex Gaussian random variables with mean 0 and variance 1 ($\mathcal{CN}(0,1)$).

The only difference between the rayleigh model and the rician model considered here is in the statistics of the fading matrix H. In the case of the rayleigh model the elements h_{mn} of H are modeled as independent $\mathcal{CN}(0,1)$ random variables. Here for the rician model, the matrix H is modeled as

$$H = \sqrt{1-r}G + \sqrt{rNM}v\alpha\beta^\dagger$$

where G consists of independent $\mathcal{CN}(0,1)$ random variables, v is a real random variable such that $E[v^2] = 1$ and α and β are independent isotropically random unit magnitude vectors of length M and N, respectively. G, α and β take on independent values every T^{th} symbol period and remain unchanging in between. The parameter r ranges between zero and one, with the limits corresponding respectively to purely rayleigh or specular propagation. Irrespective of the value of r, the average variance of the components of H is equal to one, $E[\text{tr}\{HH^\dagger\}] = M \cdot N$.

An M-dimensional unit vector α is isotropically random if its probability density is invariant to pre-multiplication by an $M \times M$ deterministic unitary matrix, that is $p(\Psi\alpha) = p(\alpha)$, $\forall\Psi : \Psi^\dagger\Psi = I_M$. The isotropic density is $p(\alpha) = \frac{\Gamma(M)}{\pi^M}\delta(\alpha^\dagger\alpha - 1)$ ([48]).

The choice of $p(v)$ the density of v is not clear. $p(v)$ can be chosen to maximize the entropy of $R = v\alpha\beta^\dagger$ which is possibly the worst case scenario for the channel H. In that regard, we have the following proposition.

Proposition IV.1. *There is no distribution $p(v)$ such that the elements of $R = v\alpha\beta^\dagger$ have a joint Gaussian distribution, where α and β are isotropically random unitary vectors, and v, α, and β are mutually independent.*

Proof: Proof is by contradiction. Consider the covariance of the elements, R_{mn} of R.

$$\begin{aligned} E[R_{m_1n_1}R^*_{m_2n_2}] &= E[v^2]E[\alpha_{m_1}\alpha^*_{m_2}]E[\beta_{n_1}\beta^*_{n_2}] \\ &= E[v^2]\frac{1}{M}\delta_{m_1m_2}\delta_{n_1n_2} \end{aligned}$$

If elements of R were jointly Gaussian then they must be independent of each other which contradicts the assumption that R is of rank one. ■

From now on we will assume that v is identically equal to 1. In that case, the conditional probability density function is given by

$$p(X|S) = E_\alpha E_\beta \left[\frac{e^{-\mathrm{tr}\{[I_T+(1-r)\frac{\rho}{M}SS^\dagger]^{-1}(X-\sqrt{\rho rN}S\alpha\beta^\dagger)(X-\sqrt{\rho rN}S\alpha\beta^\dagger)^\dagger\}}}{\pi^{TN}\det^N[I_T+(1-r)\frac{\rho}{M}SS^\dagger]} \right]$$

where E_α denotes the expectation over the density of α.

Irrespective of whether the fading is rayleigh or rician, we have $p(\Psi^\dagger H) = p(H)$ for any $M \times M$ unitary matrix Ψ. In the rest of the section we will deal with H satisfying this property and refer to rayleigh and rician fading as special cases of this channel. In that case, the condition probability density of the received signals has the following properties

1. For any $T \times T$ unitary matrix Φ
$$p(\Phi X|\Phi S) = p(X|S)$$

2. For any $M \times M$ unitary matrix Ψ
$$p(X|S\Psi) = p(X|S)$$

We state some Lemmas without proof for lack of space. However, the reader is referred to [48] for proofs of similar Lemmas.

Lemma IV.1. *If $p(X|S)$ satisfies property 2 defined above then the transmitted signal can be written as ΦV where Φ is a $T \times T$ unitary matrix and V is a $T \times M$ real nonnegative diagonal matrix.*

Proof: Let the input signal matrix S have the SVD $\Phi V \Psi^\dagger$ then the channel can be written as
$$X = \sqrt{\frac{\rho}{M}}\Phi V \Psi^\dagger H + W$$
Now, consider a new signal S_1 formed by multiplying Φ and V and let X_1 be the corresponding received signal. Then
$$X_1 = \sqrt{\frac{\rho}{M}}\Phi V H + W$$
Note that X_1 and X have exactly the same statistics since $p(\Psi^\dagger H) = p(H)$. Therefore, one might as well send ΦV instead of $\Phi V \Psi^\dagger$. ∎

Corollary IV.1. *If $M > T$ then power should be transmitted only through T of the antennas.*

Proof: Note that the V in the signal transmitted, ΦV is $T \times M$. It means that $V = [V_T|\mathbf{0}]$ where V_T is $T \times T$ and $\mathbf{0}$ is $T \times (T - M)$. That means
$$X = \sqrt{\frac{\rho}{M}}\Phi V_T H_T + W$$

where H_T is the matrix of first T rows in H. That means power is transmitted via only through T transmit antennas instead of through all M. ■

In the case of rayleigh Fading however Lemma IV.1 gives rise to a stronger result ([48])

Theorem IV.1. *For any coherence interval T and any number of receiver antennas, the capacity obtained with $M > T$ transmitter antennas is the same as the capacity obtained with $M = T$ antennas.*

4.3 Properties of Capacity Achieving Signals

Marzetta and Hochwald [48] have established these results for the case of a purely rayleigh fading channel but, the proofs were based only on the fact that the conditional probability density satisfies Properties 1 and 2 as stated in section 5.2. Therefore, these results are also applicable for the case of rician fading channel discussed in this chapter. The capacity being calculated is under the power constraint $E[\mathrm{tr}\{SS^\dagger\}] \leq TM$.

Lemma IV.2. *: Suppose that S has a probability density $p_0(S)$ that generates some mutual information I_0. Then, for any $M \times M$ unitary matrix Ψ and for any $T \times T$ unitary matrix Φ, the "rotated" probability density, $p_1(S) = p_0(\Phi^\dagger S \Psi)$, also generates I_0.*

Proof: (For more details refer to the proof for Lemma 1 in [48].) The proof hinges on the fact that Jacobian determinant of any unitary transformation is one, $p(\Phi X|\Phi S) = p(X|S)$, $p(X|S\Psi^\dagger) = p(X|S)$ and $E[\mathrm{tr}\{SS^\dagger\}]$ is invariant to pre- and post-multiplication of S by unitary matrices. ■

Lemma IV.3. *: For any transmitted signal probability density $p_0(S)$, there is a* *probability density $p_1(S)$ that generates at least as much mutual information and is unchanged by rearrangements of the rows and columns of S.*

Proof: (For more details refer to the proof for Lemma 2 in [48].) From Lemma IV.2 it is evident any density obtained from the original density on S, $p_0(S)$ by pre- and post-multiplying S by any arbitrary permutation matrices P_{Tk}, $k = 1, \ldots, T!$ (there are $T!$ permutations of the rows) and P_{Ml}, $l = 1, \ldots, M!$ (there are $M!$ permutations of the columns), generates the same mutual information. Since mutual information is a concave functional of the input signal density a mixture input density, $p_1(S)$ formed by taking the average over all densities obtained by permuting S generates a mutual information at least as large as that of the original density. Note that the new mixture density satisfies the same power constraint as the original density since $E[\mathrm{tr}\{SS^\dagger\}]$ is invariant to permutations of S. ■

Corollary IV.2. *: The following power constraints all yield the same channel capacity.*

- $E|s_{tm}|^2 = 1, \quad m = 1, \ldots, M, \; t = 1, \ldots, T$
- $\frac{1}{M}\sum_{m=1}^{M} E|s_{tm}|^2 = 1, \quad t = 1, \ldots, T$
- $\frac{1}{T}\sum_{t=1}^{T} E|s_{tm}|^2 = 1, \quad m = 1, \ldots, M$
- $\frac{1}{TM}\sum_{t=1}^{T}\sum_{m=1}^{M} E|s_{tm}|^2 = 1$

Basically, the corollary tells us that tighter power constraints as shown above don't result in a decrease of capacity.

Theorem IV.2. *The signal matrix that achieves capacity can be written as $S = \Phi V$, where Φ is a $T \times T$ isotropically distributed unitary matrix, and V is an independent $T \times M$ real, nonnegative, diagonal matrix. Furthermore, we can choose the* *joint density of the diagonal elements of V to be unchanged by rearrangements of its arguments.*

Proof: Proof is similar to the proof for Theorem 2 in [48]. ∎

4.4 Capacity Upper and Lower Bounds

First we will state the following result which has already been established in [48], and follows in a straightforward manner from [64].

Theorem IV.3. *The expression for capacity when only the receiver has complete knowledge about the channel is*

$$C_H = T \cdot E \log\det\left[I_N + \frac{\rho}{M} H^\dagger H\right]$$

The following upper bound on capacity is quite well known in the literature ([16], [48]).

Proposition IV.2. *An upper bound on capacity when neither the transmitter nor the receiver has any knowledge about the channel is given by*

$$C \leq C_H = T \cdot E \log\det\left[I_N + \frac{\rho}{M} H^\dagger H\right] \tag{4.2}$$

The one thing that the literature lacks is a tractable lower bound on capacity. In this work we establish such a lower bound when no channel information is present at either the transmitter or the receiver.

Theorem IV.4. *A lower bound on capacity when neither the transmitter nor the receiver has any knowledge about the channel is given by*

$$\begin{aligned} C &\geq TE\left[\log_2\det\left(I_N + \frac{\rho}{M} H^\dagger H\right)\right] - NE\left[\log_2\det\left(I_T + \frac{\rho}{M} SS^\dagger\right)\right] && (4.3) \\ &\geq TE\left[\log_2\det\left(I_N + \frac{\rho}{M} H^\dagger H\right)\right] - NM\log_2\left(1 + \frac{\rho}{M}T\right) && (4.4) \end{aligned}$$

Proof: First note that the capacity C is given by

$$C = \max_{p(S)} [I(X;S) = \mathcal{H}(X) - \mathcal{H}(X|S)]$$

by choosing a specific distribution on S, in this case $\mathcal{CN}(0,1)$, we get a lower bound to C. Note that $\mathcal{H}(X|H) \leq \mathcal{H}(X)$ so that we obtain

$$C \geq \mathcal{H}(X|H) - \mathcal{H}(X|S) \tag{4.5}$$

Now, the expression on the right hand side in (4.5) can be rewritten as $I(X;S|H) - I(X;H|S)$ which has a very intuitive meaning. $I(X;S|H)$ is the mutual information when H is known to the receiver and $I(X;H|S)$ is the information contained in X and S about H and therefore, $I(X;H|S)$ can be viewed as the penalty for learning H in the course of decoding S from the reception of X.

Since $p(S) = \frac{1}{\pi^{TM}} \exp - \mathrm{tr}\{SS^\dagger\}$ (the elements of S are $\mathcal{CN}(0,1)$ random variables)

$$\mathcal{H}(X|H) = TE\left[\log_2\left((\pi e)^N \det\left(I_N + \frac{\rho}{M} H^\dagger H\right)\right)\right]$$

Now we turn to evaluating $\mathcal{H}(X|S)$. Note that since $p(\Psi^\dagger H) = p(H)$ for all unitary matrices Ψ, $E[h_{m_1 n_1} h^*_{m_2 n_2}] = \delta_{m_1 m_2}\delta_{n_1 n_2}$. Therefore given S, X has covariance given by

$$E[x_{t_1 n_1} x^*_{t_2 n_2} | S] = \delta_{n_1 n_2} \cdot \left[\delta_{t_1 t_2} + \frac{\rho}{M} \sum_{m=1}^{M} s_{t_1 m} s^*_{t_2 m}\right]$$

Since $\mathcal{H}(X|S)$ is bounded above by the entropy of a Gaussian with the same mean and covariance as $X|S$ we have

$$\mathcal{H}_G(X|S) \leq NE\left[\log_2\left((\pi e)^T \det\left(I_T + \frac{\rho}{M} SS^\dagger\right)\right)\right]$$

where the expectation is over the distribution of S which gives us (4.3). Note that both the expressions in the right hand side of (4.3) can be easily evaluated using Monte Carlo simulations.

However we can simplify the expression above further to obtain a looser lower bound. We use the property that for any $T \times M$ matrix S, $\det(I_T + SS^\dagger) = \det(I_M + S^\dagger S)$. This, along with the fact that $\log_2 \det(K)$ is convex cap and Jensen's inequality, gives

$$\begin{aligned} NE\left[\log_2 \det\left(I_T + \frac{\rho}{M}SS^\dagger\right)\right] &\leq N\log_2 \det\left(I_M + \frac{\rho}{M}E[S^\dagger S]\right) \qquad (4.6) \\ &= NM\log_2(1 + \frac{\rho}{M}T) \qquad (4.7) \end{aligned}$$

Therefore, we obtain (4.4). ■

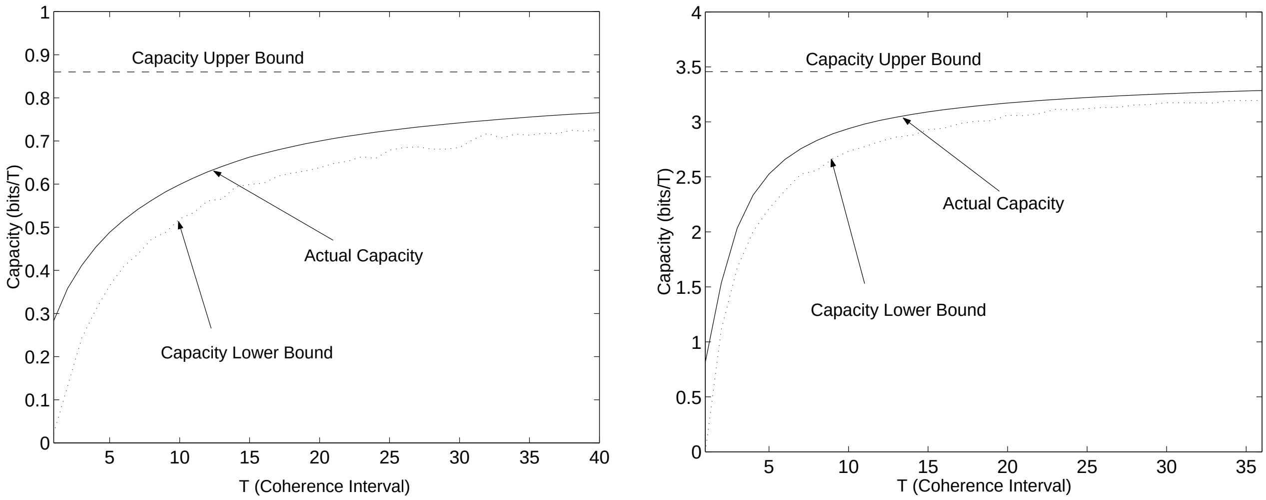
A simple improved lower bound in either (4.3) or (4.4) can be obtained by optimizing over the number of transmit and receive antennas used.

From the new lower bound we can easily prove that as $T \to \infty$ the normalized capacity, $C_n = C/T$ in bits per second per Hertz is given by $C_n = E\log\det\left[I_N + \frac{\rho}{M}H^\dagger H\right]$. This was conjectured by Marzetta and Hochwald in [48] and discussed in [16, p. 2632].

4.5 Numerical Results

First we show the utility of the lower bound by comparing it with the actual capacity curve calculated in [48] for the case of a single transmit and receive antennas in a purely rayleigh fading environment for increasing T. We plot the upper bound, the lower bound derived in this work and the actual capacity in figure 4.1. The utility of this bound is clearly evident from how well the lower bound follows the actual capacity.

In different simulations, using the upper and lower bounds, we found that the capacity variations as the channel moves from a purely rayleigh to a purely rician channel become more and more visible as we move to higher SNR regimes and larger number of transmit and receive antennas. Examples of this behaviour are shown in figures 4.2 and 4.3. Note that for the special case of $M = N = 1$ the capacity curves



(a) $\rho = 0dB$ (b) $\rho = 12dB$

Figure 4.1: Capacity and Capacity lower bound for $M = N = 1$ as $T \to \infty$

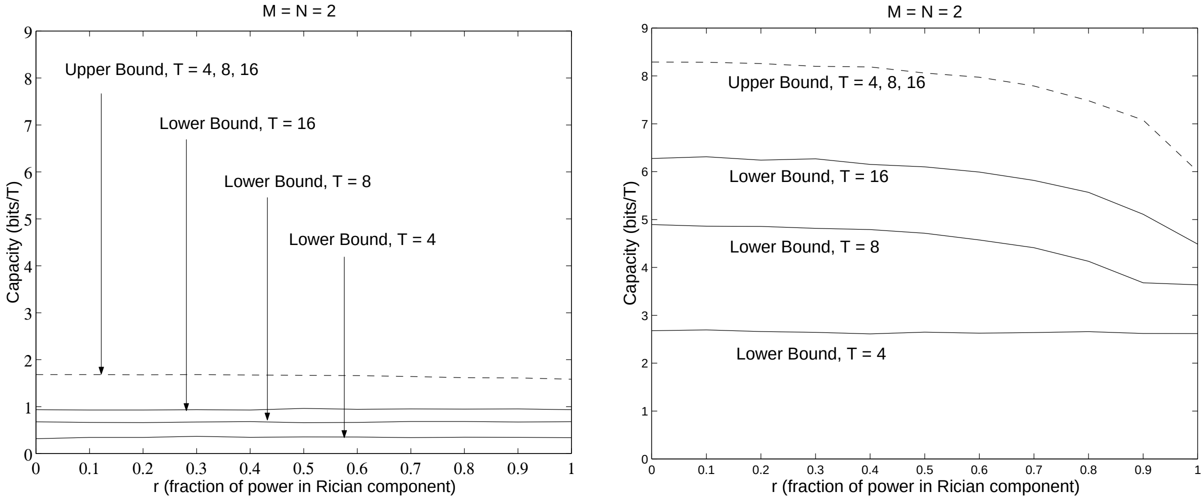
have an upward trend as r varies from 0 to 1. For all other cases the capacity curves have a downward trend, where the reduction in capacity becomes significant only for $r > 1/2$.

4.6 Analysis for High SNR

The techniques developed in [74] can be easily applied to the model in this section to conclude that the number of degrees of freedom is given by $M^*(T - M^*)$ where $M^* = \min\{M, N, T/2\}$. All the Theorems developed in [74] can be easily translated to the case of isotropically random rican fading.

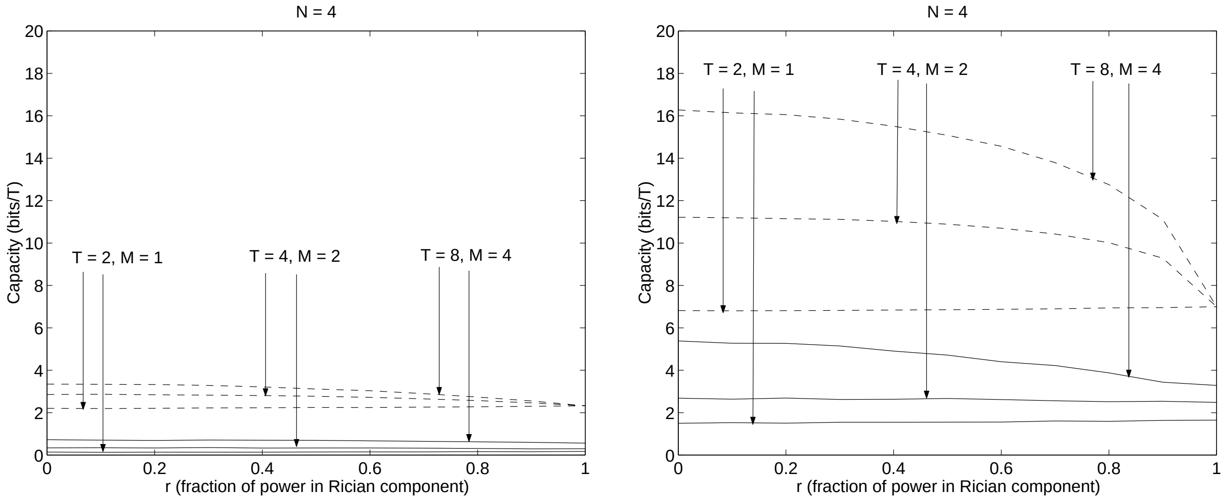
4.7 Conclusions

We have analyzed the case of rician fading channel. We have proposed a tractable model for rician fading with a stochastic isotropic specular component of rank one. Using this model we were able to establish most of the results obtained in the case of rayleigh fading. We were also able to derive a lower bound on capacity for any



(a) $\rho = 0dB$ (b) $\rho = 15dB$

Figure 4.2: Capacity upper and lower bounds as the channel moves from purely rayleigh to purely rician fading



(a) $\rho = 0dB$ (b) $\rho = 15dB$

Figure 4.3: Capacity upper and lower bounds as the channel moves from purely rayleigh to purely rician fading

number of transmit and receive antennas. This lower bound is also applicable to the case of rayleigh fading. For single transmit and receive antennas, the rician channel gives superior performance with respect to the rayleigh channel. For multiple antenna channels, rician fading tends to degrade the performance. Our numerical results indicate that the rayleigh model is surprisingly robust, however: under our rician model, up to half of the received energy can arrive via the specular component without significant reduction in capacity compared with the purely rayleigh case.

CHAPTER V

Min-Capacity: Rician Fading, Unknown Static Specular Component

5.1 Introduction

In the previous chapter we considered a non-conventional model for the case of rician fading where the fading channel consists of a rayleigh component, modeled as in [48] and an independent rank-one isotropically distributed specular component. The fading channel was assumed to remain constant over a block of T consecutive symbol periods but take a completely independent realization over each block. We derived similar results on optimal capacity achieving signal structures as in [48]. We also established a lower bound to capacity that can be easily extended to the model considered in this chapter. The model described in the previous chapter is applicable to a mobile-wireless link where both the direct line of sight component (specular component) and the diffuse component (rayleigh component) change with time.

In this chapter, we consider a quasi-static rician model where the specular component is non-changing while the rayleigh component is varying over time. This model is similar to the traditional model where the specular component is deterministic and persists over time. The model is applicable to the case where the transmitter and receiver are fixed in space or are in motion but sufficiently far apart so that the specular component is practically constant while the diffuse multipath component changes rapidly. If the specular component were known to both the transmitter and the receiver then the signaling scheme as well as the capacity would depend on the specific realization of the specular component. We however deal with the case when the transmitter has no knowledge about the specular component. In this scenario the transmitter can either maximize the worst case rate over the ensemble of values that the specular component can take or maximize the average rate by establishing a prior distribution on the ensemble. We address both approaches in this chapter.

Similarly to [19] the specular component is an outer product of two vectors of unit magnitude that are non-changing and unknown to the transmitter but known to the receiver. The difference between our approach and that of [19] is that in [19] the authors consider the channel to be known completely to the receiver. We assume that the receiver's extent of knowledge about the channel is limited to the specular component. That is, the receiver has no knowledge about the rayleigh component of the model. Considering the absence of knowledge at the transmitter it is important to design a signal scheme that guarantees the largest overall rate for communication irrespective of the value of the specular component. This is formulated as the problem of determining the worst case capacity in section 5.2. In section 5.5 we consider the average capacity instead of worst case capacity and show that both formulations imply the same optimal signal structure and the same maximum possible rate.

5.2 Signal Model and Problem Formulation

Let there be M transmit antennas and N receive antennas. We assume that the fading coefficients remain constant over a block of T consecutive symbol periods but are independent from block to block. Keeping that in mind, we model the channel as carrying a $T \times M$ signal matrix S over a $M \times N$ MIMO channel H, producing X at the receiver according to the model:

$$X = \sqrt{\frac{\rho}{M}} SH + W \tag{5.1}$$

where the elements, w_{tn} of W are independent circular complex Gaussian random variables with mean 0 and variance 1 ($\mathcal{CN}(0, 1)$).

The MIMO rician model for the matrix H is $H = \sqrt{1-r}G + \sqrt{rNM}\alpha\beta^\dagger$ where G consists of independent $\mathcal{CN}(0, 1)$ random variables and α and β are deterministic vectors of length M and N, respectively, such that $\alpha^\dagger\alpha = 1$ and $\beta^\dagger\beta = 1$. We assume α and β are known to the receiver. Since the receiver is free to apply a co-ordinate transformation by post multiplying X by a unitary matrix, without loss of generality we can take β to be identically equal to $[1\ 0\ \ldots\ 0]^T$. We will sometimes write H as H_α to highlight the dependence of H on α. G remains constant for T symbol periods and takes on a completely independent realization every T^{th} symbol period.

The problem we are investigating is to find the distribution $p^m(S)$ that attains the maximum in the following maximization defining the worst case channel capacity

$$C^m = \max_{p(S)} I^m(X; S) = \max_{p(S)} \inf_{\alpha \in A} I^\alpha(X; S)$$

and also to find the maximum value, C^m.

$$I^\alpha(X; S) = \int p(S)p(X|S, \alpha\beta^\dagger) \log \frac{p(X|S, \alpha\beta^\dagger)}{\int p(S)p(X|S, \alpha\beta^\dagger)\ dS}\ dSdX$$

is the mutual information between X and S when the specular component is given by $\alpha\beta^\dagger$ and $A \stackrel{\text{def}}{=} \{\alpha : \alpha \in \mathbb{C}^M \text{ and } \alpha^\dagger\alpha = 1\}$. Since A is compact the "inf" in the problem can be replaced by "min". For convenience we will refer to $I^m(X; S)$ as the *min-mutual information* and C^m as *min-capacity*.

The *min-capacity* defined above is just the capacity of a *compound channel*. For more information on the concept of compound channels and worst case capacity please refer to [7, chapter 5, pp. 172-178]. It is shown that [7, Prob. 13, p. 183] *min-capacity* doesn't depend on the receiver's knowledge of the channel. Hence, it is not necessary for us to assume that the specular component is known to the receiver. However, we do so because it facilitates easier description of *min-capacity* in terms of the conditional probability distribution $p(X|S)$.

5.3 Capacity Upper and Lower Bounds

Theorem V.1. *Min-capacity, C_H^m when the channel matrix H is known to the receiver but not to the transmitter is given by*

$$C_H^m = TE \log\det\left[I_N + \frac{\rho}{M} H_{e_1}^\dagger H_{e_1}\right] \tag{5.2}$$

where $e_1 = [1\ 0\ \ldots\ 0]^T$ is a unit vector in $\mathbb{C}^M$. Note that e_1 in (5.2) can be replaced by any $\alpha \in A$ without changing the answer. *Proof:* First we note that for $T > 1$, given H the channel is memoryless and hence the columns of the input signal matrix S are independent of each other. That means the mutual information $I^\alpha(X;S) = \sum_{i=1}^T I^\alpha(X_i;S_i)$ where X_i and S_i denote the i^{th} row of X and S, respectively. The maximization over each term can be done separately and it is easily seen that each term will be maximized individually for the same density on S_i. That is $p(S_i) = p(S_j)$ for $i \neq j$ and $\max_{p(S)} I^\alpha(X;S) = T \max_{p(S_1)} I^\alpha(X_1;S_1)$. Therefore, WLOG we assume $T = 1$.

Given H the channel is an AWGN channel therefore, capacity is attained by Gaussian signal vectors. Let Λ_S be the input signal covariance. Since the transmitter doesn't know α, Λ_S can not depend on α and the *min-capacity* is given by

$$\max_{\Lambda_S:\mathrm{tr}\{\Lambda_S\}\leq M} \mathcal{F}(\Lambda_S) = \max_{\Lambda_S:\mathrm{tr}\{\Lambda_S\}\leq M} \min_{\alpha\in A} E \log\det\left[I_N + \frac{\rho}{M} H_\alpha^\dagger \Lambda_S H_\alpha\right] \tag{5.3}$$

where $\mathcal{F}(\Lambda_S)$ is implicitly defined in an obvious manner. First note that $\mathcal{F}(\Lambda_S)$ in (5.3) is a concave function of Λ_S (This follows from the fact that $\log\det K$ is a concave function of K). Also, $\mathcal{F}(\Psi^\dagger\Lambda_S\Psi) = \mathcal{F}(\Lambda_S)$ for any $M \times M$ $\Psi : \Psi^\dagger\Psi = I_M$ since $\Psi^\dagger\alpha \in A$ for any $\alpha \in A$ and G has i.i.d. zero mean complex Gaussian entries. Let $Q^\dagger DQ$ be the SVD of Λ_S then we have $\mathcal{F}(D) = \mathcal{F}(Q^\dagger DQ) = \mathcal{F}(\Lambda_S)$. Therefore, we can choose Λ_S to be diagonal. Moreover, $\mathcal{F}(P_k^\dagger\Lambda_S P_k) = \mathcal{F}(\Lambda_S)$ for any permutation matrix P_k, $k = 1, \ldots, M!$. Therefore, if we choose $\Lambda'_S = \frac{1}{M!}\sum_{k=1}^{M!} P_k^\dagger\Lambda_S P_k$ then by concavity and Jensen's inequality we have

$$\mathcal{F}(\Lambda'_S) \geq \sum_{k=1}^{M!} \mathcal{F}(P_k^\dagger\Lambda_S P_k) = \mathcal{F}(\Lambda_S)$$

Therefore, we conclude that the maximizing input signal covariance Λ_S is a multiple of the identity matrix. It is quite obvious to see that to maximize the expression in (5.3) we need to choose $\mathrm{tr}\{\Lambda_S\} = M$ or $\Lambda_S = I_M$ and since $E\log\det[I_N + \frac{\rho}{M}H_{\alpha_1}^\dagger H_{\alpha_1}] = E\log\det[I_N + \frac{\rho}{M}H_{\alpha_2}^\dagger H_{\alpha_2}]$ for any $\alpha_1, \alpha_2 \in A$, (5.2) easily follows. ■

By the data processing theorem additional information at the receiver doesn't decrease capacity. Therefore:

Proposition V.1. *An upper bound on the channel min-capacity when neither the transmitter nor the receiver has any knowledge about the channel is given by*

$$C^m \leq T \cdot E\log\det\left[I_N + \frac{\rho}{M}H_{e_1}^\dagger H_{e_1}\right] \tag{5.4}$$

Now, we establish a lower bound.

Proposition V.2. *A lower bound on min-capacity when the transmitter has no knowledge about H and the receiver has no knowledge about G is given by*

$$\begin{aligned} C^m &\geq C_H^m - NE\left[\log_2\det\left(I_T + (1-r)\frac{\rho}{M}SS^\dagger\right)\right] & (5.5)\\ &\geq C_H^m - NM\log_2(1 + (1-r)\frac{\rho}{M}T) & (5.6)\end{aligned}$$

Proof: Proof is similar to that of Theorem IV.4 and won't be repeated here. ■

Combining (5.4) and (4.4) gives us the following

Corollary V.1. *The normalized min-capacity, $C_n^m = C^m/T$ in bits per second per Hertz as $T \to \infty$ is given by*

$$C_n^m = E \log\det\left[I_N + \frac{\rho}{M} H_{e_1}^\dagger H_{e_1}\right]$$

Note that this is same as the capacity when the receiver knows H, so that as $T \to \infty$ perfect channel estimation can be performed.

5.4 Properties of Capacity Achieving Signals

In this section, we derive the optimum signal structure for achieving *min-capacity*. The optimization is being done under the power constraint $E[\text{tr}\{SS^\dagger\}] \le TM$.

Lemma V.1. *$I^m(X;S)$ as a functional of $p(S)$ is concave in $p(S)$.*

Proof: First we note that $I^\alpha(X;S)$ is a concave functional of $p(S)$ for every $\alpha \in A$. Let $I^m(X;S)_{p(S)}$ denote $I^m(X;S)$ evaluated using $p(S)$ as the signal density. Then,

$$\begin{aligned}
I^m(X;S)_{\delta p_1(S)+(1-\delta)p_2(S)} &= \min_{\alpha\in A} I^\alpha(X;S)_{\delta p_1(S)+(1-\delta)p_2(S)} \\
&\ge \min_{\alpha\in A}[\delta I^\alpha(X;S)_{p_1(S)} + (1-\delta)I^\alpha(X;S)_{p_2(S)}] \\
&\ge \delta \min_{\alpha\in A} I^\alpha(X;S)_{p_1(S)} + (1-\delta)\min_{\alpha\in A} I^\alpha(X;S)_{p_2(S)} \\
&= \delta I^m(X;S)_{p_1(S)} + (1-\delta)I^m(X;S)_{p_2(S)}
\end{aligned}$$

■

Lemma V.2. *For any $T\times T$ unitary matrix Φ and any $M \times M$ unitary matrix Ψ, if $p(S)$ generates $I^m(X;S)$ then so does $p(\Phi S \Psi^\dagger)$.*

Proof: 1) Note that $p(\Phi X|\Phi S) = p(X|S)$, therefore $I^\alpha(X;\Phi S) = I^\alpha(X;S)$ for any $T \times T$ unitary matrix Φ and all $\alpha \in A$.

2) We have, $\Psi\alpha \in A$ for any $\alpha \in A$ and any $M \times M$ unitary matrix Ψ. Therefore, if $I^m(X;S)$ achieves its minimum value at $\alpha_0 \in A$ then $I^m(X;S\Psi^\dagger)$ achieves its

minimum value at $\Psi\alpha_0$ because $I^{\alpha}(X;S) = I^{\Psi\alpha}(X;S\Psi^{\dagger})$ for $\alpha \in A$ and Ψ an $M \times M$ unitary matrix.

Combining 1) and 2) we get the lemma. ∎

Lemma V.3. *The min-capacity achieving signal distribution, $p(S)$ is unchanged by any pre- and post- multiplication of S by unitary matrices of appropriate dimensions.*

Proof: We will show that for any signal density $p_0(S)$ generating *min-mutual information* I_0^m there exists a density $p_1(S)$ generating $I_1^m \geq I_0^m$ such that $p_1(S)$ is invariant to pre- and post- multiplication of S by unitary matrices of appropriate dimensions. By Lemma V.2, for any pair of permutation matrices, Φ $(T \times T)$ and Ψ $(M \times M)$ $p_0(\Phi S \Psi^{\dagger})$ generates the same *min-mutual information* as $p(S)$. Define $u_T(\Phi)$ to be the isotropically random unitary density function of a $T \times T$ unitary matrix Φ. Similarly define $u_M(\Psi)$. Let $p_1(S)$ be a mixture density given as follows

$$p_1(S) = \int\int p_0(\Phi S \Psi^{\dagger}) u(\Phi) u(\Psi) \; d\Phi d\Psi$$

It is easy to see that $p_1(S)$ is invariant to any pre- and post- multiplication of S by unitary matrices and if I_1^m is the *min-mutual information* generated by $p_1(S)$ then from Jensen's inequality and concavity of $I^m(X;S)$ we have $I_1^m \geq I_0^m$. ∎

Corollary V.2. *$p^m(S)$, the optimal min-capacity achieving signal density lies in $\mathcal{P} = \cup_{I>0} \mathcal{P}_I$ where*

$$\mathcal{P}_I = \{p(S) : I^{\alpha}(X;S) = I \quad \forall \alpha \in A\} \tag{5.7}$$

Proof: Follows immediately from Lemma V.3 because any signal density that is invariant to pre- and post- multiplication of S by unitary matrices generates the same mutual information $I^{\alpha}(X;S)$ irrespective of the value of α. ∎

Corollary V.3. *The min-capacity achieving signal distribution $p(S)$ is unchanged by rearrangements of elements in S.*

Corollary V.4. *: The following power constraints all yield the same channel min-capacity.*

- $E|s_{tm}|^2 = 1, \quad m = 1, \dots, M, \; t = 1, \dots, T$
- $\frac{1}{M}\sum_{m=1}^{M} E|s_{tm}|^2 = 1, \quad t = 1, \dots, T$
- $\frac{1}{T}\sum_{t=1}^{T} E|s_{tm}|^2 = 1, \quad m = 1, \dots, M$
- $\frac{1}{TM}\sum_{t=1}^{T}\sum_{m=1}^{M} E|s_{tm}|^2 = 1$

Theorem V.2. *The signal matrix that achieves min-capacity can be written as $S = \Phi V \Psi^\dagger$, where Φ and Ψ are $T \times T$ and $M \times M$ isotropically distributed matrices independent of each other, and V is a $T \times M$ real, nonnegative, diagonal matrix, independent of both Φ and Ψ.*

Proof: From the singular value decomposition (SVD) we can write $S = \Phi V \Psi^\dagger$, where Φ is a $T \times T$ unitary matrix, V is a $T \times M$ nonnegative real diagonal matrix, and Ψ is an $M \times M$ unitary matrix. In general, Φ, V and Ψ are jointly distributed. Suppose S has probability density $p_0(S)$ that generates *min-mutual information* I_0^m. Let Θ_1 and Θ_2 be isotropically distributed unitary matrices of size $T \times T$ and $M \times M$ independent of S and of each other. Define a new signal $S_1 = \Theta_1 S \Theta_2^\dagger$, generating *min-mutual information* I_1^m. Now conditioned on Θ_1 and Θ_2, the *min-mutual information* generated by S_1 equals I_0^m. From the concavity of the *min-mutual information* as a functional of $p(S)$, and Jensen's inequality we conclude that $I_1^m \geq I_0^m$.

Since Θ_1 and Θ_2 are isotropically distributed $\Theta_1\Phi$ and $\Theta_2\Psi$ are also isotropically distributed when conditioned on Φ and Ψ respectively. This means that both $\Theta_1\Phi$

and $\Theta_2\Psi$ are isotropically distributed making them independent of Φ, V and Ψ. Therefore, S_1 is equal to the product of three independent matrices, a $T \times T$ unitary matrix Φ, a $T \times M$ real nonnegative matrix V and an $M \times M$ unitary matrix Ψ.

Now, we will show that the density $p(V)$ on V is unchanged by rearrangements of diagonal entries of V. There are $\min\{M!, T!\}$ ways of arranging the diagonal entries of V. This can be accomplished by pre- and post-multiplying V by appropriate permutation matrices, P_{Tk} and P_{Mk}, $k = 1, \ldots, \min\{M!, T!\}$. The permutation doesn't change the *min-mutual information* because ΦP_{Tk} and ΨP_{Mk} have the same density functions as Φ and Ψ. By choosing an equally weighted mixture density for V, involving all $\min\{M!, T!\}$ arrangements we obtain a higher value of *min-mutual information* because of concavity and Jensen's inequality. This new density is invariant to the rearrangements of the diagonal elements of V. ■

5.5 Average Capacity Criterion

In this case we maximize $I_E(X;S) = E_\alpha[I^\alpha(X;S)]$, where I^α is as defined earlier and E_α denotes expectation over $\alpha \in A$ under the assumption that all α are equally likely. That is, under the assumption that α is unchanging over time, isotropically random and known to the receiver. Please note that this differs from the model considered in [29] where the authors consider the case of a piecewise constant, time varying, i.i.d. specular component.

Therefore, the problem can be stated as finding $p_E(S)$ the probability density function on the input signal S that achieves the following maximization

$$C_E = \max_{p(S)} E_\alpha[I^\alpha(X;S)] \tag{5.8}$$

and also to find the value C_E. We will refer to $I_E(X;S)$ as *avg-mutual information* and C_E as *avg-capacity*.

We will show that the signal density $p^m(S)$ that attains C^m also attains C_E. For that we need to establish the following Lemmas.

Lemma V.4. *$I_E(X;S)$ is a concave functional of the signal density $p(S)$*

Proof: First we note that $I^\alpha(X;S)$ is a concave functional of $p(S)$ for every $\alpha \in A$. Then,

$$\begin{aligned} I_E(X;S)_{\delta p_1(S)+(1-\delta)p_2(S)} &= E_\alpha[I^\alpha(X;S)_{\delta p_1(S)+(1-\delta)p_2(S)}] \\ &\geq E_\alpha[\delta I^\alpha(X;S)_{p_1(S)} + (1-\delta)I^\alpha(X;S)_{p_2(S)}] \\ &= \delta E_\alpha[I^\alpha(X;S)_{p_1(S)}] + (1-\delta)E_\alpha[I^\alpha(X;S)_{p_2(S)}] \\ &= \delta I_E(X;S)_{p_1(S)} + (1-\delta)I_E(X;S)_{p_2(S)} \end{aligned}$$

■

Lemma V.5. *For any $T \times T$ unitary matrix Φ and any $M \times M$ unitary matrix Ψ, if $p(S)$ generates $I_E(X;S)$ then so does $p(\Phi S \Psi^\dagger)$.*

Proof: We want to show if $p(S)$ generates $I_E(X;S)$ then so does $p(\Phi S \Psi^\dagger)$. Now since the density function of α, $p(\alpha) = \frac{\Gamma(M)}{\pi^M}\delta(\alpha^\dagger\alpha - 1)$ we have

$$I_E(X;S) = \frac{\pi^M}{\Gamma(M)} \int I^\alpha(X;S)\, d\alpha$$

Note that $I^\alpha(X;\Phi S) = I^\alpha(X;S)$ and $I^{\Psi\alpha}(X;S\Psi^\dagger) = I^\alpha(X;S) \Rightarrow I^{\Psi^\dagger\alpha}(X;S) = I^\alpha(X;S\Psi^\dagger)$. Therefore,

$$\begin{aligned} I'_E(X;S) &= \frac{\pi^M}{\Gamma(M)} \int I^\alpha(X;\Phi S\Psi^\dagger)\, d\alpha \\ &= \frac{\pi^M}{\Gamma(M)} \int I^\alpha(X;S\Psi^\dagger)\, d\alpha \\ &= \frac{\pi^M}{\Gamma(M)} \int I^{\Psi^\dagger\alpha}(X;S)\, d\alpha \\ &= \frac{\pi^M}{\Gamma(M)} \int I^\omega(X;S)\, d\omega \\ &= I_E(X;S) \end{aligned}$$

where the last two equalities follow from the transformation $\omega = \Psi^\dagger \alpha$ and the fact the Jacobian of the transformation is equal to 1. ∎

Lemma V.6. *The avg-capacity achieving signal distribution, $p(S)$ is unchanged by any pre- and post- multiplication of S by unitary matrices of appropriate dimensions.*

Corollary V.5. *$p^m(S)$, the optimal avg-capacity achieving signal density lies in $\mathcal{P} = \cup_{I>0}\mathcal{P}_I$ where $\mathcal{P}_I$ is as defined in (5.7).*

Based on the last corollary we conclude that for a given $p(S)$ in $\mathcal{P}$ we have $I^m(X;S) = \min_{\alpha \in A} I^\alpha(X;S) = E_\alpha[I^\alpha(X;S)] = I_E(X;S)$. Therefore, the maximizing densities for C_E and C^m are the same and also $C_E = C^m$. Therefore, designing the signal constellation with the objective of maximizing the worst case performance is not more pessimistic than maximizing the average performance.

5.6 Conclusions

We have proposed another tractable model for rician fading channel different from the one in chapter IV but, along the lines of [19]. We were able to analyze this channel and derive some interesting results on optimal signal structure. We were also able to show that the optimization effort is over a much smaller set of parameters than the set originally started with. We were also able to derive a lower bound that is very useful because capacity computations in general are not tractable.

Finally, we were able to show that the approach of maximizing the worst case scenario is not overly pessimistic in the sense that the signal density maximizing the worst case performance also maximizes the average performance and the capacity value in both formulations turns out to be the same.

CHAPTER VI

Capacity: Rician Fading, Known Static Specular Component

6.1 Introduction

In the previous chapters we had not considered the case of the standard rician model where the specular component is deterministic and known to both the transmitter and the receiver. In chapter IV, we considered a model where the specular component is rank-one and has an isotropic distribution. In chapter V, we considered a model where the specular component is deterministic and static but unknown to the transmitter. Both these models led to a tractable analysis of the capacity of a rician fading model. Also, the specular component considered in both the chapters was of rank one.

In this chapter, we analyze the standard rician fading model for capacity. Throughout the chapter, we assume that the specular component is deterministic and is known to both the transmitter and the receiver. The specular component in this chapter is of general rank except in section 6.2 where it is restricted to be of rank one. The rayleigh component is never known to the transmitter. There are some cases we consider where the receiver has complete knowledge of the channel. In such cases, the receiver has knowledge about the rayleigh as well as the specular component whereas the transmitter has knowledge only about the specular component. The capacity when the receiver has complete knowledge about the channel will be referred to as *coherent capacity* and the capacity when the receiver has no knowledge about the rayleigh component will be referred to as *non-coherent capacity*. This chapter is organized as follows. In section 6.2 we deal with the special case of a rank-one specular component with the characterization of coherent capacity in section 6.2.1. The general case of no restrictions on the rank of the specular component is dealt with in section 6.3. The coherent capacity for this case is considered in section 6.3.1, the non-coherent capacity for low SNR in section 6.3.3 and the non-coherent capacity for high SNR in section 6.3.4. Finally, in section 6.4 we consider the performance of a rician channel in terms of capacity when pilot symbol based training is used in the communication system.

6.2 Rank-one Specular Component

The model for the rician fading channel being considered here is

$$X = \sqrt{\frac{\rho}{M}} SH + W \tag{6.1}$$

where X is the $T \times N$ matrix of received signals, H is the $M \times N$ matrix of propagation coefficients, S is the $T \times M$ matrix of transmitted signals and W is the $T \times N$ matrix of additive noise components and ρ is the expected signal to noise ratio at the receivers.

For a deterministic rank one rician channel H is defined as $H = \sqrt{1-r}G + \sqrt{rNM}H_m$ where G consists of independent $\mathcal{CN}(0,1)$ random variables, H_m is $M \times N$ deterministic matrix of rank one such that $\mathrm{tr}\{H_m^\dagger H_m\} = 1$ and r is a number lying between 0 and 1 known to both the transmitter and the receiver. Therefore $H_m = \alpha\beta^\dagger$ where α is a length M vector and β is a length N vector such that $\alpha^\dagger\alpha = \beta^\dagger\beta = 1$. It is assumed that H_m is known to both the receiver and the transmitter. Therefore, WLOG we can assume that

$$H_m = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [1\ 0 \ldots\ 0] \tag{6.2}$$

where the column and row vectors are of appropriate lengths.

In this case, the conditional probability density function is given by,

$$p(X|S) = \frac{e^{-\mathrm{tr}\{[I_T+(1-r)(\rho/M)SS^\dagger]^{-1}(X-\sqrt{rNM}SH_m)(X-\sqrt{rNM}SH_m)^\dagger\}}}{\pi^{TN}\det^N[I_T+(1-r)(\rho/M)SS^\dagger]}$$

The conditional probability density enjoys the following properties

1. For any $T \times T$ unitary matrix ϕ

$$p(\phi X|\phi S) = p(X|S)$$

2. For any $(M-1)\times(M-1)$ unitary matrix ψ

$$p(X|S\Psi) = p(X|S)$$

   where

$$\Psi = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^\dagger & \psi \end{bmatrix} \tag{6.3}$$

6.2.1 Coherent Capacity

The mutual information expression under the known H case has already been derived in [21]. For known H the capacity achieving signal S is Gaussian independent from time instant to time instant. For this signal the expression is

$$I(X;S|H) = T \cdot E \log\det\left[I_N + \frac{\rho}{M}H^\dagger \Lambda H\right]$$

where $\Lambda = E[S_t^\tau S_t^*]$ for $t = 1, \ldots, T$ where S_t is the t^{th} row of the $T \times M$ matrix S. S_t^τ denotes the transpose of S_t and $S_t^* \stackrel{\text{def}}{=} (S_t^\tau)^\dagger$.

Theorem VI.1. *The capacity achieving signal covariance matrix is of the form*

$$\Lambda = \begin{bmatrix} M-(M-1)d & L \\ L^\dagger & D \end{bmatrix}$$

where d is a positive real number such that $0 \leq d \leq M/(M-1)$ and D is the product of d and the identity matrix of dimension $(M-1)\times(M-1)$ and L is the product of a complex number, l ($|l| \leq \sqrt{(\frac{M}{M-1}-d)d}$) and an all ones row vector of length $M-1$.

Proof: This proof is a modification of the proof in [64]. Using the property that $\Psi^\dagger H$ has the same distribution as H where Ψ is of the form given in 6.3 we conclude that

$$T \cdot E \log\det\left[I_N + \frac{\rho}{M}H^\dagger \Lambda H\right] = T \cdot E \log\det\left[I_N + \frac{\rho}{M}H^\dagger \Psi\Lambda\Psi^\dagger H\right]$$

If Λ is written as

$$\Lambda = \begin{bmatrix} c & A \\ A^\dagger & B \end{bmatrix}$$

where c is a positive number such that $c \geq A^\dagger B^{-1} A$ (to ensure positive semi-definiteness of the covariance matrix Λ), A is a column vector of length $M-1$ and B is a positive definite matrix of size $(M-1)\times(M-1)$. Then

$$\Psi\Lambda\Psi^\dagger = \begin{bmatrix} c & A\psi^\dagger \\ \psi A^\dagger & \psi B \psi^\dagger \end{bmatrix}$$

Since $B = UDU^\dagger$ where D is a diagonal matrix and U is a unitary matrix of size $(M-1)\times(M-1)$, choosing $\psi = \Pi U$ where Π is a permutation matrix, we obtain that

$$T \cdot E \log\det\left[I_N + \frac{\rho}{M} H^\dagger \Lambda H\right] = T \cdot E \log\det\left[I_N + \frac{\rho}{M} H^\dagger \Lambda^\Pi H\right]$$

where

$$\Lambda^\Pi = \begin{bmatrix} c & AU^\dagger\Pi^\dagger \\ \Pi U A^\dagger & \Pi D \Pi^\dagger \end{bmatrix}$$

Since $\log\det$ is a concave (convex cap) function we have

$$\begin{aligned} T \cdot E \log\det\left[I_N + \frac{\rho}{M} H^\dagger\ \overline{\Lambda^\Pi}\ H\right] &\geq T \cdot \frac{1}{(M-1)!} \sum_\Pi E \log\det\left[I_N + \frac{\rho}{M} H^\dagger \Lambda^\Pi H\right] \\ &= I(X;S) \end{aligned}$$

where $\overline{\Lambda^\Pi} = \frac{1}{(M-1)!}\sum_\Pi \Lambda^\Pi$ and the summation is over all possible permutation matrices which are $(M-1)!$ in number. Therefore, the capacity achieving Λ is given by $\frac{1}{(M-1)!}\sum_\Pi \Lambda^\Pi$ and is of the form

$$\Lambda = \begin{bmatrix} c & L \\ L^\dagger & D \end{bmatrix}$$

Now, the capacity achieving signal matrix has to satisfy $\mathrm{tr}\{\Lambda\} = M$ because if $\mathrm{tr}\{\Lambda\} < M$ then we can always multiply Λ by a constant greater than 1 which can only increase the mutual information. Therefore, $c = M - (M-1)d$. And since $c \geq L^\dagger D^{-1} L$ this implies $M - (M-1)d \geq \frac{(M-1)|l|^2}{d}$ and we obtain the desired signal covariance structure. ■

The problem is to choose l and d to maximize $I(X;S)$. This problem has an analytical solution for the special cases of 1) $r = 0$ for which $d = 1$ and $l = 0$ and 2) $r = 1$ for which $d = l = 0$. The optimization problem can be solved by using the method of steepest descent (See Appendix .7). Results for $\rho = 100, 10, 1, 0.1$ are shown in Figure 6.1. The optimum values of l for different values of ρ turned out to be zero. As can be seen from the plot the optimum value of d stays close to 1 for high

SNR and close to 0 for low SNR. That is, the optimum covariance matrix is close to an identity matrix for high SNR. For low SNR, all the energy is concentrated in the direction of the specular component. These observations are proven in section 6.3.1.

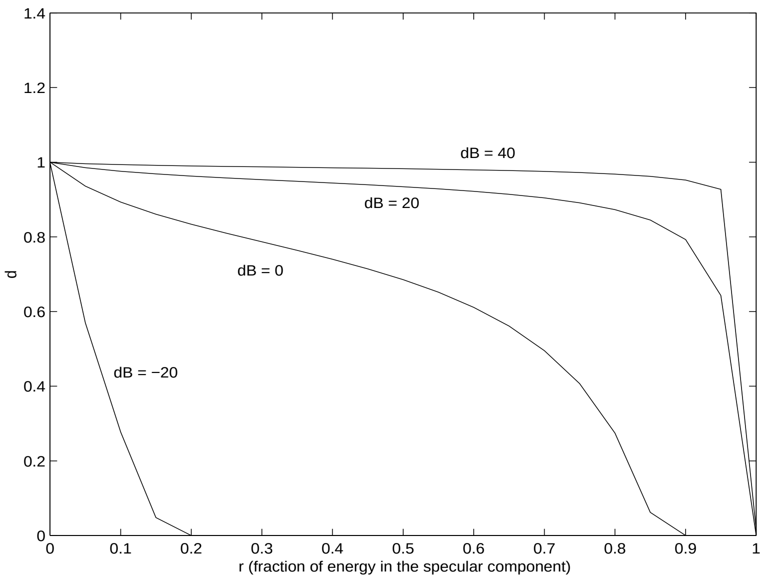


Figure 6.1: Optimum d as a function of r for different values of ρ

6.3 General Rank Specular Component

In this case the channel matrix can be written as $H = \sqrt{1-r}G + \sqrt{r}H_m$ where G is the regular rayleigh Fading channel and H_m is a deterministic matrix such that $\mathrm{tr}\{H_m H_m^\dagger\} = MN$ with no restriction on its rank. Since H_m is known to both the transmitter and receiver, WLOG, we can assume H_m to be an $M \times N$ diagonal matrix with positive real entries.

6.3.1 Coherent Capacity

For high SNR, we show that the capacity achieving signal structure basically ignores the specular component. There is no preference given to the channel directions in the specular component.

Proposition VI.1. *For high SNR, capacity, C_H in the known H case is attained by a diagonal signal covariance matrix when $M \leq N$ and*

$$C_H \approx T \cdot E \log\det[\frac{\rho}{M}H^\dagger H]$$

Proof: The expression for capacity, C_H is

$$C_H = T \cdot E \log\det[I_N + \frac{\rho}{M}H^\dagger \Lambda H]$$

Consider the following expression for a specific realization of the matrix H with SVD given by $H = \Phi\Sigma\Psi^\dagger$ then

$$\log\det[I_N + \frac{\rho}{M}H^\dagger \Lambda H] = \log\det[I_N + \frac{\rho}{M}\Sigma^\dagger\Phi^\dagger\Lambda\Phi\Sigma]$$

Let $\Phi^\dagger\Lambda\Phi = D$. Then

$$\log\det[I_N + \frac{\rho}{M}\Sigma^\dagger D\Sigma] = \log\det[I_M + \frac{\rho}{M}D\Sigma\Sigma^\dagger]$$

The term above is maximized by choosing D to be diagonal ([10, page 255]). Let $D = \text{diag}\{d_1,\ d_2,\ \ldots, d_M\}$ and σ_i be the eigenvalues of $\Sigma\Sigma^\dagger$ and $E_A[\cdot]$ denote the expectation conditioned on the event A, then for large ρ

$$\begin{aligned} E\log\det[I_M + \frac{\rho}{M}D\Sigma\Sigma^\dagger] &= \underbrace{\sum_{i=1}^M E_{\sigma_i < 1/\sqrt{\rho}} \log[1 + \frac{\rho}{M}d_i\sigma_i]}_{=K} + \\ &\quad \underbrace{\sum_{i=1}^M E_{\sigma_i \geq 1/\sqrt{\rho}} \log[1 + \frac{\rho}{M}d_i\sigma_i]}_{=L} \\ &= \log\frac{\rho}{M} + \sum_{i=1}^M \log(d_i) + \sum_{i=1}^M E_{\sigma_i > 1/\sqrt{\rho}}[\log(\sigma_i)] + \\ &\quad O(\log(\sqrt{\rho})/\sqrt{\rho}) \end{aligned}$$

since

$$K \leq \log[1+\sqrt{\rho}]\sum_{i=1}^M P(\sigma_i < 1/\sqrt{\rho}) = O(\log(\sqrt{\rho})/\sqrt{\rho})$$

and

$$L = \log \frac{\rho}{M} + \sum_{i=1}^{M} \log(d_i) + \sum_{i=1}^{M} E_{\sigma_i > 1/\sqrt{\rho}}[\log(\sigma_i)] + O(1/\sqrt{\rho})$$

On account of log being a convex cap function the first term in the expression on the last line above is maximized by choosing $d_i = d$ for $i = 1, \ldots, M$ such that $M \cdot d = M$. ■

For low SNR, we show that the rician fading channel essentially behaves like an AWGN channel in the sense that the rayleigh fading component has no effect on the structure of the optimum covariance structure.

Proposition VI.2. *For low SNR, capacity, C_H when H is known to the receiver is attained by the same signal covariance matrix that attains capacity when $H = H_m$ or when $r = 0$, irrespective of the value of M and N and*

$$C_H \approx \rho[r\lambda_{max}(H_m H_m^\dagger) + (1-r)N]$$

Proof: We have

$$\begin{aligned} C_H &= T \cdot E \log\det[I_N + \frac{\rho}{M} H^\dagger \Lambda H] \\ &= T \cdot E_{\|H\| \geq 1/\rho^\gamma} \log\det[I_N + \frac{\rho}{M} H^\dagger \Lambda H] + E_{\|H\| < 1/\rho^\gamma} \log\det[I_N + \frac{\rho}{M} H^\dagger \Lambda H] \\ &= E\mathrm{tr}\{\frac{\rho}{M} H^\dagger \Lambda H\} + O(\rho^{2-2\gamma}) \end{aligned}$$

This follows from the fact that $P(\|H\| \geq 1/\rho^\gamma) < O(e^{-\frac{1}{\rho^\gamma}/TM})$ and $\log\det[I_N + \frac{\rho}{M} H^\dagger \Lambda H] = \mathrm{tr}[\frac{\rho}{M} H^\dagger \Lambda H] + O(\rho^{2-2\gamma})$ for $\|H\| < 1/\rho^\gamma$. Since γ is arbitrary

$$E \log\det[I_N + \frac{\rho}{M} H^\dagger \Lambda H] = E\mathrm{tr}[\frac{\rho}{M} H^\dagger \Lambda H] + O(\rho^2)$$

Now

$$\begin{aligned} E\mathrm{tr}[H^\dagger \Lambda H] &= \mathrm{tr}\{(1-r)E[G^\dagger \Lambda G] + r H_m^\dagger \Lambda H_m]\} \\ &= \mathrm{tr}\{(1-r)\Lambda E[GG^\dagger] + r\Lambda H_m H_m^\dagger\} \end{aligned}$$

Therefore, we have to choose Λ to maximize $\mathrm{tr}\{(1-r)N\Lambda + r\Lambda H_m H_m^\dagger\}$. Since H_m is diagonal tr depends only on the diagonal elements of Λ. Therefore, Λ can be chosen to be diagonal. Also, because of the power constraint, $\mathrm{tr}\{\Lambda\} \leq M$, to maximize the expression we choose $\mathrm{tr}\{\Lambda\} = M$. The maximizing Λ has as many non-zero elements as the multiplicity of the maximum eigenvalue of $(1-r)NI + rH_m H_m^\dagger$. The non-zero elements of Λ multiply the maximum eigenvalues of $(1-r)NI + rH_m H_m^\dagger$ and can be chosen to be of equal magnitude summing up to N. The Λ so chosen is the same Λ maximizing the capacity for low SNR, additive white Gaussian noise channel with channel H_m. ∎

6.3.2 Non-Coherent Capacity Upper and Lower Bounds

It is quite well known that the non-coherent capacity. C can never be greater than coherent capacity C_H, that is, when the channel is known to the receiver.

Proposition VI.3.

$$C \leq C_H$$

Now, we establish a lower bound which is similar in flavour to those derived in chapters IV and V.

Proposition VI.4. *A lower bound on capacity when the transmitter has no knowledge about H and the receiver has no knowledge about G is given by*

$$\begin{aligned} C &\geq C_H - NE\left[\log_2 \det\left(I_T + (1-r)\frac{\rho}{M}SS^\dagger\right)\right] && (6.4) \\ &\geq C_H - NM\log_2(1 + (1-r)\frac{\rho}{M}T) && (6.5) \end{aligned}$$

Proof: Proof is similar to that of Proposition V.2 and won't be repeated here. ∎

We notice that the second term in the lower bound goes to zero when $r = 1$. This agrees very well with the observation that as the channel becomes purely Gaussian the capacity can be easily calculated analytically and there is no need for lower and upper bounds on capacity.

6.3.3 Non-Coherent Capacity: Expressions for Low SNR

In this section, we introduce some new notation for ease of description. if X is a $T \times N$ matrix then let $\tilde{X}$ denote the "unwrapped" $NT \times 1$ vector formed by placing the transposed rows of X in a single column in an increasing manner. That is, if $X_{i,j}$ denotes the element of X in the i^{th} row and j^{th} column then $\tilde{X}_{i,1} = X_{\lfloor i/N \rfloor, i\%N}$. The channel model $X = \sqrt{\frac{\rho}{M}}SH + W$ can now be written as $\tilde{X} = \sqrt{\frac{\rho}{M}}\hat{H}\tilde{S} + \tilde{W}$. $\hat{H}$ is given by $\hat{H} = I_T \otimes H^\tau$ where H^τ denotes the transpose of H. The notation $A \otimes B$ denotes the Kronecker product of the matrices A and B and is defined as follows. If A is a $I \times J$ matrix and B a $K \times L$ matrix then $A \otimes B$ is a $IK \times JL$ matrix given by

$$A \otimes B = \begin{bmatrix} (A)_{11}B & (A)_{12}B & \dots & (A)_{1J}B \\ (A)_{21}B & (A)_{22}B & \dots & (A)_{2J}B \\ \vdots & \vdots & \ddots & \vdots \\ (A)_{I1}B & (A)_{I2}B & \dots & (A)_{IJ}B \end{bmatrix}$$

This way, we can describe the conditional probability density function $p(X|S)$ as follows

$$p(X|S) = \frac{1}{\pi^{TN}|\Lambda_{\tilde{X}|\tilde{S}}|} e^{-(\tilde{X} - \sqrt{r\frac{\rho}{M}}\hat{H}_m\tilde{S})^\dagger \Lambda^{-1}_{\tilde{X}|\tilde{S}} (\tilde{X} - \sqrt{r\frac{\rho}{M}}\hat{H}_m\tilde{S})}$$

where $|\Lambda_{\tilde{X}|\tilde{S}}| = \det(I_{TN} + (1-r)SS^\dagger \otimes I_N)$.

Peak Power Constraint

Here we concentrate on calculating the capacity under the constraint $\text{tr}\{SS^\dagger\} \leq TM$.

Theorem VI.2. *For the peak power constraint*

$$C = rT\rho\lambda_{max}(H_m H_m^\dagger) + O(\rho^{3/2})$$

and the capacity achieving signal satisfies $\text{tr}\{SS^\dagger\} = TM$.

Proof: First, we have $p(\tilde{X}) = E[p(\tilde{X}|\tilde{S})]$ where

$$p(\tilde{X}|\tilde{S}) = \frac{1}{\pi^{TN}\Lambda_{\tilde{X}|\tilde{S}}} e^{-(\tilde{X}-\sqrt{r\frac{\rho}{M}}\hat{H}_m\tilde{S})^\dagger \ \Lambda^{-1}_{\tilde{X}|\tilde{S}} \ (\tilde{X}-\sqrt{r\frac{\rho}{M}}\hat{H}_m\tilde{S})}$$

Now

$$\mathcal{H}(\tilde{X}) = E_{\|\tilde{X}\|<(\frac{M}{\rho})^\gamma}[\log p(\tilde{X})] + E_{\|\tilde{X}\|\geq(\frac{M}{\rho})^\gamma}[\log p(\tilde{X})]$$

Since $P(\|\tilde{X}\| \geq (\frac{M}{\rho})^\gamma) < O(e^{-(\frac{M}{\rho})^\gamma/TM})$ where we have chosen γ such that $1-2\gamma > 1/2$ or $\gamma < 1/4$. We have

$$\mathcal{H}(\tilde{X}) = E_{\|\tilde{X}\|<(\frac{M}{\rho})^\gamma}[\log p(\tilde{X})] + O(e^{-(\frac{M}{\rho})^\gamma/TM})$$

For $\|\tilde{X}\| < (\frac{M}{\rho})^\gamma$

$$\begin{aligned} p(\tilde{X}|\tilde{S}) &= \frac{1}{\pi^{TN}} e^{-\tilde{X}^\dagger\tilde{X}} \Big[1 + \sqrt{r\frac{\rho}{M}}(\tilde{X}^\dagger\hat{H}_m\tilde{S} + \tilde{S}^\dagger\hat{H}_m^\dagger\tilde{X}) - \\ & \frac{\rho}{M}\left(\text{tr}\{(1-r)SS^\dagger \otimes I_N\} + \text{tr}\{r\tilde{S}^\dagger\hat{H}_m^\dagger\hat{H}_m\tilde{S}\}\right) + \\ & (1-r)\frac{\rho}{M}\tilde{X}^\dagger SS^\dagger \otimes I_N\tilde{X} + r\frac{1}{2}\frac{\rho}{M}\Big(\tilde{S}^\dagger\hat{H}_m^\dagger\tilde{X}\tilde{S}^\dagger\hat{H}_m^\dagger\tilde{X} + \tilde{S}^\dagger\hat{H}_m^\dagger\tilde{X}\tilde{X}^\dagger\hat{H}_m\tilde{S} + \\ & \tilde{X}^\dagger\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger\tilde{X} + \tilde{X}^\dagger\hat{H}_m\tilde{S}\tilde{X}^\dagger\hat{H}_m\tilde{S}\Big) + O(\rho^{3/2-3\gamma})\Big] \end{aligned}$$

Since the capacity achieving signal has zero mean, for $\|\tilde{X}\| < (\frac{M}{\rho})^\gamma$

$$\begin{aligned} p(\tilde{X}) &= \frac{1}{\pi^{TN}} e^{-\tilde{X}^\dagger\tilde{X}} \Big[1 - \frac{\rho}{M}\left(\text{tr}\{(1-r)E[SS^\dagger] \otimes I_N\} + \text{tr}\{r\hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger\}\right) + \\ & \frac{\rho}{M}((1-r)\tilde{X}^\dagger E[SS^\dagger] \otimes I_N\tilde{X} + r\tilde{X}^\dagger\hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger\tilde{X} + O(\rho^{3/2-3\gamma})\Big] \\ &= \frac{1}{\pi^{TN}\det(\Lambda_{\tilde{X}})} e^{-\tilde{X}^\dagger\Lambda_{\tilde{X}}^{-1}\tilde{X}} + \frac{1}{\pi^{TN}} e^{-\tilde{X}^\dagger\tilde{X}}[O(\rho^{3/2-3\gamma})] \end{aligned}$$

where $\Lambda_{\tilde{X}} = I_{TN} + \frac{\rho}{M}(1-r)E[SS^\dagger] \otimes I_N + \frac{\rho}{M} r \hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger$

$$\begin{aligned}\mathcal{H}(\tilde{X}) &= \log\det(I_{TN} + \frac{\rho}{M}(1-r)E[SS^\dagger] \otimes I_N + \frac{\rho}{M} r \hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger) + O(\rho^{3/2-3\gamma}) \\ &= \frac{\rho}{M}\mathrm{tr}\{(1-r)E[SS^\dagger] \otimes I_N + r\hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger\} + O(\rho^{3/2-3\gamma})\end{aligned}$$

Since $P(\|S\|^2 > TM) = 0$ we can show $\mathcal{H}(\tilde{X}|\tilde{S}) = (1-r)\frac{\rho}{M}\mathrm{tr}\{E[SS^\dagger] \otimes I_N\} + O(\rho^2)$. Since γ is arbitrary, $C = r\frac{\rho}{M}\mathrm{tr}\{\hat{H}_m E[\tilde{S}\tilde{S}^\dagger]\hat{H}_m^\dagger\} + O(\rho^{3/2})$. It is very clear that to maximize C we need to choose $E[\tilde{S}\tilde{S}^\dagger]$ in such a way that all the energy is concentrated in the direction of the maximum eigenvalues of $H_m H_m^\dagger$. So that we obtain, $C = r\frac{\rho}{M}\lambda_{max}(H_m H_m^\dagger)\mathrm{tr}E[\tilde{S}\tilde{S}^\dagger] + O(\rho^{3/2})$. $\mathrm{tr}E[\tilde{S}\tilde{S}^\dagger]$ can be easily maximized by choosing $\mathrm{tr}\{\tilde{S}\tilde{S}^\dagger\}$ to be the maximum possible which is TM. Therefore,

$$C = r\rho T\lambda_{max}(H_m H_m^\dagger) + O(\rho^{3/2})$$

∎

Corollary VI.1. *For purely rayleigh fading channels* $\lim_{\rho\to 0} C/\rho = 0$

Average Power Constraint

For low SNR, it will be shown that the channel behaves as well as an AWGN channel.

Theorem VI.3. *For a fixed M, N and T if S is Gaussian distributed input then as* $\rho \to 0$

$$C \approx rT\rho\lambda_{max}(H_m H_m^\dagger)$$

Proof: First, $I(X;S) = \mathcal{H}(X) - \mathcal{H}(X|S)$. Since S is Gaussian distributed, $E[\log\det(I_N + \frac{\rho}{M}\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger)] \leq \mathcal{H}(X) \leq \log\det(I_N + \frac{\rho}{M}\Lambda_{\tilde{X}})$ where the expectation is taken over the distribution of H and $\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger = \Lambda_{\tilde{X}|H}$ is the covariance of $\tilde{X}$

for a particular H. Next, we show that $\mathcal{H}(X) = \frac{\rho}{M}\mathrm{tr}\{\Lambda_{\tilde{X}}\} + O(\rho^2)$. First, the upper bound to $\mathcal{H}(X)$ can be written as $\frac{\rho}{M}\mathrm{tr}\{\Lambda_{\tilde{X}}\} + O(\rho^2)$ because H is Gaussian distributed and the probability that $\|H\| > R$ is of the order e^{-R^2}. Second, $E[\log\det(I_{TN} + \frac{\rho}{M}\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger)] = E_{\|H\|<(\frac{M}{\rho})^\gamma}[\cdot] + E_{\|H\|\geq(\frac{M}{\rho})^\gamma}[\cdot]$ where γ is a number such that $2-\gamma > 1$ or $\gamma < 1$. Then

$$\begin{aligned} E[\log\det(I_{TN} + \frac{\rho}{M}\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger)] &= \frac{\rho}{M}E_{\|H\|<(\frac{M}{\rho})^\gamma}[\mathrm{tr}\{\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger\}] + \\ &\quad O(\rho^{2-\gamma}) + O(\log((\frac{M}{\rho})^\gamma)e^{-(\frac{M}{\rho})^\gamma}) \\ &= \frac{\rho}{M}E[\mathrm{tr}\{\hat{H}\Lambda\hat{H}^\dagger\}] + O(\rho^{2-\gamma}) \end{aligned}$$

Since γ is arbitrary, we have $\mathcal{H}(X) = \frac{\rho}{M}E[\mathrm{tr}\{\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger\}] + O(\rho^2)$. Note that $\Lambda_{\tilde{X}} = E[\Lambda_{\tilde{X}|H}]$ and since $\mathcal{H}(X)$ is sandwiched between two expressions of the form $\frac{\rho}{M}\mathrm{tr}\{\Lambda_{\tilde{X}}\} + O(\rho^2)$ the assertion follows.

Now, calculate $\mathcal{H}(X|S) = E[\log\det(I_{TN} + (1-r)\frac{\rho}{M}SS^\dagger \otimes I_N)]$. We have S to be Gaussian distributed therefore in a similar manner it can be shown that $\mathcal{H}(X|S) = (1-r)\frac{\rho}{M}\mathrm{tr}\{E[SS^\dagger \otimes I_N]\} + O(\rho^2)$.

Now, $H = \sqrt{r}H_m + \sqrt{1-r}G$ therefore, $\Lambda_{\tilde{X}} = E[\hat{H}\Lambda_{\tilde{S}}\hat{H}^\dagger] = r\hat{H}_m\Lambda_{\tilde{S}}\hat{H}_m^\dagger + (1-r)E[SS^\dagger] \otimes I_N$. Therefore, we have for a Gaussian distributed input $C = r\frac{\rho}{M}\mathrm{tr}\{\hat{H}_m\Lambda_{\tilde{S}}\hat{H}_m^\dagger\} + O(\rho^2)$. Since H_m is a diagonal matrix only the diagonal elements of $\Lambda_{\tilde{S}}$ matter. Therefore, we can choose the signals to be independent from time instant to time instant. Also, to maximize $\mathrm{tr}\{\hat{H}_m\Lambda_{\tilde{S}}\hat{H}_m^\dagger\}$ under the condition $\mathrm{tr}\{\Lambda\} \leq TM$ it is best to concentrate all the available energy on the largest eigenvalues of H_m. Therefore, we obtain

$$C = r\frac{\rho}{M}TM\lambda_{max}(H_m H_m^\dagger) + O(\rho^2)$$

$\blacksquare$

Corollary VI.2. *For purely rayleigh fading channels Gaussian input gives* $\lim_{\rho\to 0} C/\rho = 0$.

The peak constraint results and the Gaussian input results give an indication that for low SNR rayleigh fading channels are at a disadvantage compared to rician fading channels. But, it has been shown in [40, 67] for single antenna transmit and receive channel rayleigh fading provides as much capacity as a Gaussian channel with the same energy for low SNR. We will extend that result to multiple transmit and receive antenna channel for the general case of rician fading. The result for rayleigh fading will follow as a special case.

Theorem VI.4. *As* $\rho \to 0$

$$C \approx \rho T \left[r\lambda_{max}(H_m H_m^\dagger) + N(1-r) \right]$$

Proof: First, we rewrite the channel as

$$\tilde{X} = \hat{H}\tilde{S} + W$$

with the average power constraint on the signal $\tilde{S}$ to be given by $E[\mathrm{tr}\{\tilde{S}\tilde{S}^\dagger\}] \leq \frac{\rho}{M}TM = \rho T$. In other words, we have absorbed $\sqrt{\frac{\rho}{M}}$ into $\tilde{S}$.

It has been effectively shown ([67]) that if the input alphabet (the set of values the input signal S, can take) includes the value "0" (symbol with 0 power) for a channel with output X, with condition probability given by $p(X|S)$ then

$$\lim_{P_C\to 0} \frac{C}{P_C} = \sup_{s\in\mathcal{S}} \frac{D(\ p(X|S=s) \ \|\ p(X|S=0)\ )}{P_s}$$

where $\mathcal{S}$ is the set of values that the input can take, P_C is the power constraint on the input (in our case, $P_C = \rho T$) and P_s is the energy in the specific realization of the input $S = s$ and $D(p_A\|p_B)$ is the Kullback-Leibler distance for continuous density functions with argument x defined as

$$D(p_A \| p_B) = \int p_A(x) \log \frac{p_A(x)}{p_B(x)} dx$$

Applying the above result to the case of rician fading channels, we obtain

$$\lim_{\rho \to 0} \frac{C}{\rho T} = \sup_{\tilde{S}} \frac{D(\ p(\tilde{X}|\tilde{S})\ \|\ p(\tilde{X}|0)\ )}{\mathrm{tr}\{\tilde{S}\tilde{S}^\dagger\}}$$

First, we have

$$p(\tilde{X}|\tilde{S}) = \frac{1}{\pi^{TN}|\Lambda_{\tilde{X}|\tilde{S}}|} e^{-(\tilde{X}-\sqrt{r}\hat{H}_m\tilde{S})^\dagger\ \Lambda^{-1}_{\tilde{X}|\tilde{S}}\ (\tilde{X}-\sqrt{r}\hat{H}_m\tilde{S})}$$

where $\Lambda_{\tilde{X}|\tilde{S}} = I_{TN} + (1-r)SS^\dagger \otimes I_N$. Second, we have

$$p(\tilde{X}|0) = \frac{1}{\pi^{TN}} e^{-\tilde{X}^\dagger \tilde{X}}$$

Therefore,

$$\begin{aligned}
D(\ p(\tilde{X}|\tilde{S})\ \|\ p(\tilde{X}|0)\ ) &= \int p(\tilde{X}|\tilde{S}) \left[\log \frac{1}{|\Lambda_{\tilde{X}|\tilde{S}}|} \ +\ \tilde{X}^\dagger\tilde{X} - \right. \\
&\quad \left. \left(\tilde{X} - \sqrt{r}\hat{H}_m\tilde{S}\right)^\dagger\ \Lambda^{-1}_{\tilde{X}|\tilde{S}}\ \left(\tilde{X} - \sqrt{r}\hat{H}_m\tilde{S}\right)\right] d\tilde{X} \\
&= \log \frac{1}{|\Lambda_{\tilde{X}|\tilde{S}}|} \ +\ \mathrm{tr}\left\{r\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger + \Lambda_{\tilde{X}|\tilde{S}}\right\} - TN \\
&= \log \frac{1}{\det(I_{TN} + (1-r)SS^\dagger \otimes I_N)} \ + \\
&\quad \mathrm{tr}\left\{r\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger + (1-r)SS^\dagger \otimes I_N\right\}
\end{aligned}$$

This gives,

$$\begin{aligned}
\frac{D(\ p(\tilde{X}|\tilde{S})\ \|\ p(\tilde{X}|0)\ )}{\mathrm{tr}\{\tilde{S}\tilde{S}^\dagger\}} &= -N\frac{\sum_{i=1}^T \log(1+\lambda_i(SS^\dagger))}{\sum_{i=1}^T \lambda_i(SS^\dagger)} \ +\ \frac{\mathrm{tr}\{r\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger\}}{\sum_{i=1}^T \mathrm{tr}\{S_iS_i^\dagger\}} \ + \\
&\quad N(1-r)
\end{aligned}$$

where we have used the facts that $\det(I_{TN}+(1-r)SS^\dagger \otimes I_N) = \det(I_T + (1-r)SS^\dagger)^N$, $\mathrm{tr}\{\tilde{S}\tilde{S}^\dagger\} = \mathrm{tr}\{SS^\dagger\} = \sum_{i=1}^T \mathrm{tr}\{S_i^\tau S_i^*\}$ where S_i is the i^{th} row in the matrix S.

Since,

$$\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger = \begin{bmatrix} H_m^\tau S_1^\tau S_1^* H_m^* & H_m^\tau S_1^\tau S_2^* H_m^* & \dots & H_m^\tau S_1^\tau S_T^* H_m^* \\ H_m^\tau S_2^\tau S_1^* H_m^* & H_m^\tau S_2^\tau S_2^* H_m^* & \dots & H_m^\tau S_2^\tau S_T^* H_m^* \\ \vdots & \vdots & \ddots & \vdots \\ H_m^\tau S_T^\tau S_1^* H_m^* & H_m^\tau S_T^\tau S_2^* H_m^* & \dots & H_m^\tau S_T^\tau S_T^* H_m^* \end{bmatrix}$$

we have $\text{tr}\{\hat{H}_m\tilde{S}\tilde{S}^\dagger\hat{H}_m^\dagger\} = \sum_{i=1}^T \text{tr}\{H_m^\tau S_i^\tau S_i^* H_m^*\} = \text{tr}\{H_m^* H_m^\tau \sum_{i=1}^T S_i^\tau S_i^*\}$. Therefore,

$$\frac{D(\ p(\tilde{X}|\tilde{S}) \parallel p(\tilde{X}|0)\ )}{\text{tr}\{\tilde{S}\tilde{S}^\dagger\}} = -N\frac{\sum_{i=1}^N \log(1+\lambda_i(S^\dagger S))}{\sum_{i=1}^N \lambda_i(S^\dagger S)} + r\frac{\text{tr}\{H_m^* H_m^\tau \sum_{i=1}^T S_i^\tau S_i^*\}}{\sum_{i=1}^T \text{tr}\{S_i^\tau S_i^*\}} + N(1-r)$$

Note that since H_m is a diagonal matrix only the diagonal elements of $S_i S_i^\dagger$ affect the second term. Therefore, for a given $\sum_{i=1}^T S_i^\tau S_i^*$ the second term in the expression above can be maximized by choosing S_i such that $S_i^\tau S_i^*$ is diagonal with only as many non-zero values which correspond to the multiplicity of the maximum entry in the diagonal of $H_m^* H_m^\tau$. In such a case the expression above evaluates to

$$\frac{D(\ p(\tilde{X}|\tilde{S}) \parallel p(\tilde{X}|0)\ )}{\text{tr}\{\tilde{S}\tilde{S}^\dagger\}} = -N\frac{\log(1+\text{tr}\{S^\dagger S\})}{\text{tr}\{S^\dagger S\}} + r\lambda_{max}(H_m^* H_m^\tau) + N(1-r)$$

The first term can be made arbitrarily small by letting $\text{tr}\{S^\dagger S\} \rightarrow \infty$. Therefore, we have $\lim_{\rho\rightarrow 0}\frac{C}{\rho T} = r\lambda_{max}(H_m H_m^\dagger) + N(1-r)$. ∎

This result suggests that at low SNR all the energy has to be concentrated in the strongest directions of the specular component which agrees quite well with the results obtained so far. In [40] it is shown that the optimum signaling scheme for rayleigh fading channels is an "on-off" signaling scheme. It can be safely conjectured that the capacity achieving signaling scheme for low SNR in the case of the rician fading is also a similar "on-off" signaling scheme.

6.3.4 Non-Coherent Capacity: Expressions for High SNR

In this section we apply the method developed in [74] for the analysis of rayleigh fading channels. The only difference in the models considered in [74] and here is that we assume H has a deterministic non-zero mean. For convenience, we use a different notation for the channel model. We rewrite the channel model as

$$X = SH + W$$

with $H = \sqrt{r} + \sqrt{1-r}G$ where H_m is the specular component of H and G denotes the rayleigh component. G and W consist of Gaussian circular independent random variables and the covariance matrices of G and W are given by $(1-r)I_{MN}$ and $\sigma^2 I_{TN}$, respectively. H_m is such that $E[\mathrm{tr}\{HH^\dagger\}] = MN$ and r is a number between 0 and 1. Both H_m and r are known to both the transmitter and the receiver.

Lemma VI.1. *The capacity achieving signal, S can be written as $S = \Phi V \Psi^\dagger$ where Φ is a $T \times M$ unitary matrix independent of V and Ψ. V and Ψ are $M \times M$.*

Proof: Follows from the fact that $p(\Phi X|\Phi S) = p(X|S)$. ∎

In [74] the requirement for $X = SH + W$ was that X had to satisfy the property that in the singular value decomposition of X, $X = \Phi V \Psi^\dagger$ Φ be independent of V and Ψ. This property holds for the case of rician fading too because the density functions of X, SH and S are invariant to pre-multiplication by a unitary matrix. Therefore, the leading unitary matrix in the SVD decomposition of any of X, SH and S is independent of the other two components in the SVD and isotropically distributed.

Therefore, Lemma 6 in [74] still holds

Lemma VI.2. *Let $R = \Phi_R \Sigma_R \Psi_R^\dagger$ have the property described above. Then*

$$\mathcal{H}(R) = \mathcal{H}(Q\Sigma_R\Psi_R^\dagger) + \log|G(T,M)| + (T-M)E[\log\det\Sigma_R^2]$$

where Q is $M \times M$ unitary matrix independent of V and Ψ and $|G(T,M)|$ is the volume of the Grassmann Manifold.

$M = N$, $T \geq 2M$

For a given distribution of $\|S_i\|$

$$I(X;S) = \mathcal{H}(X) - \mathcal{H}(X|S)$$

To compute $\mathcal{H}(X|S)$ we note that given S, X is a Gaussian random vector with columns of X independent of each other. Each row has the common covariance matrix given by $(1-r)SS^\dagger + \sigma^2 I_T = \Phi V^2 \Phi^\dagger + \sigma^2 I_T$. Therefore

$$\mathcal{H}(X|S) = ME[\sum_{i=1}^{M} \log(\pi e((1-r)\|s_i\|^2 + \sigma^2)] + M(T-M)\log(\pi e \sigma^2)$$

To compute $\mathcal{H}(X)$, we write the SVD $X = \Phi_X \Sigma_X \Psi_X^\dagger$. Note that Φ_X is isotropically distributed and independent of $\Sigma_X \Psi_X^\dagger$, therefore from Lemma VI.2 we have

$$\mathcal{H}(X) = \mathcal{H}(Q\Sigma_X \Psi_X^\dagger) + \log|G(T,M)| + (T-M)E[\log\det \Sigma_X^2]$$

We first characterize the optimal input distribution in the following lemma.

Lemma VI.3. *Let $(s_i^\sigma, i = 1, \ldots, M)$ be the optimal input signal of each antenna at noise level σ. If $T \geq 2M$,*

$$\frac{\sigma}{\|s_i^\sigma\|} \xrightarrow{P} 0, \; for \; i = 1, \ldots, M \tag{6.6}$$

where $\xrightarrow{P}$ denotes convergence in probability.

Proof: See Appendix .8. ■

Lemma VI.4. *The maximal rate of increase of capacity with SNR is $M(T-M)\log\rho$ and the constant norm input where $\|s_i\|^2 = T$ for $i = 1, \ldots, M$ attains this rate.*

Proof: See Appendix .8. ∎

Lemma VI.5. *As $T \to \infty$ the optimal input is the constant norm input*

Proof: See Appendix .8. ∎

From now on, we assume that the optimal input signal is the constant norm input. For the constant norm input $\Phi V \Psi^\dagger = \Phi V$ since Φ is isotropically distributed.

Theorem VI.5. *For the constant norm input, as $\sigma^2 \to 0$ the capacity is given by*

$$\begin{aligned} C &= \log|G(T,M)| + (T-M)E[\log\det H^\dagger H] - M(T-M)\log \pi e\sigma^2 - \\ &\quad M^2 \log \pi e + \mathcal{H}(QVH) + (T-2M)M\log T - M^2\log(1-r) \end{aligned}$$

Proof: Since $\|s_i^2\| \gg \sigma^2$ for all $i = 1, \dots, M$

$$\begin{aligned} \mathcal{H}(X|S) &= ME[\sum_{i=1}^M \log \pi e((1-r)\|s_i\|^2 + \sigma^2)] + M(T-M)\log(\pi e \sigma^2) \\ &\approx ME[\sum_{i=1}^M \log \pi e(1-r)\|s_i\|^2] + M(T-M)\log \pi e\sigma^2 \\ &= ME[\log\det(1-r)V^2] + M^2\log \pi e + M(T-M)\log \pi e\sigma^2 \end{aligned}$$

and from Appendix .10

$$\begin{aligned} \mathcal{H}(X) &\approx \mathcal{H}(SH) \\ &= \mathcal{H}(QVH) + \log|G(T,M)| + (T-M)E[\log\det(H^\dagger V^2 H)] \\ &= \mathcal{H}(QVH) + \log|G(T,M)| + (T-M)E[\log\det V^2] + \\ &\quad (T-M)E[\log\det HH^\dagger] \end{aligned}$$

Combining the two equations

$$\begin{aligned} I(X;S) &\approx \log|G(T,M)| + (T-M)E[\log\det H^\dagger H] - M(T-M)\log \pi e\sigma^2 + \\ &\quad \mathcal{H}(QVH) - M^2\log \pi e + (T-2M)E[\log\det V^2] - M^2\log(1-r) \end{aligned}$$

Now, since the optimal input signal is $\|s_i\|^2 = T$ for $i = 1, \ldots, M$, we have

$$\begin{aligned} C &= I(X;S) \\ &\approx \log|G(T,M)| + (T-M)E[\log\det H^\dagger H] - M(T-M)\log \pi e \sigma^2 - \\ &\quad M^2 \log \pi e + \mathcal{H}(QVH) + (T-2M)M\log T - M^2\log(1-r) \end{aligned}$$

$\blacksquare$

Theorem VI.6. *As $T \to \infty$ the normalized capacity $C/T \to E[\log\det \frac{\rho}{M} H^\dagger H]$ where $\rho = M/\sigma^2$.*

Proof: First, a lower bound to capacity as $\sigma^2 \to 0$ is given by

$$\begin{aligned} C &\geq \log|G(T,M)| + (T-M)E[\log\det H^\dagger H] + M(T-M)\log \frac{T\rho}{M\pi e} - \\ &\quad M^2 \log T - M^2 \log(1-r) - M^2 \log \pi e \end{aligned}$$

In [74] it's already been shown that $\lim_{T\to\infty}(\frac{1}{T}\log|G(T,M)| + M(1-\frac{M}{T})\log\frac{T}{\pi e}) = 0$ Therefore we have as $T \to \infty$

$$C/T \geq ME[\log\det \frac{\rho}{M} H^\dagger H]$$

Second, since $\mathcal{H}(QVH) \leq M^2 \log(\pi e T)$ an asymptotic upper bound to capacity is given by

$$\begin{aligned} C &\leq \log|G(T,M)| + (T-M)E[\log\det H^\dagger H] + M(T-M)\log \frac{T\rho}{M\pi e} - \\ &\quad M^2 \log(1-r) \end{aligned}$$

Therefore, we have as $T \to \infty$

$$C \leq E[\log\det \frac{\rho}{M} H^\dagger H]$$

$\blacksquare$

$M < N$ $T \geq M + N$

In this case we show that the optimal rate of increase is given by $M(T - M)\log\rho$. The higher number of receive antennas can provide only a finite increase in capacity for all SNRs.

Theorem VI.7. *The maximum rate of increase is given by $M(T - M)\log\rho$.*

Proof: See Appendix .8. ∎

6.4 Training in Non-Coherent Communications

It is important to know whether training based signal schemes are practical and if they are how much time can be spent in learning the channel and what the optimal training signal is like. Hassibi and Hochwald [33] have addressed these issues for the case of rayleigh fading channels. They showed that 1) pilot symbol training based communication schemes are highly suboptimal for low SNR and 2) when practical the optimal amount of time devoted to training is equal to the number of transmitters, M when the fraction of power devoted to training is allowed to vary and 3) the orthonormal signal is the optimal signal for training.

In [74] the authors demonstrate a very simple training method that achieves the optimal rate of increase with SNR. The same training method can also be easily applied to the rician fading model with known specular component. The training signal is the $M \times M$ diagonal matrix DI_M. D is chosen such that the same power is used in the training and the communication phase. Therefore, $D = \sqrt{M}$. In this case the output of the multiple-antenna channel in the training phase is given by

$$X = \sqrt{M}\sqrt{r}H_m + \sqrt{M}\sqrt{1-r}G + W$$

The rayleigh channel coefficients G can be estimated separately using scalar MMSE since the elements of W and G are both Gaussian distributed

$$\hat{G} = \frac{\sqrt{1-r}\sqrt{M}}{(1-r)M+\sigma^2}[X - \sqrt{M}\sqrt{r}H_m]$$

The elements of the estimate $\hat{G}$ are independent of each other and are Gaussian with variance $\frac{(1-r)M}{(1-r)M+\sigma^2}$. Similarly, the estimation errors $G-\hat{G}$ are Gaussian distributed with zero mean and variance $\frac{\sigma^2}{(1-r)M+\sigma^2}$ and are independent of each other.

The output of the channel in the communication phase is given by

$$\begin{aligned} X &= SH + W \\ &= \sqrt{r}SH_m + \sqrt{1-r}S\hat{G} + \sqrt{1-r}S(G-\hat{G}) + W \end{aligned}$$

where S consists of zero mean i.i.d circular Gaussian random variables with zero mean and unit variance. Let $\hat{W} = \sqrt{1-r}S(G-\hat{G}) + W$. The entries of $\hat{W}$ are uncorrelated with each other and also with $S(\sqrt{r}H_m + \sqrt{1-r}\hat{G})$. The variance of the entries of $\hat{W}$ is given by $\sigma^2 + (1-r)M\frac{\sigma^2}{(1-r)M+\sigma^2}$. If $\hat{W}$ is replaced with a white Gaussian noise with the same covariance matrix then the resulting mutual information is a lower bound on the actual mutual information ([10, p. 263]). This result is formally stated in Proposition VI.5. The lower bound is given by

$$C \geq \frac{T-T_t}{T} E \log\det\left(I_M + \frac{\rho_{eff}}{M}H_1H_1^\dagger\right)$$

where H_1 is a rician channel with new r given by $r_{new} = \frac{r}{r+(1-r)\frac{(1-r)M}{(1-r)M+\sigma^2}}$. This lower bound can be easily calculated because the lower bound is essentially the coherent capacity with H replaced by $\sqrt{r_{new}}H_m + \sqrt{1-r_{new}}\hat{G}$. The signal covariance structure was chosen to be an identity matrix as this is the optimum covariance matrix for high SNR. The effective SNR is now given by the ratio of the energy of the elements of $S(\sqrt{r}H_m + \sqrt{1-r}\hat{G})$ to the energy of the elements of $\hat{W}$. The energy in the elements of $S(\sqrt{r}H_m + \sqrt{1-r}\hat{G})$ is given by $M(r + (1-r)^2\frac{M}{(1-r)M+\sigma^2})$ and the energy in the elements of $\hat{W}$ are given by $\sigma^2 + \frac{(1-r)M\sigma^2}{(1-r)M+\sigma^2}$. Therefore, the effective SNR is given by $\frac{\rho[r+r(1-r)\rho+(1-r)^2\rho]}{[1+2(1-r)\rho]}$ where $\rho = \frac{M}{\sigma^2}$ is the actual SNR. Note, for $r = 1$ no training is required since the channel is completely known.

This simple scheme achieves the the optimum increase of capacity with SNR and uses only M of the T symbols for training.

We will compare using LMS to estimate the channel coefficients to this simple scheme. LMS is obviously at a disadvantage because when $\sigma^2 = 0$ the simple scheme outlined above has a perfect channel estimate just after M training symbols whereas the LMS algorithm requires much more than M training symbols to obtain an estimate very close to the actual value. This obviously shows the LMS algorithm is a very suboptimal algorithm which can't attain the optimal degrees of freedom. The performance of the simple training scheme is plotted with respect to different SNR values for comparison with the asymptotic upper bound to capacity in the proof of Theorem VI.6. The plot also verifies the result of Theorem VI.6. The plots are for $M = N = 5$, $r = 0.9$ and $T = 50$ in Figure 6.2 the specular component is a rank-one specular component given by (6.2).

We can do a much better analysis of the amount of training required using the techniques in [33]. In [33], the authors use the optimization of the lower bound to capacity stated above to find the best amount of training required and optimum power allocation between training and communication phases. Let T_t denote the amount of time devoted to training and T_c the amount of time devoted to actual communication. Let S_t be the $T_t \times M$ signal used for training and S_c the $T_c \times M$ signal used for communication. Let κ denote the fraction of the energy used for

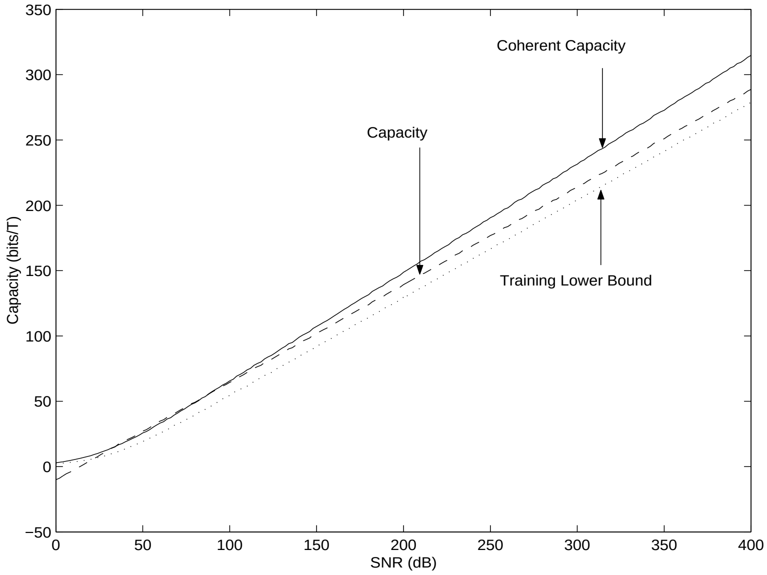


Figure 6.2: Asymptotic capacity upper bound, Capacity Upper and Lower bounds for different values of SNR

communication. Then $T = T_t + T_c$ and $\mathrm{tr}\{S_t S_t^\dagger\} = (1-\kappa)TM$ and $\mathrm{tr}\{S_c S_c^\dagger\} = \kappa TM$.

$$\begin{aligned} X_t &= S_t(\sqrt{r}H_m + \sqrt{1-r}G) + W_t \\ X_c &= S_c(\sqrt{r}H_m + \sqrt{1-r}G) + W_c \end{aligned}$$

where X_t is $T_t \times N$ and X_c is $T_c \times N$. H is estimated from the training phase. For that we need $T_t \geq M$. Since, G and W_t are Gaussian the MMSE estimate of G is also the Linear MMSE estimate. The estimate is given by

$$\hat{G} = \sqrt{1-r}(\sigma^2 I_M + (1-r)S_t^\dagger S_t)^{-1} S_t^\dagger (X_t - \sqrt{r} S_t H)$$

Let $\bar{G} = G - \hat{G}$. Now

$$X_c = S_t(\sqrt{r}H_m + \sqrt{1-r}\hat{G}) + \sqrt{1-r}S_t\bar{G} + W_c$$

Let $\hat{W}_c = \sqrt{1-r}S_t\bar{G} + W$. If we replace $\hat{W}_c$ with an additive white Gaussian noise that is zero-mean spatially and white temporally the elements of which have the same variance as the elements of $\hat{W}_c$ then the resulting mutual information is a lower bound to the actual mutual information in the above channel. This is stated formally in the following proposition.

Proposition VI.5 (Theorem 1 in [33]). *Let*

$$X = SH + W$$

be a rician fading channel with H known to the receiver. Let S and W satisfy $\frac{1}{M}E[SS^\dagger] = 1$ and $\frac{1}{M}E[WW^\dagger] = \sigma^2$ and be uncorrelated with each other. Then the worst case noise has i.i.d. zero mean Gaussian distribution, $W \sim \mathcal{CN}(0, I_N)$. Moreover, this distribution has the following minimax property

$$I_{W\sim\mathcal{CN}(0,\sigma^2 I_N),S}(X;S) \leq I_{V\sim\mathcal{CN}(0,\sigma^2 I_N),S\sim\mathcal{CN}(0,I_M)}(X;S) \leq I_{V,S\sim\mathcal{CN}(0,I_M)}(X;S)$$

The variance of the elements of $\hat{W}_c$ is given by

$$\begin{aligned}
\sigma^2_{w_c} &= \sigma^2 + \frac{1-r}{NT_c}\mathrm{tr}\{E[\bar{G}\bar{G}^\dagger]\kappa T I_M\} \\
&= \sigma^2 + \frac{(1-r)\kappa TM}{T_c}\frac{1}{NM}E[\bar{G}\bar{G}^\dagger] \\
&= \sigma^2 + \frac{(1-r)\kappa TM}{T_c}\sigma^2_{\bar{G}}
\end{aligned}$$

and the lower bound is

$$C_t \geq \frac{T-T_t}{T}E\log\det\left(I_M + \frac{\rho_{eff}}{M}H_1\Lambda H_1^\dagger\right)$$

where ρ_{eff} is the ratio of the energies in the elements of $S_c\hat{H}$ and energies in the elements of $\hat{W}_c$ and $H_1 = \sqrt{r_{new}}H_m + \sqrt{1-r_{new}}\hat{G}$ where $r_{new} = \frac{r}{r+(1-r)\sigma^2_{\hat{G}}}$. Λ is the optimum signal correlation matrix the form of which depends on the distribution of H_1. This problem has been solved when H_1 is a rician channel for low values of ρ_{eff} and high values of ρ_{eff} with $M \leq N$ in section 6.3.1.

To calculate ρ_{eff}, the energy in the elements of $S\hat{H}$ is given by

$$\begin{aligned}
\sigma^2_{SH} &= \frac{1}{NT_c}[r\mathrm{tr}\{H_m H_m^\dagger \kappa T I_M\} + (1-r)\mathrm{tr}\{\hat{G}\hat{G}^\dagger \kappa T I_M\}] \\
&= \frac{\kappa T M}{T_c}\frac{1}{NM}[rNM + (1-r)\mathrm{tr}\{\hat{G}\hat{G}^\dagger\}] \\
&= \frac{\kappa T M}{T_c}[r + (1-r)\sigma^2_{\hat{G}}]
\end{aligned}$$

which gives us

$$\rho_{eff} = \frac{\kappa T \rho[r + (1-r)\sigma^2_{\hat{G}}]}{T_c + (1-r)\kappa T \rho \sigma^2_{\tilde{G}}}$$

6.4.1 Optimization of S_t, κ and T_t

We will optimize S_t, κ and T_t to maximize the lower bound. In this section we merely state the main results and their interpretations. Derivations and details are given in the Appendices.

Optimization over S_t is not very practical as S_t effects the distribution of $\hat{H}$, the form of Λ as well as ρ_{eff}. To make the problem simpler we will just find the value of S_t that maximizes ρ_{eff}.

Theorem VI.8. *The signal S_t that maximizes ρ_{eff} satisfies the following condition*

$$S_t^\dagger S_t = (1-\kappa) T I_M$$

Proof: See Appendix .11. ∎

Optimizing over S_t gives

$$\rho_{eff} = \frac{\kappa T \rho[Mr + \rho(1-\kappa)T]}{T_c(M + \rho(1-\kappa)T) + (1-r)\kappa T \rho M}$$

and the corresponding lower bound is given by

$$C_t \geq \frac{T - T_t}{T} E \log\det\left(I_M + \frac{\rho_{eff}}{M} H_1 \Lambda H_1^\dagger\right)$$

where ρ_{eff} is as given above and $H_1 = \sqrt{r_{new}}H_m + \sqrt{1-r_{new}}G$ where $r_{new} = r\frac{1+(1-r)(1-\kappa)\frac{\rho}{M}T}{r+(1-r)(1-\kappa)\frac{\rho}{M}T}$ and G is a matrix consisting of i.i.d. Gaussian circular random variables with mean zero and unit variance. Now, Λ is the input signal covariance matrix when the channel is rician and known to the receiver. The form of Λ is known to us for $\rho_{eff} \to 0$ and $\rho_{eff} \to \infty$.

Optimization over κ is straighforward as κ affects the lower bound only through ρ_{eff} and can be stated in the following proposition.

Theorem VI.9. *The optimal power allocation κ in a training based scheme is given by*

$$\kappa = \begin{cases} \min\{\gamma - \sqrt{\gamma(\gamma-1-\eta)}, 1\} & \textit{for } T_c > (1-r)M \\ \min\{\frac{1}{2} + \frac{rM}{2T\rho}, 1\} & \textit{for } T_c = (1-r)M \\ \min\{\gamma + \sqrt{\gamma(\gamma-1-\eta)}, 1\} & \textit{for } T_c < (1-r)M \end{cases}$$

where $\gamma = \frac{MT_c+T\rho T_c}{T\rho[T_c-(1-r)M]}$ and $\eta = \frac{rM}{T\rho}$. The corresponding lower bound is given by

$$C_t \geq \frac{T-T_t}{T} E \log\det\left(I_M + \frac{\rho_{eff}}{M} H_1 \Lambda H_1^\dagger\right)$$

where for $T_c > (1-r)M$

$$\rho_{eff} = \begin{cases} \frac{T\rho}{T_c-(1-r)M}(\sqrt{\gamma} - \sqrt{\gamma-1-\eta})^2 & \textit{when } \kappa = \gamma - \sqrt{\gamma(\gamma-1-\eta)} \\ \frac{r\rho}{1+(1-r)\rho} & \textit{when } \kappa = 1 \end{cases}$$

for $T_c = (1-r)M$

$$\rho_{eff} = \begin{cases} \frac{T^2\rho^2}{4(1-r)M(M+T\rho)}(1 + \frac{rM}{T\rho})^2 & \textit{when } \kappa = \frac{1}{2} + \frac{rM}{2T\rho} \\ \frac{rT\rho}{(1-r)(M+T\rho)} & \textit{when } \kappa = 1 \end{cases}$$

and for $T_c < (1-r)M$

$$\rho_{eff} = \begin{cases} \frac{T\rho}{(1-r)M-T_c}(\sqrt{-\gamma} - \sqrt{-\gamma+1+\eta})^2 & \textit{when } \kappa = \gamma + \sqrt{\gamma(\gamma-1-\eta)} \\ \frac{r\rho}{1+(1-r)\rho} & \textit{when } \kappa = 1 \end{cases}$$

and r_{new} is given by substituting the appropriate value of κ in the expression

$$r\frac{1+(1-r)(1-\kappa)\frac{\rho}{M}T}{r+(1-r)(1-\kappa)\frac{\rho}{M}T}$$

Proof: See Appendix .12. ∎

For optimization over T_t we draw somewhat similar conclusions as in [33]. We show that $T_t = M$ is the optimal setting for "large" SNR and $T_t = 0$ for small SNR. The intuition is that increasing T_t linearly decreases the capacity through the term $(T - T_t)/T$, but only logarithmically increases the capacity through the higher effective SNR ρ_{eff} [33]. Therefore, it makes sense to make T_t as small as possible. For small SNR we show that $\kappa = 1$. It is clear that optimization of T_t makes sense only when κ is strictly less than 1. When $\kappa = 1$ no power is devoted to training and T_t can be made as small as possible which is zero. When $\kappa < 1$ the smallest value T_t can be is M since it takes alteast that many intervals to completely determine the unknowns.

Theorem VI.10. *The optimal length of the training interval is $T_t = M$ whenever $\kappa < 1$ for all values of ρ and T, and the capacity lower bound is*

$$C_t \geq \frac{T-M}{T} E \log\det\left(I_M + \frac{\rho_{eff}}{M} H_1 \Lambda H_1^\dagger\right) \tag{6.7}$$

where

$$\rho_{eff} = \begin{cases} \frac{T\rho}{T-(2-r)M}(\sqrt{\gamma} - \sqrt{\gamma - 1 - \eta})^2 & \text{for } T > (2-r)M \\ \frac{T^2\rho^2}{4(1-r)M(M+T\rho)}(1 + \frac{rM}{T\rho})^2 & \text{for } T = (2-r)M \\ \frac{T\rho}{T-(2-r)M}(\sqrt{-\gamma} - \sqrt{-\gamma + 1 + \eta})^2 & \text{for } T < (2-r)M \end{cases}$$

The optimal power allocations are easily obtained from Theorem VI.9 by simply setting $T_c = T - M$.

Proof: See Appendix .13 ∎

6.4.2 Equal training and data power

As stated in [33] sometimes it is difficult for the transmitter to assign different powers for training and communication phases. In this section, we will concentrate on setting the training and communication powers equal to each other or on the following restriction

$$\frac{(1-\kappa)T}{T_t} = \frac{\kappa T}{T_c} = \frac{\kappa T}{T - T_t} = 1$$

this means $\kappa = 1 - T_t/T$.

In this case,

$$\rho_{eff} = \frac{\rho[r + \rho\frac{T_t}{M}]}{1 + \rho[\frac{T_t}{M} + (1-r)]}$$

and the lower bound is

$$C_t \geq \frac{T - T_t}{T} E \log\det(I_M + \frac{\rho_{eff}}{M} H_1 \Lambda H_1^\dagger)$$

where ρ_{eff} is as given above and $H_1 = \sqrt{r_{new}} H_m + \sqrt{1 - r_{new}} G$ where $r_{new} = r\frac{1+(1-r)\frac{\rho}{M}T_t}{r+(1-r)\frac{\rho}{M}T_t}$ and G is a matrix consisting of i.i.d. Gaussian circular random variables with mean zero and unit variance.

6.4.3 Plots of Optimization Parameters and Capacities

In this section, H_m used is an identity matrix of appropriate dimension. Throughout the section we have chosen the number of transmit and receive antennas to be equal.

The figures 6.3 and 6.4 show r_{new} and κ respectively as a function of r for different values of SNR. The plots have been calculated for a block length given by $T = 40$ and the number of transmit and receive antennas given by $M = N = 5$. Figure 6.3 shows that for low SNR values the channel behaves like a purely AWGN channel given by $\sqrt{r} H_m$ and for high SNR values the channel behaves exactly like the original rician fading channel. Figure 6.4 shows that as the SNR goes to zero less and less power is allocated for training. This agrees fairly well with the plot in figure 6.3.

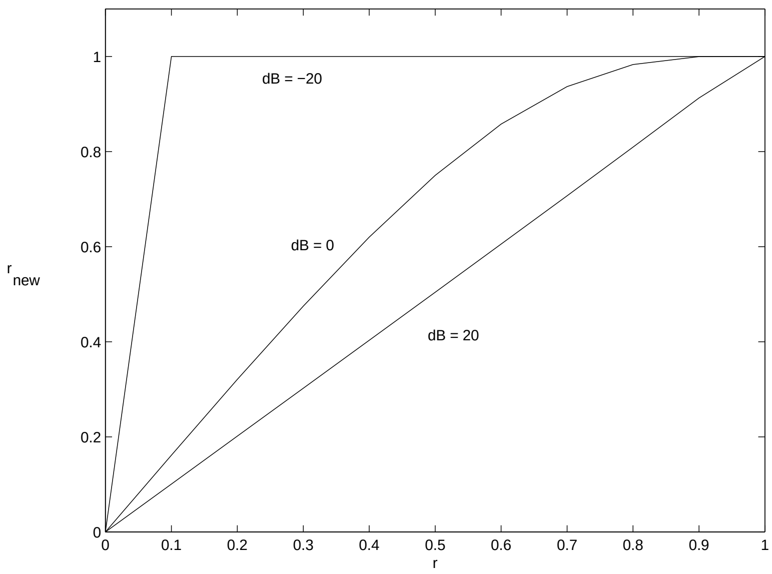


Figure 6.3: Plot of r_{new} as a function of r

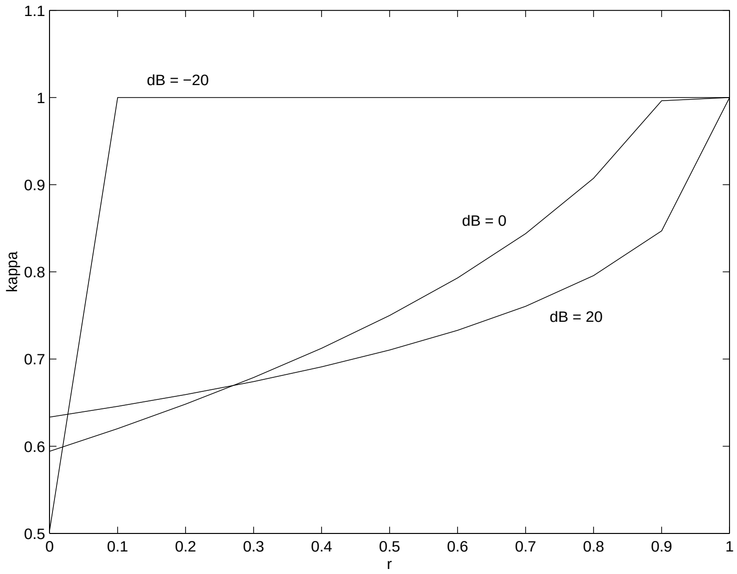


Figure 6.4: Plot of optimal energy allocation κ as a function of r

In figure 6.5 we plot the training and communication powers for $M = N = 10$ and $dB = 18$ for different values of r. We see that as r goes to 1 less and less power is allocated for training purposes. This makes sense as the proportion of the

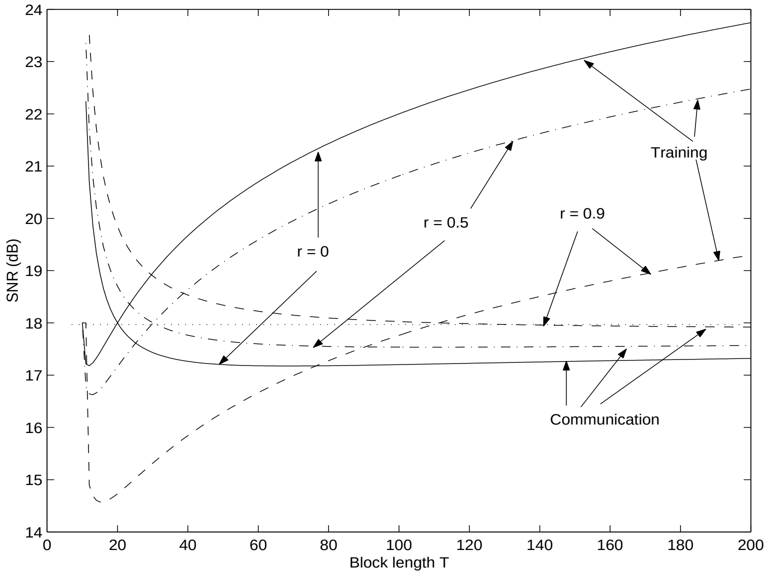


Figure 6.5: Plot of optimal power allocation as a function of T

energy through the specular component increases there is less need for the system to estimate the unknown rayleigh component.

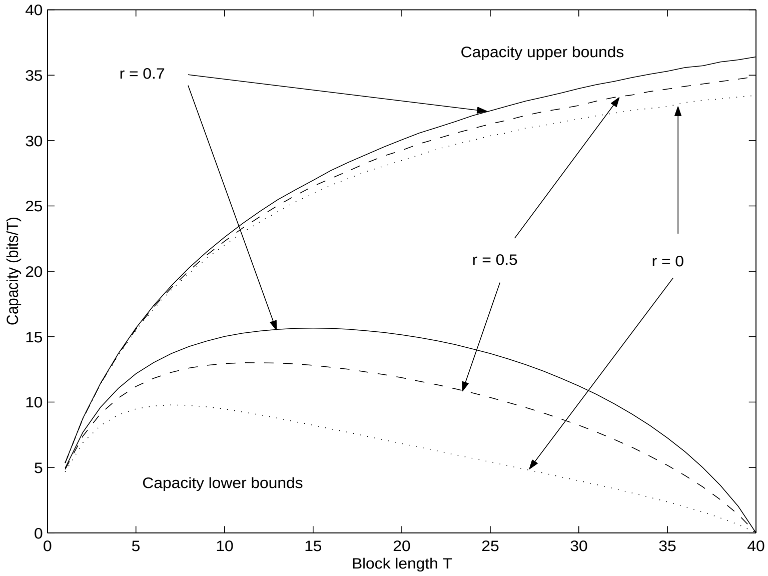


Figure 6.6: Plot of capacity as a function of number of transmit antennas for a fixed T

Figure 6.6 shows the optimum number of transmit antennas for a fixed block length $T = 40$ for $dB = 0$ and $N = 40$. In this case, we see that for a fixed T the optimum number of transmit antennas increases as as r increases. This shows that as r goes to 1 there is a lesser need to estimate the unknown rayleigh part of the channel and this agrees very well with figure 6.5 and figure 6.7 as well which shows that the optimal amount of training decreases as r increases. Figure 6.7 shows the optimal training period as a function of the block length for the case of equal transmit and training powers.

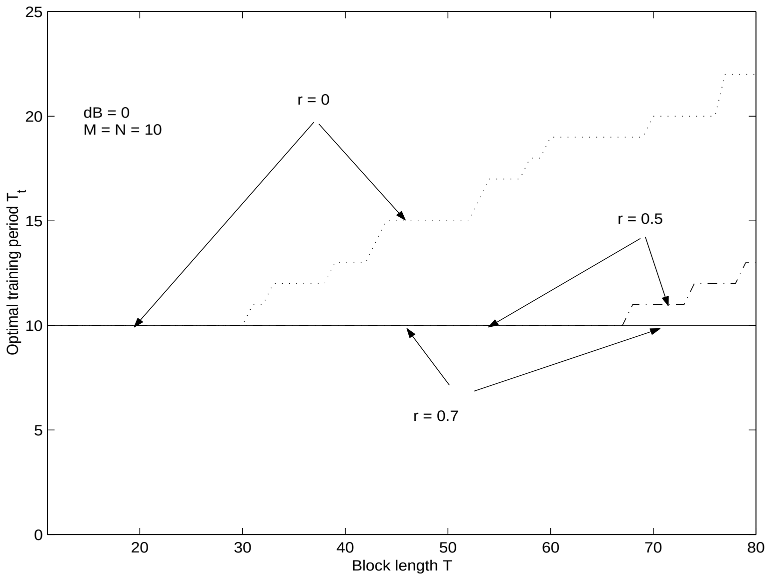


Figure 6.7: Plot of capacity as a function of number of transmit antennas for a fixed T

6.4.4 Effect of Low SNR on Capacity Lower Bound

Let's consider the effect of low SNR on the optimization of κ when $r \neq 0$. For $T_c > (1-r)M$, as $\rho \to 0$ it is easy to see that $\gamma - \sqrt{\gamma(\gamma - 1 - \eta)} \to \infty$. Therefore, we conclude that for small ρ we have $\kappa = 1$. Similarly, for $T_c = (1-r)M$ and $T_c < (1-r)M$. Therefore, the lower bound tells us that no energy need be spent on training for small ρ. Also, the form of Λ is known from section 6.3.1.

Evaluating the case where the training and transmission powers are equal we come to a similar conclusion. For small ρ, $\rho_{eff} \approx r\rho$ which is independent of T_t. Therefore, the best value of T_t is $T_t = 0$. Which also means that we spend absolutely no time on training. This is in stark contrast to the case when $r = 0$. In this case, for low

SNR $T_t = T/2$ [33] and ρ_{eff} behaves as $O(\rho^2)$.

Note that in both cases of equal and unequal power distribution between training and communication phases we have the signal distribution during data transmission phase to be Gaussian. Therefore, the lower bound behaves as $r\rho\lambda_{max}\{H_m H_m^\dagger\}$. Also, $r_{new} = 1$ for small ρ showing that the channel behaves as a purely Gaussian channel.

All the conclusions above mimic those of the capacity results with Gaussian input in section 6.3.3. The low SNR non-coherent capacity results for the case of a Gaussian input tell us that the capacity behaves as $r\rho\lambda_{max}$ with Gaussian input. Moreover, the results in [33] also agree quite well the results derived in section 6.3.3. We showed that for purely rayleigh fading channels with Gaussian input the capacity behaves as ρ^2 which is what the lower bound results in [33] also show. This makes sense because the capacity lower bound assumes that the signaling input during communication period is Gaussian. This shows that the lower bound derived in [33] and extended here is quite tight for low SNR values.

6.4.5 Effect of High SNR on Capacity Lower Bound

For high SNR, γ becomes $\frac{T_c}{T_c-(1-r)M}$ and the optimal power allocation κ becomes

$$\kappa = \frac{\sqrt{T_c}}{\sqrt{T_c} + \sqrt{(1-r)M}}$$

and

$$\rho_{eff} = \frac{T}{(\sqrt{T_c} + \sqrt{(1-r)M})^2}\rho$$

In the case of equal training and transmit powers, we have for high ρ

$$\rho_{eff} = \rho\frac{T_t}{T_t + M(1-r)}$$

For high SNR, the channel behaves as if it is completely known to the receiver. Note that in this case $r_{new} = r$ and Λ is an identity matrix for the case $M \leq N$.

From the expressions for ρ_{eff} given above we conclude that unlike the case of low SNR the value of r affects the amount of time and power devoted for training.

Let's look at the capacity lower bound for high SNR. The optimizing Λ in this regime is an identity matrix. We know that at high SNR the optimal training period is M therefore, the resulting lower bound is given by

$$C_t \geq \frac{T-M}{T} E \log\det\left(I_M + \frac{\rho}{\left(\sqrt{1-\frac{M}{T}}+\sqrt{\frac{(1-r)M}{T}}\right)^2}\frac{HH^\dagger}{M}\right)$$

Note that the lower bound has H figuring in it instead of H_1. That is so because for high SNR, $r_{new} = r$. This lower bound can be optimized over the number of transmit antennas used in which case the lower bound can be rewritten as

$$C_t \geq \max_{M' \leq M} \max_{n \leq \begin{pmatrix} M \\ M' \end{pmatrix}} \frac{T-M'}{T} E \log\det\left(I_{M'} + \frac{\rho}{\left(\sqrt{1-\frac{M'}{T}}+\sqrt{\frac{(1-r)M'}{T}}\right)^2}\frac{H_n H_n^\dagger}{M'}\right)$$

where now H_n is the n^{th} matrix out of a possible M choose M' (the number of ways to choose M' transmit elements out of a maximum M elements) matrices of size $M' \times N$. Let $Q = \min\{M', N\}$ and λ_i be an arbitrary nonzero eigenvalue of $\frac{1}{\left(\sqrt{1-\frac{M'}{T}}+\sqrt{\frac{(1-r)M'}{T}}\right)^2}\frac{H_n H_n^\dagger}{M'}$ then we have

$$C_t \geq \max_{M' \leq M} \max_{n \leq \begin{pmatrix} M \\ M' \end{pmatrix}} \left(1 - \frac{M'}{T}\right) \sum_{i=1}^{Q} E \log(1+\rho\lambda_i)$$

At high SNR, the leading term involving ρ in $\sum_{i=1}^{Q} E\log(1+\rho\lambda_i)$ is $Q\log\rho$ which is independent of n. Therefore,

$$C_t \geq \max_{M' \leq M} \begin{cases} (1-\frac{M'}{T})M'\log\rho & \text{if } M' \leq N \\ (1-\frac{M'}{T})N\log\rho & \text{if } M > N \end{cases}$$

The expression $(1-\frac{M'}{T})M'$, is maximized by choosing $M' = T/2$ when $\min\{M,N\} \geq T/2$ and by choosing $M' = \min\{M,N\}$ when $\min\{M,N\} \leq T/2$. This means that the expression is maximized when $M' = \min\{M,N,T/2\}$. This is a similar conclusion drawn in [33] and [74]. Also, the leading term in ρ for high SNR in the lower bound is given by

$$C_t \geq (1-\frac{K}{T})K\log\rho$$

where $K = \min\{M,N,T/2\}$. This result suggests that the number of degrees of freedom available for communication is limited by the minimum of the number of transmit antennas, receive antennas and half the coherence interval. Moreover, from the results in section 6.3.4 we see that the lower bound is tight for the case when $M \leq N$ and large T in the sense that the leading term involving ρ in the lower bound is the same as the one in the expression for capacity.

6.4.6 Comparison of the training based lower bound (6.7) with the lower bound derived in section 6.3.2

It is quite conceivable to use the lower bound to training based techniques as the lower bound to the overall capacity of the system. Actually, using this "training" based lower bound it can be shown that the capacity as $T \to \infty$ converges to the capacity as if the receiver knows the channel. We will see how the new lower bound derived in this work compares with this training based lower bound. The three figures below show that the new lower bound is indeed useful as it does better than the training based lower bound for $r = 0$. The plots are for $M = N = 1$ for different values of SNR.

However, we note that for $r = 1$ the training based lower bound and the lower bound derived in section 6.3.2 agree perfectly with each other and are equal to the upper bound.

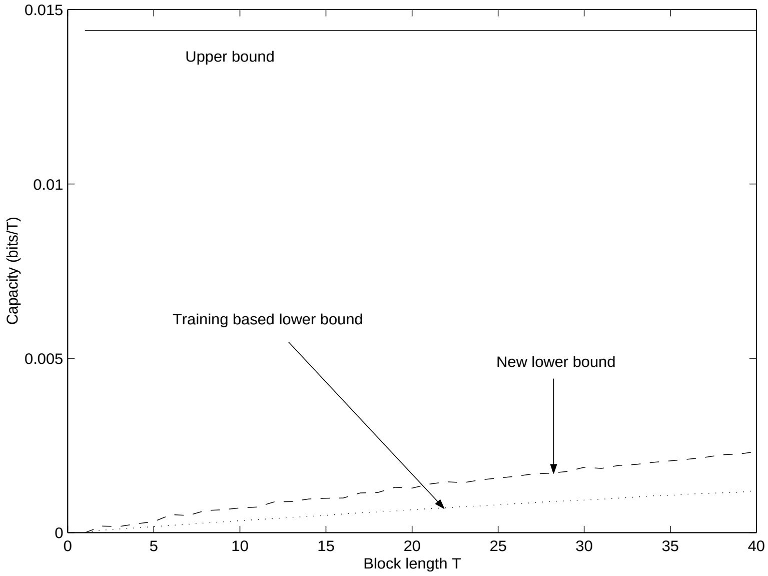


Figure 6.8: Comparison of the two lower bounds for $dB = -20$

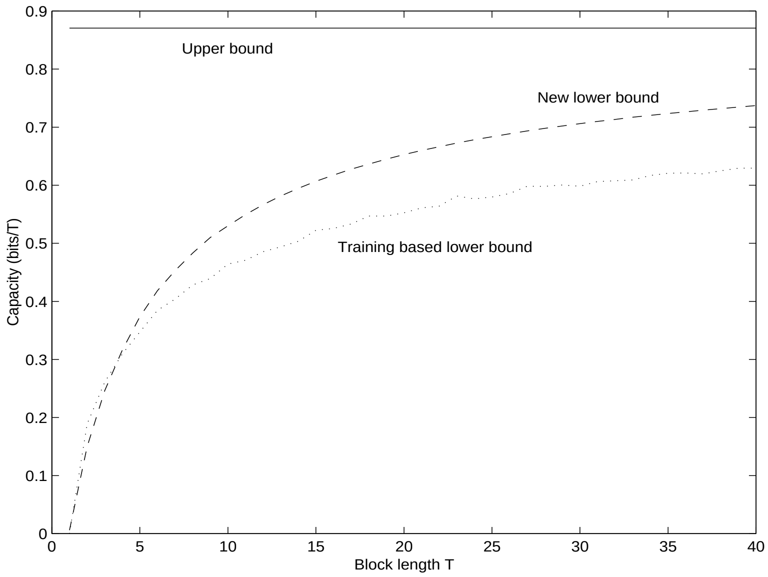


Figure 6.9: Comparison of the two lower bounds for $dB = 0$

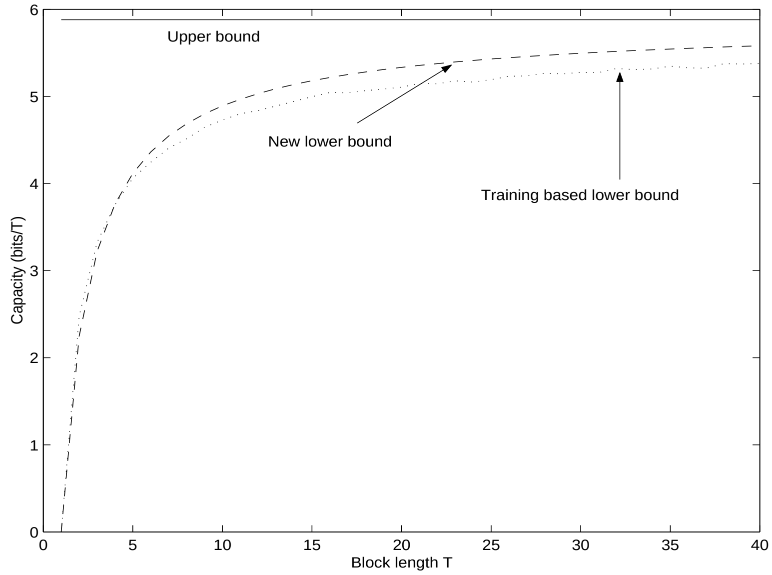


Figure 6.10: Comparison of the two lower bounds for $dB = 20$

6.5 Conclusions and Future Work

In this chapter, we have analyzed the standard rician fading channel for capacity. Most of the analysis was for a general specular component but, for the special case of a rank-one specular component we were able to show more structure on the signal input. For the case of general specular component, we were able to derive asymptotic closed form expressions for capacity for low and high SNR scenarios.

A big part of the analysis e.g. the non-coherent capacity expression and training based lower bounds can be very easily extended to the non-standard rician models considered in the previous two chapters.

One important result of the analysis is that for low SNRs beamforming is very desirable whereas for high SNR scenarios it is not. This result is very useful in designing space-time codes. For high SNR scenarios, one could wager that the standard codes designed for rayleigh fading can work for the case of rician fading as well.

A lot more work needs to be done such as for case of $M > N$. We believe that more work along the lines of [74] is possible for the case of rician fading. We conclude as in [74] that at least for the case $M = N$ the number of degrees of freedom is given by $M\frac{T-M}{T}$. The training based lower bound gives an indication that the number of degrees of freedom of a rician channel is the same as that of a rayleigh fading channel $\min\{M, N, T/2\}$ (derived in [74] and [33]). It also seems reasonable that the work in [4] can be extened to the case of rician fading.

APPENDICES

.1 Derivation of Stability Condition (3.6)

We will follow the Z-transform method of [37]. Let $\tilde{\xi}(z)$ donate the Z-transform of ξ_k and $\tilde{G}_i(z)$ donate the Z-transform of the i^{th} component of G_k. Then we have the following

$$\begin{aligned}\tilde{\xi}(z) &= \xi_{min}\frac{1}{1-z^{-1}} + \sum_{i=1}^{N}\tilde{G}_i(z)\\ \tilde{G}_i(z) &= (1-\frac{2\mu}{P}\lambda_i + \frac{2\mu^2}{P}\lambda_i^2)\tilde{G}_i(z) + \frac{\mu^2}{P}\lambda_i^2 z^{-1}\tilde{\xi}(z) + \\ &\quad G_i(0)\end{aligned}$$

which leads to

$$\tilde{\xi}(z) = \frac{\xi_{min}\frac{1}{1-z^{-1}} + \sum_{i=1}^{N}\frac{G_i(0)}{1-z^{-1}(1-\frac{2\mu}{P}\lambda_i+\frac{2\mu^2}{P}\lambda_i^2)}}{1-\sum_{i=1}^{N}\frac{\frac{\mu^2}{P}\lambda_i^2 z^{-1}}{1-z^{-1}(1-\frac{2\mu}{P}\lambda_i+\frac{2\mu^2}{P}\lambda_i^2)}} \qquad (.8)$$

and

$$\tilde{G}_i(z) = \frac{1}{D(z)}\frac{\frac{\mu^2}{P}\lambda_i^2 N(z) + G_i(0)D(z)}{1-z^{-1}(1-\frac{2\mu}{P}\lambda_i+\frac{2\mu^2}{P}\lambda_i^2)} \qquad (.9)$$

where $N(z)$ and $D(z)$ denote the numerator and the denominator in (.8). Therefore, the condition for stability is that the roots of

$$z-(1-\frac{2\mu}{P}\lambda_i+\frac{2\mu^2}{P}\lambda_i^2)$$

for $i = 1, \ldots, N$ and

$$\begin{gathered}\prod_{i=1}^{N}\left[z-(1-\tfrac{2\mu}{P}\lambda_i+\tfrac{2\mu^2}{P}\lambda_i^2)\right] - \\ \sum_{i=1}^{N}\tfrac{\mu^2}{P}\lambda_i^2\prod_{k\neq i}\left[z-(1-\tfrac{2\mu}{P}\lambda_k+\tfrac{2\mu^2}{P}\lambda_k^2)\right]\end{gathered}$$

should lie within the unit circle.

Please note that (.9) should be used to determine the stability of $\tilde{G}_i(z)$ and not

$$\tilde{G}_i(z) = \frac{\frac{\mu^2}{P}\lambda_i^2 z^{-1}\left[\xi_{min}\frac{1}{1-z^{-1}} + \sum_{k\neq i}\tilde{G}_k(z)\right] + G_i(0)}{1-z^{-1}(1-\frac{2\mu}{P}\lambda_i+\frac{3\mu^2}{P}\lambda_i^2)}$$

that was used in [37].

Following the rest of the procedure as outlined in [37] exactly, we obtain the conditions for stability to be (3.6).

.2 Derivation of expression (3.8)

Here we follow the procedure in [20]. Assuming G_k converges we have the expression for G_∞ to be

$$G_\infty = P\ [2\mu\Lambda - 2\mu^2\Lambda - \mu^2\Lambda^2\mathbf{1}\mathbf{1}^T]^{-1}\frac{\mu^2}{P}\Lambda^2\mathbf{1}\xi_{min}$$

Then we have

$$G_{k+1} - G_\infty = F(G_k - G_\infty)$$

where $F = I - \frac{2\mu}{P}\Lambda + \frac{2\mu^2}{P}\Lambda + \frac{\mu^2}{P}\Lambda^2\mathbf{1}\mathbf{1}^T$. Since $\xi_k = \mathrm{tr}\{G_k\}$ we have

$$\begin{aligned}
\sum_{k=0}^{\infty}(\xi_k - \xi_\infty) &= \mathrm{tr}\{\sum_{k=0}^{\infty}(G_k - G_\infty)\} \\
&= \mathrm{tr}\{\sum_{k=0}^{\infty} F^k(G_0 - G_\infty)\} \\
&= \mathrm{tr}\{(I-F)^{-1}(G_0 - G_\infty)\}
\end{aligned}$$

from which (3.8) follows.

.3 Derivation of the misadjustment factor (3.7)

Here we follow the approach of [37]. The misadjustment factor is defined as $M(\mu) = \frac{\xi_\infty - \xi_{min}}{\xi_{min}}$. Since $\xi_\infty = \lim_{z\to 1}(1 - z^{-1})\tilde{\xi}(z)$ we have

$$\xi_\infty = \frac{\lim_{z\to 1}\left[\xi_{min} + (1 - z^{-1})\sum_{i=1}^{N}\frac{G_i(0)}{1 - z^{-1}(1 - \frac{2\mu}{P}\lambda_i + \frac{2\mu^2}{P}\lambda_i^2)}\right]}{\lim_{z\to 1}\left[1 - \sum_{i=1}^{N}\frac{\frac{\mu^2}{P}\lambda_i^2 z^{-1}}{1 - z^{-1}(1 - \frac{2\mu}{P}\lambda_i + \frac{2\mu^2}{P}\lambda_i^2)}\right]}$$

That is

$$\begin{aligned}\xi_{\infty} &= \frac{\xi_{min}}{1-\frac{1}{2}\sum_{i=1}^{N}\frac{\mu\lambda_i}{1-\mu\lambda_i}} \\ &= \frac{\xi_{min}}{1-\eta(\mu)}\end{aligned}$$

from which 3.7 follows.

.4 Proof of Theorem III.1 in Section 3.5.1

For the proof, we need some definitions first. We define for $p \geq 1$ the set $\mathcal{M}_p$ of $F = \{F_i\}$ as

$$\mathcal{M}_p = \left\{ F : \sup_i \|S_i^{(T)}\|_p = o(T), \text{ as } T \to \infty \right\} \tag{.10}$$

where $S_i^{(T)} = \sum_{j=iT}^{(i+1)T-1}(F_j - E[F_j])$.

The proof is just a slightly modified version of the proof to Theorem 2 derived in Section IV in [45, pp. 766-769]. The modification takes into account that F_k is no longer $F_k = X_k X_k^H$ but, $F_k = I_k X_k X_k^H$. The Theorem III.1 is proved in a step by step manner using different lemmas. First, we rewrite

$$X_k = \sum_{j=-\infty}^{\infty} a_j \epsilon(k,j) + \xi_k, \quad \sum_{j=-\infty}^{\infty} a_j < \infty$$

where by definition

$$a_j \stackrel{\text{def}}{=} \sup_k \|A(k,j)\|, \;\; \epsilon(k,j) = a_j^{-1} A(k,j)\epsilon_{k-j}$$

The new process has some simple properties as listed in [45].

Lemma .6. *if $\{G_k\}$ is a ϕ-mixing $d \times d$-dimensional matrix then so is $\{F_k = I_k G_k\}$.*

Lemma .7. *Let $\{F_k\}$ be a ϕ-mixing $d \times d$-dimensional matrix process with mixing rate $\{\phi(m)$. Then*

$$\sup_i \|S_i^{(T)}\|_2 \leq 2cd \left\{ T \sum_{m=0}^{T-1} \sqrt{\phi(m)} \right\}^{1/2}, \;\; \forall T \geq 1$$

where $S_i^{(T)}$ *is as defined earlier and* c *is defined by* $c \stackrel{\text{def}}{=} \sup_i \|F_i - EF_i\|_2$.

Proof: Proof is the same as the proof of Lemma 1 in [45]. ■

Lemma .8. *Let* $F_k = I_k X_k X_k^H$*, where* $\{X_k\}$ *is defined by (3.11) with* $\sup_k \|\epsilon_k\|_4 < \infty$*. Then* $\{F_k\} \in \mathcal{M}_2$*, where* $\mathcal{M}_2$ *is defined by (.10).*

Proof: Proof is practically the same as the proof for Lemma 2 in [45]. All we need to say is that if $\{G_k\}$ is ϕ-mixing then so is $\{I_k G_k\}$. ■

Lemma .9. *Let* $\sup_k E\|X_k\|_2 < \infty$*. Then* $\{I_k X_k X_k^H\} \in \mathcal{S}$ *if and only if (3.13) holds, where* $\mathcal{S}$ *is defined by (3.10).*

Proof: Let us first assume that (3.13) is true. Take $\mu^* = (1 + \sup_k E\|X_k\|^2)^{-1}$. Then applying Theorem 2.1 in [32] to the deterministic sequence $A_k = \mu E[I_k X_k X_k^H]$ for any $\mu \in (0, \mu^*]$, it is easy to see that $\{I_k X_k X_k^H\} \in \mathcal{S}(\mu^*)$.

Conversely, if $\{X_k X_k^H\} \in \mathcal{S}$, then there exists $\mu^* \in (0, (1 + \sup_k E\|X_k\|^2)^{-1}]$ such that $\{X_k X_k^H\} \in \mathcal{S}(\mu^*)$. Now, applying Theorem 2.2 in [32] to the deterministic sequence $A_k = \mu^* E[I_k X_k X_k^H]$, it is easy to see that (3.13) holds. This completes the proof. ■

Lemma .10. *Let* $F_k = I_k X_k X_k^H$*, where* $\{X_k\}$ *is defined by (3.11) with (3.12) satisfied. Then* $\{F_k\}$ *satisfies condition 1) of Theorem 1.*

Proof: From Lemma 4 in [45] we know that $G_k = X_k X_k^H$ satisfies condition 1) of Theorem 1. Since $\|F_k\| \leq \|G_k\|$ it follows that $\{F_k\}$ satisfies condition 1) of Theorem 1. ■

Lemma .11 (Lemma 5 in [45]). *Let* $\{z_k\}$ *be a nonnegative random sequence such that for some* $a > 0, b > 0$ *and for all* $i_1 < i_2 < \ldots < i_n$, $\forall n \geq 1$

$$E \exp\left\{\sum_{k=1}^{n} z_{i_k}\right\} \leq \exp\{an + b\} \tag{.11}$$

Then for any $L > 0$ and any $n \geq i \geq 0$

$$E \exp\left\{\frac{1}{2}\sum_{j=i+1}^{n} z_j I(z_j \geq L)\right\} \leq \exp\{e^{a-L/2}(n-i)+b\}$$

where $I(\cdot)$ is the indicator function.

Proof: This lemma has been proved in [45]. ■

Lemma .12. *Let $F_k = I_k X_k X_k^H$ where $\{X_k\}$ is defined by (3.11) with (3.12) satisfied. Then $\{F_k\}$ satisfies condition 2) of Theorem 1.*

Proof: Set for any fixed k and l

$$z_j \stackrel{\text{def}}{=} z_j(k,l) = \left\|\sum_{t=jT}^{(j+1)T-1}[I_t\epsilon(t,k)\epsilon(t,l)^H - EI_t\epsilon(t,k)\epsilon(t,l)^H]\right\|$$

Then, we have

$$\sum_{j=i+1}^{n}\|S_j^(T)\| \leq \sum_{k,l=-\infty}^{\infty} a_k a_l \sum_{j=i+1}^{n} z_j + 2\sum_{k=-\infty}^{\infty} a_k \sum_{j=i+1}^{n}\left\|\sum_{t=jT}^{(j+1)T-1} I_t\epsilon(t,k)\xi_t^H\right\| \qquad (.12)$$

We first consider the second to the last term in the previous equation. By the Holder inequality

$$E\exp\left\{\mu\sum_{k,l=-\infty}^{\infty} a_k a_l \sum_{j=i+1}^{n} z_j\right\} \leq \prod_{k,l=-\infty}^{\infty}\left\{E\exp\left\{\mu A^2\sum_{j=i+1}^{n} z_j\right\}\right\}^{\frac{a_k a_l}{A^2}}$$

where $A \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} a_j$.

Now, let $c = \sum_k E\|\epsilon_k\|^2$, and note that

$$\begin{aligned}\|I_t\epsilon(t,k)\epsilon(t,l)^H\| &\leq \|\epsilon(t,k)\epsilon(t,l)^H\| \\ &\leq \frac{1}{2}[\|\epsilon(t,k)\|^2 + \|\epsilon(t,l)\|^2] \\ &\leq \frac{1}{2}(\|\epsilon_{t-k}\|^2 + \|\epsilon_{t-l}\|^2)\end{aligned}$$

and we have

$$z_j \leq \frac{1}{2}\sum_{t=jT}^{(j+1)T-1}(\|\epsilon_{t-k}\|^2 + \|\epsilon_{t-l}\|^2) + \frac{c}{m}T$$

By this and (3.12) it is easy to prove that the sequence $\{\alpha z_j\}$ satsifies (.11) with $a = (K + c/m)T$ and $b = logM$, where α is defined as in (3.12). Consequently, by Lemma 5 in [45] we have for any $L > 0$

$$E \exp\left\{\frac{\alpha}{2} \sum_{j=i+1}^{n} z_j I(z_j \geq LT)\right\} \leq M \exp\left\{e^{(K+c-\frac{\alpha L}{2})T}(n-i)\right\}$$

Now, in view of the above, taking $\mu < \frac{\alpha A^{-2}}{4}$ and $L > 2\alpha^{-1}(K + c)$, and applying the Holder inequality, we have

$$E \exp\left\{2\mu A^2 \sum_{j=i+1}^{n} z_j I(z_j \geq LT)\right\} \leq M exp\{\mu\delta(T)(n-i)\}$$

where $\delta(T) \to 0$ as $T \to \infty$, which is defined by

$$\delta(T) = 4\alpha^{-1} A^2 \exp\left\{(K + c - \frac{\alpha L}{2})T\right\}$$

Next, we consider the term $x_j \stackrel{\text{def}}{=} z_j I(z_j \leq LT)$.

By the inequality $e^x \leq 1 + 2x$, $0 \leq x \leq \log 2$, we have for small $\mu > 0$

$$\exp\left\{2\mu A^2 \sum_{j=i+1}^{n} x_j\right\} \leq \prod_{j=i+1}^{n} (1 + 4\mu A^2 x_j)$$

As noted before, for any fixed k and l, the process $\{\epsilon(t,k)\epsilon(t,l)^H\}$ is ϕ-mixing with mixing rate $\phi(m - |k-l|)$. Consequently, for any fixed k and l, both $\{z_j\}$ and $\{x_j\}$ are also ϕ-mixing with mixing rate $\phi((m-1)T + 1 - |k-l|)$. Note also that by Lemma 1 in [45]

$Ex_j \leq Ez_j \leq \|z_j\|_2 \leq f_{kl}(T)$ where $f_{kl}(T) = 2cd\left\{T \sum_{m=0}^{T-1} \sqrt{\phi(m - |k-j|)}\right\}^{\frac{1}{2}}$

Therefore, applying Lemma 6.2 in [45], we have

$$\begin{aligned} E \prod_{j=i+1}^{n} (1 + 4\mu A^2 x_j) &\leq 2\{1 + 8\mu A^2[f_{kl}(T) + 2LT\phi(T + 1 - |k-l|)]\}^{n-i} \\ &\leq 2\exp\{8\mu A^2[f_{kl}(T) + 2LT\phi(T + 1 - |k-l|)](n-i)\} \end{aligned}$$

Finally, using the Schwarz inequality we get

$$
\begin{aligned}
E\exp\left\{\mu A^2\sum_{j=i+1}^{n} z_j\right\} &\le \left\{E\exp\left\{2\mu A^2\sum_{j=i+1}^{n} z_j I(z_j\ge LT)\right\}\right\}^{1/2}\times \\
&\quad \left\{E\exp\left\{2\mu A^2\sum_{j=i+1}^{n} x_j\right\}\right\}^{1/2} \\
&\le \sqrt{2M}\exp\{\mu[\delta(T)+8A^2 f_{kl}(T)+ \\
&\quad 16LTA^2\phi(T+1-|k-l|)](n-i)\}
\end{aligned}
$$

Therefore, it is not difficult to see that there exists a function $g(T)=o(T)$ such that for all small $\mu>0$

$$
E\exp\left\{\mu\sum_{k,l=-\infty}^{\infty} a_k a_l \sum_{j=i+1}^{n} z_j\right\}\le\sqrt{2M}\exp\{\mu g(T)(n-i)\}.
$$

We can similarly bound the second term in (.12) and we are done. ■

Now we will highlight some of the remarks and corollaries to Theorem 2 in [45]. The remarks and corollaries and pertinent to this Theorem too.

Remark .1. *By taking $A(k,0)=I$, $A(k,j)=0$, $\forall k,\forall j\neq 0$, and $\xi_k=0$, $\forall k$ in (3.11), we see that $\{X_k\}$ is the same as ϵ_k, which means that Theorem III.1 is applicable to any ϕ-mixing sequences.*

Corollary .3. *Let the signal process be generated by (3.11), where $\{\xi_k\}$ is a bounded deterministic sequence, and $\{\epsilon_k\}$ is an independent sequence satisfying*

$$
\sup_k E[\exp(\alpha\|\epsilon_k\|^2)]<\infty,\ \textit{for some } \alpha>0
$$

Then $\{X_k X_k^H\}\in\mathcal{S}_p$ for some $p\ge 1$ if and only if there exists an integer $h>0$ and a constant $\delta>0$ such that (3.13) holds.

Proof: Similar to the proof of Corollary 1 in [45]. ■

.5 Derivation of Expressions in Section 3.7.2

In this section, we will need the following identity

$$\sum_{s=0}^{\infty} s(1-\alpha\mu)^{2s} = \frac{(1-\alpha\mu)^2}{\alpha^2\mu^2(2-\alpha\mu)^2}$$

First, we have the following expressions for LMS

$$\begin{aligned}
J_{k+1}^0 &= \sum_{s=0}^{k}(1-\mu\sigma^2)^{k-s}X_s n_s \\
J_{k+1}^1 &= \mu\sum_{s=0}^{k}(1-\mu\sigma^2)^{k-s-1}D_1(k,s+1)X_s n_s \\
J_{k+1}^2 &= \mu^2\sum_{s=0}^{k}(1-\mu\sigma^2)^{k-s-2}D_2(k,s+1)X_s n_s
\end{aligned}$$

where

$$\begin{aligned}
D_1(k,s) &= \sum_{u=s}^{k} Z_u \quad k \geq s \quad D_1(k,s) = 0 \quad s > k \\
D_2(k,s) &= \sum_{u=s}^{k} D_1(k,u+1)Z_u
\end{aligned}$$

and $Z_u = E[X_u X_u^H] - X_u X_u^H$

This leads to

$$\lim_{k\to\infty} E[J_{k+1}^0(J_{k+1}^0)^H] = \lim_{k\to\infty}\sigma_v^2\sum_{s=0}^{k}(1-\mu\sigma^2)^{2(k-s)}E[X_0X_0^H]$$

and the expression follows.

$$\lim_{k\to\infty} E[J_{k+1}^0(J_{k+1}^1)^H] = \mu\sigma_v^2\sum_{s=0}^{k}(1-\mu\sigma^2)^{2s-1}E[D_1(s,1)X_0X_0^H]$$

Now, $E[Z_u X_0 X_0^H] = E[X_u X_u^H]E[X_0X_0^H] - E[X_u X_u^H X_0 X_0^H] = 0$ which gives

$$E[D_1(s,1)X_0X_0^H] = 0$$

$$\lim_{k\to\infty} E[J^1_{k+1}(J^1_{k+1})^H] = \sigma_v^2\mu^2\sum_{u=0}^{\infty}(1-\mu\sigma^2)^{2u-2}E[D_1(u,1)X_0X_0^HD_1(u,1)^H]$$

Now

$$\begin{aligned}
E[Z_vX_0X_0^HZ_u^H] &= \sigma^6 I - \sigma^2E[X_vX_v^HX_0X_0^H] - \sigma^2E[X_0X_0^HX_uX_u^H] + \\
&\quad E[X_vX_v^HX_0X_0^HX_uX_u^H] \\
&= 0 \ \text{ if } v \neq u \\
&= N\sigma^6 I \ \text{ if } v = u
\end{aligned}$$

Therefore, $E[D_1(u,1)X_0X_0^HD_1(u,1)^H] = uN\sigma^6 I$ and the expression follows.

$\lim_{k\to\infty} E[J^0_{k+1}(J^2_{k+1})^H] = \mu^2\sigma_v^2\sum_{s=0}^{\infty}E[D_2(s,1)X_0X_0^H]$. Now

$$\begin{aligned}
E[Z_vZ_uX_0X_0^H] &= \sigma^6 I - \sigma^2E[X_vX_v^HX_0X_0^H] - \sigma^2E[X_uX_u^HX_0X_0^H] + \\
&\quad E[X_vX_v^HX_uX_u^HX_0X_0^H] \\
&= 0 \ \text{ if } v \neq u
\end{aligned}$$

Therefore, $E[D_2(s,1)X_0X_0^H] = 0$.

Second, we have the following expressions for SPULMS

$$\begin{aligned}
J^0_{k+1} &= \sum_{s=0}^{k}(1-\frac{\mu}{P}\sigma^2)^{k-s}I_sX_sn_s \\
J^1_{k+1} &= \mu\sum_{s=0}^{k}(1-\frac{\mu}{P}\sigma^2)^{k-s-1}D_1(k,s+1)I_sX_sn_s \\
J^2_{k+1} &= \mu^2\sum_{s=0}^{k}(1-\frac{\mu}{P}\sigma^2)^{k-s-2}D_2(k,s+1)I_sX_sn_s
\end{aligned}$$

where

$$\begin{aligned} D_1(k,s) &= \sum_{u=s}^{k} Z_u \quad k \geq s \quad D_1(k,s) = 0 \quad s > k \\ D_2(k,s) &= \sum_{u=s}^{k} D_1(k,u+1) Z_u \end{aligned}$$

and $Z_u = I_u X_u X_u^H - \frac{1}{P} E[X_u X_u^H]$

This leads to

$$\lim_{k\to\infty} E[J_{k+1}^0 (J_{k+1}^0)^H] = \lim_{k\to\infty} \sigma_v^2 \sum_{s=0}^{k} (1-\mu\sigma^2)^{2(k-s)} E[I_0 X_0 X_0^H I_0]$$

and the expression follows.

$$\lim_{k\to\infty} E[J_{k+1}^0 (J_{k+1}^1)^H] = \mu\sigma_v^2 \sum_{s=0}^{k} (1-\mu\sigma^2)^{2s-1} E[D_1(s,1) I_0 X_0 X_0^H I_0]$$

Now, $E[Z_u I_0 X_0 X_0^H I_0] = E[I_u X_u X_u^H] E[I_0 X_0 X_0^H I_0] - E[I_u X_u X_u^H I_0 X_0 X_0^H I_0] = 0$ which gives $E[D_1(s,1) X_0 X_0^H] = 0$.

$$\lim_{k\to\infty} E[J_{k+1}^1 (J_{k+1}^1)^H] = \sigma_v^2 \mu^2 \sum_{u=0}^{\infty} (1-\mu\sigma^2)^{2u-2} E[D_1(u,1) I_0 X_0 X_0^H I_0 D_1(u,1)^H]$$

Now

$$\begin{aligned} E[Z_v I_0 X_0 X_0 I_0 Z_u^H] &= \sigma^6 I - \sigma^2 E[I_v X_v X_v^H I_0 X_0 X_0^H I_0] - \sigma^2 E[I_0 X_0 X_0^H I_0 X_u X_u^H I_u] \\ &\quad + E[I_v X_v X_v^H I_0 X_0 X_0^H I_0 X_u X_u^H I_u] \\ &= 0 \quad \text{if } v \neq u \\ &= \frac{(N+1)P-1}{P^3} \sigma^6 I \quad \text{if } v = u \end{aligned}$$

Therefore, $E[D_1(u,1) X_0 X_0^H D_1(u,1)^H] = u\frac{(N+1)P-1}{P^3}\sigma^6 I$ and the expression follows.

$\lim_{k\to\infty} E[J^0_{k+1}(J^2_{k+1})^H] = \mu^2\sigma_v^2 \sum_{s=0}^{\infty} E[D_2(s,1)X_0X_0^H]$. Now

$$
\begin{aligned}
E[Z_vZ_uX_0X_0^H] &= \sigma^6 I - \sigma^2 E[I_vX_vX_v^H I_0X_0X_0^H I_0] - \sigma^2 E[I_uX_uX_u^H I_0X_0X_0^H I_0] \\
&\quad + E[I_vX_vX_v^H I_uX_uX_u^H I_0X_0X_0^H I_0] \\
&= 0 \text{ if } v \neq u
\end{aligned}
$$

Therefore, $E[D_2(s,1)X_0X_0^H] = 0$.

.6 Derivation of Expressions in Section 3.7.1

In this section, we will need the following identities

$$
\begin{aligned}
\sum_{v,w=1}^{s} a^{2|v-w|} &= \frac{s(1-a^4) - 2a^2 + 2a^{2(s+1)}}{(1-a^2)^2} \\
\sum_{v,w=1}^{s} a^{|v-w|}a^{v+w} &= \frac{a^2}{(1-a^2)^2}[1 + a^2 - (2s+1)a^{2s} + (2s-1)a^{2s+2}] \\
\sum_{s=0}^{\infty} s(1-\alpha\mu)^{2s} &= \frac{(1-\alpha\mu)^2}{\alpha^2\mu^2(2-\alpha\mu)^2}
\end{aligned}
$$

First, we have the following expressions for LMS

$$
\begin{aligned}
J^0_{k+1} &= \sum_{s=0}^{k}(1-\mu)^{k-s}X_s n_s \\
J^1_{k+1} &= \mu\sum_{s=0}^{k}(1-\mu)^{k-s-1}D_1(k,s+1)X_s n_s \\
J^2_{k+1} &= \mu^2\sum_{s=0}^{k}(1-\mu)^{k-s-2}D_2(k,s+1)X_s n_s
\end{aligned}
$$

where

$$
\begin{aligned}
D_1(k,s) &= \sum_{u=s}^{k} Z_u \quad k \geq s \quad D_1(k,s) = 0 \quad s > k \\
D_2(k,s) &= \sum_{u=s}^{k} D_1(k,u+1)Z_u
\end{aligned}
$$

and $Z_u = E[X_uX_u^H] - X_uX_u^H$

This leads to

$$\lim_{k\to\infty} E[J^0_{k+1}(J^0_{k+1})^H] = \lim_{k\to\infty} \sigma_v^2 \sum_{s=0}^{k}(1-\mu)^{2(k-s)} E[X_0 X_0^H]$$

and the expression follows.

$$\lim_{k\to\infty} E[J^0_{k+1}(J^1_{k+1})^H] = \mu\sigma_v^2 \sum_{s=0}^{k}(1-\mu)^{2s-1} E[D_1(s,1)X_0X_0^H]$$

Now, $E[Z_u X_0 X_0^H] = E[X_u X_u^H]E[X_0X_0^H] - E[X_uX_u^HX_0X_0^H] = -\frac{N}{P^3}\kappa^{2u}$ which gives $E[D_1(s,1)X_0X_0^H] = -\frac{N}{P^3}\kappa^2\frac{1-\kappa^{2s}}{1-\kappa^2}$.

$$\lim_{k\to\infty} E[J^1_{k+1}(J^1_{k+1})^H] = \sigma_v^2\mu^2 \sum_{u=0}^{\infty}(1-\mu)^{2u-2} E[D_1(u,1)X_0X_0^H D_1(u,1)^H]$$

Now

$$\begin{aligned} E[Z_vX_0X_0^HZ_u^H] &= I - E[X_vX_v^HX_0X_0^H] - E[X_0X_0^HX_uX_u^H] + E[X_vX_v^HX_0X_0^HX_uX_u^H] \\ &= [(N^2+1)\kappa^{v+u}\kappa^{|v-u|} + N\kappa^{2|v-u|} \end{aligned}$$

Therefore,

$$E[D_1(u,1)X_0X_0^HD_1(u,1)^H] = (N^2+1)\sum_{s=1}^{u}\sum_{t=1}^{u}\kappa^{|v-u|}\kappa^{v+u} + N\sum_{s=1}^{u}\sum_{t=1}^{u}\kappa^{2|v-u|}$$

and the expression follows.

$\lim_{k\to\infty} E[J^0_{k+1}(J^2_{k+1})^H] = \mu^2\sigma_v^2\sum_{s=0}^{\infty} E[D_2(s,1)X_0X_0^H]$. Now

$$\begin{aligned} E[Z_vZ_uX_0X_0^H] &= I - E[X_vX_v^HX_0X_0^H] - E[X_uX_u^HX_0X_0^H] + E[X_vX_v^HX_uX_u^HX_0X_0^H] \\ &= [(N^2+1)\kappa^{v+u}\kappa^{|v-u|} + N\kappa^{2|v-u|} \end{aligned}$$

Therefore,

$$E[D_2(s,1)X_0X_0^H] = (N^2+1)\sum_{s=1}^{u}\sum_{t=s+1}^{u}\kappa^{|v-u|}\kappa^{v+u} + N\sum_{s=1}^{u}\sum_{t=s+1}^{u}\kappa^{2|v-u|}$$

Second, we have the following expressions for SPULMS

$$\begin{aligned} J_{k+1}^0 &= \sum_{s=0}^{k}(1-\frac{\mu}{P})^{k-s} I_s X_s n_s \\ J_{k+1}^1 &= \mu\sum_{s=0}^{k}(1-\frac{\mu}{P})^{k-s-1} D_1(k,s+1) I_s X_s n_s \\ J_{k+1}^2 &= \mu^2\sum_{s=0}^{k}(1-\frac{\mu}{P})^{k-s-2} D_2(k,s+1) I_s X_s n_s \end{aligned}$$

where

$$\begin{aligned} D_1(k,s) &= \sum_{u=s}^{k} Z_u \quad k \geq s \quad D_1(k,s) = 0 \quad s > k \\ D_2(k,s) &= \sum_{u=s}^{k} D_1(k,u+1) Z_u \end{aligned}$$

and $Z_u = I_u X_u X_u^H - \frac{1}{P} E[X_u X_u^H]$

This leads to

$$\lim_{k\to\infty} E[J_{k+1}^0 (J_{k+1}^0)^H] = \lim_{k\to\infty} \sigma_v^2 \sum_{s=0}^{k}(1-\mu)^{2(k-s)} E[I_0 X_0 X_0^H I_0]$$

and the expression follows.

$$\lim_{k\to\infty} E[J_{k+1}^0 (J_{k+1}^1)^H] = \mu\sigma_v^2 \sum_{s=0}^{k}(1-\mu)^{2s-1} E[D_1(s,1) I_0 X_0 X_0^H I_0]$$

Now, $E[Z_u I_0 X_0 X_0^H I_0] = E[I_u X_u X_u^H] E[I_0 X_0 X_0^H I_0] - E[I_u X_u X_u^H I_0 X_0 X_0^H I_0] = -\frac{N}{P^3}\kappa^{2u}$ which gives $E[D_1(s,1) X_0 X_0^H] = -\frac{N}{P^3}\kappa^2 \frac{1-\kappa^{2s}}{1-\kappa^2}$.

$$\lim_{k\to\infty} E[J_{k+1}^1 (J_{k+1}^1)^H] = \sigma_v^2 \mu^2 \sum_{u=0}^{\infty}(1-\mu)^{2u-2} E[D_1(u,1) I_0 X_0 X_0^H I_0 D_1(u,1)^H]$$

Now

$$
\begin{aligned}
E[Z_v I_0 X_0 X_0 I_0 Z_u^H] &= I - E[I_v X_v X_v^H I_0 X_0 X_0^H I_0] - E[I_0 X_0 X_0^H I_0 X_u X_u^H I_u] \\
&\quad + E[I_v X_v X_v^H I_0 X_0 X_0^H I_0 X_u X_u^H I_u] \\
&= \frac{1}{P^3}[(\frac{N^2}{P}+1)\kappa^{v+u}\kappa^{|v-u|} + N\kappa^{2|v-u|}]I \quad \text{if } v \neq u \\
&= \frac{1}{P^3}[(\frac{N^2}{P}+1)\kappa^{v+u}\kappa^{|v-u|} + N\kappa^{2|v-u|}]I \\
&\quad + \frac{P-1}{P^3}[(N+1) + \frac{N^2+2N+1}{P}\kappa^{2u}]I \quad \text{if } v = u
\end{aligned}
$$

$\lim_{k\to\infty} E[J_{k+1}^0 (J_{k+1}^2)^H] = \mu^2 \sigma_v^2 \sum_{s=0}^{\infty} E[D_2(s,1) X_0 X_0^H]$. Now

$$
\begin{aligned}
E[Z_v Z_u X_0 X_0^H] &= \sigma^6 I - \sigma^2 E[I_v X_v X_v^H I_0 X_0 X_0^H I_0] - \sigma^2 E[I_u X_u X_u^H I_0 X_0 X_0^H I_0] \\
&\quad + E[I_v X_v X_v^H I_u X_u X_u^H I_0 X_0 X_0^H I_0] \\
&= \frac{1}{P^3}[(\frac{N^2}{P}+1)\kappa^{v+u}\kappa^{|v-u|} + N\kappa^{2|v-u|}]I \quad \text{if } v \neq u
\end{aligned}
$$

.7 Capacity Optimization in Section 6.2.1

We have the following expression for the capacity

$$C = E \log\det(I_N + \frac{\rho}{M} H^\dagger \Lambda H)$$

where Λ is of the form

$$\Lambda = \begin{bmatrix} M-(M-1)d & L \\ L^\dagger & D \end{bmatrix}$$

Let l^r denote the real part of l and l^i the imaginary part. We can find the optimal value of d and l iteratively by using the method of steepest descent as follows

$$\begin{aligned} d_{k+1} &= d_k + \mu \frac{\partial C}{\partial d_k} \\ l^r_{k+1} &= l^r_k + \mu \frac{\partial C}{\partial l^r_k} \\ l^i_{k+1} &= l^i_k + \mu \frac{\partial C}{\partial l^i_k} \end{aligned}$$

where d_k, l^r_k and l^i_k are the values of d, l^r and l^i respectively at the k^{th} iteration. We use the following identity (Jacobi's formula) to calculate the partial derivatives (REFERENCE???).

$$\frac{\partial \log \det A}{\partial d} = \mathrm{tr}\{A^{-1} \frac{\partial A}{\partial d}\}$$

Therefore, we obtain

$$\frac{\partial C}{\partial d} = E\mathrm{tr}\{[I_N + \frac{\rho}{M} H^\dagger \Lambda H]^{-1} \frac{\rho}{M} H^\dagger \frac{\partial \Lambda}{\partial d} H\}$$

and similarly for l^r and l^i where

$$\begin{aligned} \frac{\partial \Lambda}{\partial d} &= \begin{bmatrix} -(M-1) & \mathbf{0} \\ \mathbf{0}^T & I_{M-1} \end{bmatrix} \\ \frac{\partial \Lambda}{\partial l^r} &= \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^T & \mathbf{0}_{M-1} \end{bmatrix} \\ \frac{\partial \Lambda}{\partial l^i} &= \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1}^T & \mathbf{0}_{M-1} \end{bmatrix} \end{aligned}$$

The derivative can be evaluated using monte carlo simulation.

.8 Proof of Lemma VI.3 in Section 6.3.4

In this section we will show that for the optimal input $(s_i^{(\sigma)}, i = 1, \dots, M)$, $\forall \delta, \epsilon > 0$, $\exists \sigma_0$ such that for all $\sigma < \sigma_0$

$$P(\frac{\sigma}{\|s_i^\sigma\|} > \delta) < \epsilon \tag{.13}$$

for $i = 1, \dots, M$.

This proof in this section has basically been reproduced from [74] except for some minor changes to account for the determinstic specular component (H_m) present in the channel.

The proof is by contradiction. We need to show that if an input distribution P satisfies $P(\frac{\sigma}{\|s_i\|} > \delta) > \epsilon$ for some ϵ and δ and for arbitrarily small σ^2, it is not optimal for small enough noise level σ^2. That is, we can construct another input distribution that satisfies the same power constraint, but achieves higher mutual information. The steps in the proof are as follows

1. We show that in a system with M transmit and N receive antennas, coherence time $T \geq 2N$, if $M \leq N$, there exists a finite constant $k_1 < \infty$ such that for any fixed input distribution of S, $I(X; S) \leq k_1 + M(T - M)\log\rho$. That is, the mutual information increases with SNR at a rate no higher than $M(T-M)\log\rho$.

2. For a system with M transmit and receive antennas, if we only choose to send signals with strong power in M' of the transmit antennas, that is $\|s_i\| \leq C\sigma$ for $i = M' + 1, \dots, M$ and some constant C, we show that the mutual information increases with SNR at rate no higher than $M'(T - M')\log\rho$.

3. We show that for a system with M transmit and receive antennas if the input distribution doesn't satisfy (.13), that is, has a positive probability that $\|s_i\| \leq$

$C\sigma$, the mutual information achieved increases with SNR at rate strictly lower than $M(T-M)\log\rho$.

4. We show that in a system with M transmit and receive antennas for constant equal norm input $P(\|s_i\| = \sqrt{T}) = 1$, for $i = 1, \ldots, M$, the mutual information increases with SNR at rate $M(T-M)\log\rho$. Since $M(T-M) \geq M'(T-M')$ for any $M' \leq M$ and $T \geq 2M$, any input distribution that doesn't satisfy .13 yields a mutual information that increases at lower rate than constant equal norm input, and thus is not optimal at high enough SNR level.

Step 1 For a channel with M transmit and N receive antennas, if $M < N$ and $T \geq 2N$, we write the conditional differential entropy as

$$\mathcal{H}(X|S) = N\sum_{i=1}^{M} E[\log((1-r)\|s_i\|^2 + \sigma^2)] + N(T-M)\log\pi e\sigma^2$$

Let $X = \Phi_X \Sigma_X \Psi_X^\dagger$ be the SVD for X then

$$\begin{aligned}
\mathcal{H}(X) &\leq \mathcal{H}(\Phi_X) + \mathcal{H}(\Sigma_X|\Psi) + \mathcal{H}(\Psi) + E[\log J_{T,M}(\sigma_1, \ldots, \sigma_N)] \\
&\leq \mathcal{H}(\Phi_X) + \mathcal{H}(\Sigma_X) + \mathcal{H}(\Psi) + E[\log J_{T,N}(\sigma_1, \ldots, \sigma_N)] \\
&= \log|S(N,N)| + \log|S(T,N)| + \mathcal{H}(\Sigma_X) + E[\log J_{T,N}(\sigma_1, \ldots, \sigma_N)]
\end{aligned}$$

Now

$$\begin{aligned}
\Sigma_X &= \mathcal{H}(\sigma_1, \ldots, \sigma_M, \sigma_{M+1}, \ldots, \sigma_N) \\
&\leq \mathcal{H}(\sigma_1, \ldots, \sigma_M) + \mathcal{H}(\sigma_{M+1}, \ldots, \sigma_N)
\end{aligned}$$

Now

$$
\begin{aligned}
E[\log J_{T,N}(\sigma_1,\ldots,\sigma_N)] &= \log\frac{1}{(2\pi)^N} + \sum_{i=1}^{N} E[\log\sigma_i^{2(T-N)+1}] + \sum_{i<j\leq N} E[\log(\sigma_i^2-\sigma_j^2)^2] \\
&= \log\frac{1}{(2\pi)^M} + \sum_{i=1}^{M} E[\log\sigma_i^{2(T-N)+1}] + \sum_{i<j\leq M} E[\log(\sigma_i^2-\sigma_j^2)^2] + \\
&\quad \sum_{i\leq M, M<j\leq N} E[\underbrace{\log(\sigma_i^2-\sigma_j^2)^2}_{\leq\log\sigma_i^4}] + \\
&\quad \log\frac{1}{(2\pi)^{N-M}} + \\
&\quad \sum_{i=M+1}^{N} E[\log\sigma_i^{2(T-N)+1}] + \sum_{M<i<j\leq N} E[\log(\sigma_i^2-\sigma_j^2)^2] \\
&\leq E[\log J_{N,M}(\sigma_1,\ldots,\sigma_M)] + E[\log J_{T-M,N-M}(\sigma_{M+1},\ldots,\sigma_N)] \\
&\quad +2(T-M)\sum_{i=1}^{M} E[\log\sigma_i^2]
\end{aligned}
$$

Now define $C_1 = \Phi_1\Sigma_1\Psi_1^\dagger$ where $\Sigma_1 = \mathrm{diag}(\sigma_1,\ldots,\sigma_M)]$, Φ_1 is a $N\times M$ unitary matrix, Ψ_1 is a $M\times M$ unitary matrix. Σ_1, Φ_1 and Ψ_1 are all independent of each other. Similarly define C_2 from the rest of the eigenvalues. Now

$$
\begin{aligned}
\mathcal{H}(C_1) &= \log|S(M,M)| + \log|S(N,M)| + \mathcal{H}(\sigma_1,\ldots,\sigma_M) + E[\log J_{N,M}(\sigma_1,\ldots,\sigma_M)] \\
\mathcal{H}(C_2) &= \log|S(N-M,N-M)| + \log|S(T-M,N-M)| \\
&\quad +\mathcal{H}(\sigma_{M+1},\ldots,\sigma_N) + E[\log J_{T-M,N-M}(\sigma_{M+1},\ldots,\sigma_N)]
\end{aligned}
$$

Subsituting in the formula for $\mathcal{H}(X)$, we obtain

$$
\begin{aligned}
\mathcal{H}(X) &\leq \mathcal{H}(C_1) + \mathcal{H}(C_2) + (T-M)\sum_{i=1}^{M} E[\log\sigma_i^2] + \log|S(T,N)| + \log|S(N,N)| \\
&\quad -\log|S(N,M)| - \log|S(M,M)| - \log|S(N-M,N-M)| - \\
&\quad \log|S(T-M,N-M)| \\
&= \mathcal{H}(C_1) + \mathcal{H}(C_2) + (T-M)\sum_{i=1}^{M} E[\log\sigma_i^2] + \log|G(T,M)|
\end{aligned}
$$

Now note that C_1 has bounded total power

$$
\mathrm{tr}\{E[C_1C_1^\dagger]\} = \mathrm{tr}\{E[\sigma_i^2]\} = \mathrm{tr}\{E[XX^\dagger]\} \leq NT(M+\sigma^2)
$$

Therefore, the differential entropy of C_1 is bounded by the entropy of a random matrix with entries iid Gaussian distributed with variance $\frac{T(M+\sigma^2)}{M}$. That is

$$\mathcal{H}(C_1) \le NM \log\left[\pi e \frac{T(M+\sigma^2)}{M}\right]$$

Similarly, we bound the total power of C_2. Since $\sigma_{M+1}, \ldots, \sigma_N$ are the $N-M$ least singular values of X, for any $(N-M)\times N$ unitary matrix Q.

$$\mathrm{tr}\{E[C_2 C_2^\dagger]\} \le (N-M)T\sigma^2$$

Therefore, the differential entropy is maximized if C_2 has independent iid Gaussian entries and

$$\mathcal{H}(C_2) \le (N-M)(T-M)\log\left[\pi e \frac{T\sigma^2}{T-M}\right]$$

Therefore, we obtain

$$\begin{aligned}\mathcal{H}(X) \;\; \le \;\; & \log|G(T,M)| + NM\log\left[\pi e \frac{T(M+\sigma^2)}{M}\right] + (T-M)\sum_{i=1}^{M} E[\log \sigma_i^2] \\ & +(N-M)(T-M)\log \pi e \sigma^2 + (N-M)(T-M)\log\frac{T}{T-M}\end{aligned}$$

Combining with $\mathcal{H}(X|S)$, we obtain

$$\begin{aligned}I(X;S) \;\; \le \;\; & \underbrace{\log|G(T,M)| + NM\log\frac{T(M+\sigma^2)}{M} + (N-M)(T-M)\log\frac{T}{T-M}}_{\alpha} \\ & + \underbrace{(T-M-N)\sum_{i=1}^{M} E[\log\sigma_i^2]}_{\beta} + \\ & \underbrace{N\left(\sum_{i=1}^{M} E[\log\sigma_i^2] - \sum_{i=1}^{M} E[\log((1-r)\|s_i\|^2+\sigma^2)]\right)}_{\gamma} \\ & -M(T-M)\log \pi e \sigma^2\end{aligned}$$

It has been shown in [74] that β is upper bounded since $\sum_{i=1}^{M}\sigma_i^2 = \mathrm{tr}\{XX^\dagger\}$, therefore

$$\begin{aligned}\sum_{i=1}^{M} E[\log \sigma_i^2] &\leq M \log(\frac{1}{M}\sum_{i=1}^{M} E[\sigma_i^2]) \\ &= M \log \frac{NT(M+\sigma^2)}{M}\end{aligned}$$

For γ it will be shown that

$$\sum_{i=1}^{M} E[\log \sigma_i^2] - \sum_{i=1}^{M} E[\log((1-r)\|s_i\|^2 + \sigma^2)] \leq k$$

where k is some finite constant.

Given S, X has mean $\sqrt{r}SH_m$ and covariance matrix $I_N \otimes ((1-r)SS^\dagger + \sigma^2 I_T)$. We will try to find given S what the distribution of $X^\dagger X$ looks like If $S = \Phi V \Psi^\dagger$ then

$$\begin{aligned}X^\dagger X &= H^\dagger S^\dagger S H + W^\dagger S H + H^\dagger S^\dagger W + W^\dagger W \\ &\stackrel{d}{=} H_1^\dagger V^2 H_1 + W^\dagger V H_1 + H_1^\dagger V W + W^\dagger W\end{aligned}$$

where H_1 has the covariance matrix as H but mean is given by $\sqrt{r}\Psi^\dagger H_m$. Therefore, we can write $X_1 = VH_1 + W$

Now, X_1 has the same distribution as $((1-r)VV^\dagger + \sigma^2 I_T)^{1/2} Z$ where Z is a random Gaussian matrix with mean $\sqrt{r}((1-r)VV^\dagger + \sigma^2 I_T)^{-1/2}\Psi^\dagger H_m$ and covariance I_{NT}. Therefore,

$$X^\dagger X \stackrel{d}{=} Z^\dagger((1-r)VV^\dagger + \sigma^2 I_T)Z$$

$$(X^\dagger X|S) \stackrel{d}{=} Z^\dagger \begin{bmatrix} (1-r)\|s_1\|^2+\sigma^2 & & & & & \\ & \ddots & & & & \\ & & (1-r)\|s_M\|^2+\sigma^2 & & & \\ & & & \sigma^2 & & \\ & & & & \ddots & \\ & & & & & \sigma^2 \end{bmatrix} Z$$

Let $Z = [Z_1|Z_2]$ be the partition of Z such that

$$(X^\dagger X|S) \stackrel{d}{=} Z_1^\dagger((1-r)V^2 + \sigma^2 I_M)Z_1 + \sigma^2 Z_2^\dagger Z_2$$

where Z_1 has mean $\sqrt{r}((1-r)V^2+\sigma^2 I_M)^{-1/2}V\Psi^\dagger H_m$ and covariance I_{NM} and Z_2 has mean 0 and covariance $I_{N(T-M)}$

We use the following Lemma from [36]

Lemma .13. *If C and B are both Hermitian matrices, and if their eigenvalues are both arranged in decreasing order, then*

$$\sum_{i=1}^{N}(\lambda_i(C) - \lambda_i(B))^2 \leq \|C-B\|_2^2$$

where $\|A\|_2^2 \stackrel{\text{def}}{=} \sum A_{ij}^2$, $\lambda_i(A)$ denotes the i^{th} eigenvalue of matrix A.

Applying this Lemma with $C = (X^\dagger X|S)$ and $B = Z_1^\dagger(V^2 + \sigma^2 I_M)Z_1$ we obtain

$$\lambda_i(C) \leq \lambda_i(B) + \sigma^2\|Z_2^\dagger Z_2\|_2$$

for $i = 1, \ldots, M$ Note that $\lambda_i(B) = \lambda_i(B')$ where $B' = ((1-r)V^2+\sigma^2 I_M)Z_1Z_1^\dagger$. Let $k = E[\|Z_2^\dagger Z_2\|_2]$ be a finite constant. Now

$$\begin{aligned}
\sum_{i=1}^{M} E[\log \sigma_i^2|S] &\leq \sum_{i=1}^{M} E[\log(\lambda_i(((1-r)V^2 + \sigma^2 I_M)Z_1Z_1^\dagger) + \sigma^2\|Z_2^\dagger Z_2\|_2)] \\
&\leq \sum_{i=1}^{M} E[\log(\lambda_i(((1-r)V^2 + \sigma^2 I_M)Z_1Z_1^\dagger) + \sigma^2 k)] \\
&= E[\log\det(((1-r)V^2 + \sigma^2 I_M)Z_1Z_1^\dagger + k\sigma^2 I_M)] \\
&= E[\log\det Z_1Z_1^\dagger] + E[\log\det((1-r)V^2 + \sigma^2 I_M + k\sigma^2(Z_1Z_1^\dagger)^{-1})]
\end{aligned}$$

where the second inequality follows from Jensen's inequality and taking expectation over Z_2. Using the Lemma again on the second term, we have

$$\begin{aligned}
\sum_{i=1}^{M} E[\log \sigma_i^2|S] &\leq E[\log\det Z_1Z_1^\dagger] + E[\log\det((1-r)V^2 + \sigma^2 I_M \\
&\qquad + k\sigma^2\|(Z_1Z_1)^{-1}\|_2 I_M)] \\
&\leq E[\log\det Z_1Z_1^\dagger] + E[\log\det((1-r)V^2 + k'\sigma^2 I_M)]
\end{aligned}$$

where $k' = 1 + kE[\|Z_1 Z_1^\dagger\|_2]$ is a finite constant. Now we have

$$\begin{aligned}\sum_{i=1}^{M} E[\log \sigma_i^2 | S] - \sum_{i=1}^{M} \log((1-r)\|s_i\|^2 + \sigma^2) &\leq E[\log\det Z_1 Z_1^\dagger] + \\ &\quad \sum_{i=1}^{M} \log \frac{(1-r)\|s_i\|^2 + k'\sigma^2}{(1-r)\|s_i\|^2 + \sigma^2} \\ &\leq E[\log\det Z_1 Z_1^\dagger] + k''\end{aligned}$$

where k'' is another constant. Taking Expectation over S, we have shown that $\sum_{i=1}^{M} E[\log \sigma_i^2] - \sum_{i=1}^{M} E[\log((1-r)\|s_i\|^2 + \sigma^2)]$ is bounded above by a constant.

Note that as $\|s_i\| \to \infty$, $Z_1 \to \sqrt{\frac{1}{1-r}} H_1$ so that $E[Z_1 Z_1^\dagger] \to \frac{1}{1-r} E[H_1 H_1^\dagger] = \frac{1}{1-r} E[HH^\dagger]$.

Step 2 Now assume that there are M transmit and receive antennas and that for $N - M' > 0$ antennas, the transmitted signal has bounded SNR, that is, $\|s_i\|^2 < C\sigma^2$ for some constant C. Start from a system with only M' transmit antennas, the extra power we send on the rest $M - M'$ antennas will get only a limited capacity gain since the SNR is bounded. Therefore, we conclude that the mutual information must be no more than $k_2 + M'(T - M') \log \rho$ for some finite k_2 that is uniform for all SNR level and all input distributions.

Particularly, if $M' = M - 1$, ie we have at least 1 transmit antenna to transmit signal with finite SNR, under the assumption that $T \geq 2M$ (T greater than twice the number of receivers), we have $M'(T - M') < M(T - M)$. This means that the mutual information achieved has an upper bound that increases with log SNR at rate $M'(T - M') \log \rho$, which is a lower rate than $M(T - M) \log \rho$.

Step 3 Now we further generalize the result above to consider the input which on at least 1 antennas, the signal transmitted has finite SNR with a positive probability, that is $P(\|s_M\|^2 < C\sigma^2) = \epsilon$. Define the event $E = \{\|s_M\|^2 < C\sigma^2\}$, then the mutual information can be written as

$$
\begin{aligned}
I(X;S) &\le \epsilon I(X;S|E) + (1-\epsilon)I(X;S|E^c) + I(E;X) \\
&\le \epsilon(k_1 + (M-1)(T-M+1)\log\rho) + (1-\epsilon)(k_2 + M(T-M)\log\rho) + \\
&\quad \log 2
\end{aligned}
$$

where k_1 and k_2 are two finite constants. Under the assumption that $T \ge 2M$, the resulting mutual information thus increases with SNR at rate that is strictly less than $M(T-M)\log\rho$.

Step 4 Here we will show that for the case of M transmit and receive antennas, the constant equal norm input $P(\|s_i\| = \sqrt{T}) = 1$ for $i = 1, \ldots, M$, achieves a mutual information that increases at a rate $M(T - M\log\rho$.

Lemma .14. *For the constant equal norm input,*

$$\lim\inf_{\sigma^2 \to 0} [I(X;S) - f(\rho)] \ge 0$$

where $\rho = M/\sigma^2$*, and*

$$f(\rho) = \log|G(T,M)| + (T-M)E[\log\det HH^\dagger] + M(T-M)\log\frac{T\rho}{M\pi e} - M^2\log[(1-r)T]$$

Proof: Consider

$$
\begin{aligned}
\mathcal{H}(X) &\ge \mathcal{H}(SH) \\
&= \mathcal{H}(QVH) + \log|G(T,M)| + (T-M)E[\log\det H^\dagger\Psi V^2\Psi^\dagger H] \\
&= \mathcal{H}(QVH) + \log|G(T,M)| + M(T-M)\log T + (T-M)E[\log\det HH^\dagger] \\
\mathcal{H}(X|S) &\le \mathcal{H}(QVH) + M\sum_{i=1}^{M} E[\log((1-r)\|s_i\|^2 + \sigma^2)] + M(T-M)\log\pi e\sigma^2 \\
&\approx \mathcal{H}(QVH) + M^2\log[(1-r)T] + M^2\frac{\sigma^2}{(1-r)T} + M(T-M)\log\pi e\sigma^2
\end{aligned}
$$

Therefore,

$$\begin{aligned}
I(X;S) &\geq \log|G(T,M)| + (T-M)E[\log\det HH^\dagger] - M(T-M)\log\pi e\sigma^2 + \\
&\quad M(T-M)\log T - M^2\log[(1-r)T] - M^2\frac{\sigma^2}{(1-r)T} \\
&= f(\rho) - M^2\frac{\sigma^2}{(1-r)T} \to f(\rho)
\end{aligned}$$

■

Combine with the results in Step 3, for any input that doesn't satisfy (.13), since the mutual information increases at a strictly lower rate, thus at high SNR, they are not optimal, and this completes the proof of Lemma VI.3.

.9 Convergence of Entropies

The main results in this section are Theorems .11 and .12. Lemma .15 is useful for applying Theorem .11 to applications in which it is known that the cummulative distribution functions are converging. Lemma .16 is only useful in establishing the proof of Theorem .11 and is not significant otherwise.

Lemma .15. *Let a sequence of cummulative distribution functions $F_n(x)$ having continuous derivatives converge to a cummulative distribution function $F(x)$ which also has a continuous derivative. If $f_n(x)$ are uniformly continuous then $f_n(x)$ converge to $f(x)$.*

Proof: Since $F_n(x)$ is absolutely continous and converging to $F(x)$, we have for all y

$$P_n(A_y = \{x : |y-x| < \delta\}) = \int_{A_y} f_n(x)dx \to \int_{A_y} f(x)dx = P(A_y)$$

Since $f_n(x)$ are uniformly continous, given ϵ there exists a single δ for all n such that $|f_n(x+\Delta x) - f_n(x)| < \epsilon$ for all $|\Delta x| < \delta$.

We have $|f_n(x)-f_n(y)| < \epsilon\ \forall x \in A_y$ and $|\int_{A_y} f_n(x)dx - f_n(y)\mathrm{vol}(A_y)| < \epsilon\ \mathrm{vol}(A_y)$. Since $|\int_{A_y} f(x)dx - f(y)\mathrm{vol}(A_y)| < \epsilon\ \mathrm{vol}(A_y)$ and $\int_{A_y} f_n(x)dx \to \int_{A_y} f(x)dx$ we have $|\lim f_n(y) - f(y)| < 2\epsilon\ \mathrm{vol}(A_y)$. Since ϵ is arbitrary we have $\lim f_n(y) = f(y)$ for all y. ∎

Let $\chi_P(x)$ denote the characteristic function over a set P defined as $\chi_P(x) = 0$ if $x \notin P$ and $\chi_P(x) = 1$ if $x \in P$.

Lemma .16. *Let $g : \mathcal{C}^P \to R$ be a positive bounded function whose region of support, support(g), is compact. If $\int g(x)dx < 1/e$ then $|\int g(x)\log g(x)dx| \leq \max\{|L\log L| + |L\log vol(support(g))|, |L\log A|\}$ where $A = \sup g(x)$ and L is any positive real number such that $\int g(x)dx \leq L < 1/e$.*

Proof: First, $\int g(x)\log g(x)dx \leq \int g(x)\log A dx \leq L\log A$. Let $\int g(x)dx = I_g$. Consider the function $g(x)/I_g$ a probability density function. We know that $\int \frac{g(x)}{I_g}\log\frac{g(x)}{I_g f(x)}dx \geq 0$ for all probability density functions $f(x)$. If

$$f(x) = 1/\mathrm{vol}(\mathrm{support}(g))\chi_{\mathrm{support}(g)}$$

then

$$\int g(x)\log g(x)dx \geq \int g(x)\log(I_g f(x)) = I_g \log\frac{I_g}{\mathrm{vol}(\mathrm{support}(g))}$$

That implies

$$\begin{aligned}
|\int g(x)\log g(x)| &\leq \max\{|L\log A|, |I_g\log\frac{I_g}{\mathrm{vol}(\mathrm{support}(g))}|\} \\
&\leq \max\{|L\log A|, |I_g\log I_g| + |I_g\log\mathrm{vol}(\mathrm{support}(g))|\} \\
&\leq \max\{|L\log A|, |L\log L| + |L\log\mathrm{vol}(\mathrm{support}(g))|\}
\end{aligned}$$

The last inequality follows from the fact that for $x < 1/e$, $|x\log x|$ is an increasing function of x. ∎

Theorem .11. *Let $\{X_i \in \mathbb{C}^P\}$ be a sequence of continuous random variables with probability density functions, f_i and $X \in \mathbb{C}^P$ be a continuous random variable with probability density function f such that $f_i \to f$ pointwise. If 1) $0 \le \max\{f_i(x), f(x)\} \le A < \infty$ for all i and 2) $\max\{\int |x|^\kappa f_i(x)dx, \int |x|^\kappa f(x)dx\} \le L < \infty$ for some $\kappa > 1$ and all i then $\mathcal{H}(X_i) \to \mathcal{H}(X)$. $|x|$ denotes the euclidean norm of x.*

proof: The proof is based on showing that given an $\epsilon > 0$ there exists an R for all i such that

$$|\int_{|x|>R} f_i(x) \log f_i(x)dx| < \epsilon$$

This R also works for $f(x)$.

Since $y \log y \to 0$ as $y \to 0$ we have $\max_{f(x)\le A} |f(x) \log f(x)| \le \max\{A \log A, e\} \stackrel{\text{def}}{=} K$. Therefore, $f_i(x) \log f_i(x)$ is bounded above by an L^1 function ($g = K\chi_{|x|\le R}$) and by the dominated convergence theorem we have

$$-\int_{|x|\le R} f_i(x) \log f_i(x)dx \to -\int_{|x|\le R} f(x) \log f(x)dx$$

Now, to show that the integral outside of $|x| \le R$ is uniformly bounded for all f_i and f. Let g denote either f_i or f. We have $\int |x|^\kappa g(x)dx \le L$. Therefore, by Markov's inequality $\int_{R<|x|\le R+1} g(x)dx = I^R \le L/R^\kappa$. Choose R large enough so that for all $l > R$: $I^l < 1/e$. Now

$$|\int_{|x|>R} g(x) \log g(x)dx| \le \int_{|x|>R} |g(x) \log g(x)|dx = \sum_{l=R}^{\infty} \int_{B_l} |g(x) \log g(x)|dx$$

where $B_l = \{x : l < |x| \le l + 1\}$.

Consider the term $\int_{B_l} |g(x) \log g(x)|dx = G_l$. Also, define $A_+ = \{x : -\log g(x) >$

$0\}$ and $A_- = \{x : -\log g(x) < 0\}$ Now,

$$\begin{aligned} G_l &= \int_{A_+\cap B_l} |g(x)\log g(x)|dx + \int_{A_-\cap B_l} |g(x)\log g(x)|dx \\ &= |\int_{A_+\cap B_l} g(x)\log g(x)dx| + |\int_{A_-\cap B_l} g(x)\log g(x)dx| \end{aligned}$$

From Lemma .16, we have

$$G_l \le 2\max\{|I^l \log I^l| + |I^l \log \mathrm{vol}(\{B_l\})|, |I^l \log A|\}$$

We know $\mathrm{vol}(\{x : B_l\}) = o(l^{2P})$. Therefore,

$$\int_{B_l} |g(x)\log g(x)|dx < \frac{Q}{l^\kappa}\log l$$

where Q is some sufficiently large constant. Therefore, we have

$$\int_{|x|>R} |g(x)\log g(x)|dx < \sum_{l=R}^{\infty} \frac{Q}{l^\kappa}\log l = O(\log R/R^{\kappa-1})$$

Therefore, we can choose R sufficiently large to have $|\int_{|x|>R} g(x)\log g(x)dx| < \epsilon$. ■

Theorem .12. *Let $\{X_i \in \mathbb{C}^P\}$ be a sequence of continuous random variables with probability density functions, f_i and $X \in \mathbb{C}^P$ be a continuous random variable with probability density function f. Let $X_i \xrightarrow{P} X$. If 1) $\int |x|^\kappa f_n(x)dx \le L$ and $\int |x|^\kappa f(x)dx \le L$ for some $\kappa > 1$ and $L < \infty$ 2) $f(x)$ is bounded then $\limsup_{i\to\infty} \mathcal{H}(X_i) \le \mathcal{H}(X)$.*

Proof: We will prove this by constructing a density function g_i corresponding to f_i that maximizes the entropy at stage i then show that $\limsup_{i\to\infty} \mathcal{H}_{g_i} \le \mathcal{H}(X)$ thus concluding $\limsup \mathcal{H}(X_i) \le \mathcal{H}(X)$ where $\mathcal{H}_{g_i} \stackrel{\text{def}}{=} -\int g_i(x)\log g_i(x)dx$.

First we will show that for all g_i defined above there exists a single real number $R > 0$ such that $-\int_{|x|>R} g_i(x)\log g_i(x)dx \le \epsilon$. Note that this is different from the condition in Theorem .11 where we show $|\int_{|x|>R} g_i(x)\log g_i(x)dx| \le \epsilon$. As in Theorem .11 choose R large enough so that $I^l < 1/e$. Also define the two sets A_+ and A_- as in Theorem .11 then

$$\begin{aligned} -\int_{|x|>R} g(x)\log g(x)dx &= -\int_{A_+} g(x)\log g(x)dx - \int_{A_-} g(x)\log g(x)dx \\ &= -\sum_{l=R}^{\infty}\int_{B_l\cap A_+} g(x)\log g(x)dx - \int_{A_-} g(x)\log g(x)dx \end{aligned}$$

where B_l is as defined in Theorem .11. The last line follows from the Monotone Convergence Theorem. From the proof of Lemma .16 we have $-\int_{B_l\cap A_+} g(x)\log g(x)dx \le -I^l\log I^l + I^l\log \mathrm{vol}(B_l)$ Therefore

$$\begin{aligned} -\int_{|x|>R} g(x)\log g(x)dx &\le \sum_{l=R}^{\infty}[-I^l\log I^l + I^l\log\mathrm{vol}(B_l)] - \int_{A_-} g(x)\log g(x)dx \\ &\le \sum_{l=R}^{\infty}[-I^l\log I^l + I^l\log\mathrm{vol}(B_l)] \end{aligned}$$

and the sum in the last line is bounded above by $\sum_{l=R}^{\infty}\frac{Q}{l^\kappa}\log l = O(\log R/R^{\kappa-1})$. Therefore,

$$\max_g\{-\int_{|x|>R} g(x)\log g(x)dx \quad \} \le O(\log R/R^{\kappa-1})$$

From the proof of Theorem .11, $|\int_{|x|>R} f(x)\log f(x)dx| = O(\log R/R^{\kappa-1})$.

Now let's concentrate on upperbounding $-\int_{|x|\le R} f_i(x)\log f_i(x)dx$. Let $A = \sup f(x)$. For each n divide the region $\{|x|\le R\}$ into n parts $P_m, m = 1,\ldots,n$ such that $A\frac{m-1}{n} \le f(x) < A\frac{m}{n}$ for $x\in P_m$, $m<n$ and $A\frac{n-1}{n}\le f(x)\le A$ for $x\in P_n$. Now for each n, there exists a number M_n such that $\max_m|\int_{P_m}(f_i(x)-f(x))dx| < \frac{1}{n}\min_m\int_{P_m} f(x)dx$ for all $i\ge M_n$. If $M_n\le M_{n-1}$ set $M_n = M_{n-1}+1$. Now, define the function $M(i)$ such that

$$M(i) = \begin{cases} 1 & 1 \leq i \leq M_2 \\ 2 & M_2 < i \leq M_3 \\ 3 & M_3 < i \leq M_4 \\ \vdots & \end{cases}$$

For each i, divide the region $\{|x| \leq R\}$ into $M(i)$ parts as defined in the previous paragraph $P_n, n = 1, \dots, M(i)$ and define $g_i(x)$ over $\{|x| \leq R\}$ as

$$g_i(x) = \sum_{n=1}^{M(i)} \chi_{P_n}(x) I_{n,i}/V_n$$

where $I_{n,i} = \int_{P_n} f_i(x)dx$, $V_n = \text{vol}(P_n)$.

Now, it is easy to see that $-\int_{|x|\leq R} f_i(x) \log f_i(x) \leq -\int_{|x|\leq R} g_i(x) \log g_i(x)$. Also, note that $g_i(x) \to f(x)$ pointwise. Since $f(x)$ is bounded there exists a number N and a constant K such that $g_i(x) \leq K$ for all values of $i > N$, also $f(x) \leq K$. Therefore, using Theorem .11 we conclude that $\lim -\int_{|x|\leq R} g_i(x) \log g_i(x)dx \to -\int_{|x|\leq R} f(x) \log f(x)dx$.

Therefore, $\limsup \mathcal{H}(X_i) \leq \limsup \mathcal{H}_{g_i} \leq \mathcal{H}(X)$. ■

.10 Convergence of $\mathcal{H}(X)$ for $T > M = N$ in Section 6.3.4

First, we will show convergence for the case $T = M = N$ and then use the result to to show convergence for the case $T > M = N$. Let's consider the $T \times N$ random matrix $X = SH + W$. The entries of $M \times N$ matrix H, $T = M = N$, are independent circular complex Normal random variables with non-zero mean and unit variance whereas the entries of W are independent circular complex Normal random variables with zero-mean and variance σ^2.

Let S be a random matrix such that $\lambda_{min}(SS^\dagger) \geq \lambda > 0$ with distribution chosen in such a way to maximize $I(X; S)$. If $f(X)$ denotes the corresponding density function of X then it is easy to see that $f(X)$ as a function of σ^2 is a continuous function of σ^2. As $\lim_{\sigma^2\to 0} f(X)$ exists, let's call this limit $g(X)$.

For each value of $\sigma^2 = 1/n$, n an integer $\to\infty$, the density of X is

$$f(X) = E_S\left[\frac{\exp -\mathrm{tr}\{[\sigma^2 I_T + (1-r)SS^\dagger]^{-1}(X-\sqrt{rNM}SH_m)(X-\sqrt{rNM}SH_m)^\dagger\}}{\pi^{TN}\det^N[\sigma^2 I_T + (1-r)SS^\dagger]}\right]$$

If we impose the condition that $\lambda_{min}(SS^\dagger) \geq \lambda > 0$ w.p. 1 then $f(X)$ is bounded above by $\frac{1}{(\lambda\pi)^{TN}}$ which satisfies the condition for the Theorem .11. We will show that under this condition we also have that for all n there exists a common δ such that $|f(X) - f(Z)| < \epsilon$ for all $|X - Z| < \delta$.

Lemma .17. *If $\lambda_{min}(SS^\dagger) \geq \lambda > 0$ then $\forall n$ there exists an $\mathcal{M}$ such that $|f(X) - f(Z)| < \mathcal{M}\delta$ if $|X - Z| < \delta$.*

Proof: Let $Z = X + \Delta X$ with $|\Delta X| < \delta$ and $[\sigma^2 I_T + (1-r)SS^\dagger] = D$. First, we will fix S and show that for all S, $f(X|S)$ satisfies the above property. Therefore, it will follow that $f(X)$ also satsifies the same property. Consider $f^0(X|S)$ the density defined with zero mean which is just a translated version of $f(X|S)$.

$$f(X+\Delta X|S) = f(X|S)[1 - \mathrm{tr}[D^{-1}(\Delta X X^\dagger + X\Delta X^\dagger + O(\|\Delta X\|_2^2))]]$$

then

$$|f(X+\Delta X|S) - f(X|S)| \leq f(X|S)|\mathrm{tr}[D^{-1}(\Delta X X^\dagger + X\Delta X^\dagger)] + \mathrm{tr}[D^{-1}\|\Delta X\|_2^2]|$$

Now

$$f(X|S) \leq \frac{1}{\pi^{TN}\det^N[D]}\cdot\min\{\frac{1}{\sqrt{\mathrm{tr}[D^{-1}XX^\dagger]}}, 1\}$$

Now, make use of the following inequalities

$$\begin{aligned}\mathrm{tr}\{D^{-1}XX^\dagger\} &\geq \mathrm{tr}\{\lambda_{min}(D^{-1})XX^\dagger\} \\ &\geq \lambda_{min}(D^{-1})\lambda_{max}(XX^\dagger) = \lambda_{min}(D^{-1})\|X\|_2^2\end{aligned}$$

Also,

$$\begin{aligned}
|\mathrm{tr}\{D^{-1}(X\Delta X^\dagger + \Delta X X^\dagger + O(\|\Delta X\|_2^2)\}| &\leq \sum_i |\lambda_i(D^{-1}[\Delta X X^\dagger + X\Delta X^\dagger])| + \\
&\quad \|D^{-1}\|_2\|\Delta X\|_2^2 \\
&\leq T\|D^{-1}\|_2\|X\|_2\|\Delta X\|_2 + T\|D^{-1}\|_2\|\Delta X\|_2^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
|f(X+\Delta X|S) - f(X|S)| &\leq \frac{1}{\pi^{TN}\det^N[D]} \cdot \min\{\frac{1}{\sqrt{\lambda_{min}(D^{-1})}\|X\|_2}, 1\} \cdot \\
&\quad T\|D^{-1}\|_2\|\Delta X\|_2(\|X\|_2 + \|\Delta X\|_2)
\end{aligned}$$

Since, we have restricted $\lambda_{min}(SS^\dagger) \geq \lambda > 0$ we have for some constant $\mathcal{M}$

$$|f(X+\Delta X|S) - f(X|S)| \leq \mathcal{M}\|\Delta X\|_2$$

From which the Lemma follows. Note that $\det[D]$ compensates for $\sqrt{\lambda_{min}(D^{-1})}$ in the denominator. ∎

Using the above Lemma we conclude that if S is restricted to have $\lambda_{min}(SS^\dagger) \geq \lambda > 0$ then $\mathcal{H}(X) \to \mathcal{H}_g$. Since λ is arbitrary we conclude that for all optimal signals with the restriction $\lambda_{min}(SS^\dagger) > 0$ $\mathcal{H}(X) \to \mathcal{H}_g$. Now, we claim that the condition $\lambda_{min} > 0$ covers all optimal signals because for optimal signals it is not possible for $\lambda_{min}(SS^\dagger) = 0$ with finite probability because this means that for all σ^2 we have $\min\|s_i\|^2 \leq L\sigma^2$ for some constant L with finite probability which is a contradiction of the condition (6.6). This completes the proof of convergence of $\mathcal{H}(X)$ for $T = M = N$.

Now, we will show convergence of $\mathcal{H}(X)$ for $T > M = N$. We will show that $\mathcal{H}(X) \approx \mathcal{H}(SH)$ for small values of σ where $S = \Phi V \Psi^\dagger$ with Φ independent of V and Ψ.

Let $S_0 = \Phi_0 V_0 \Psi_0^\dagger$ denote a signal with its density set to the limiting optimal density of S as $\sigma^2 \to 0$.

$$\mathcal{H}(X) \geq \mathcal{H}(Y) = \mathcal{H}(Q\Sigma_Y \Psi_Y^\dagger) + \log|G(T,M)| + (T-M)E[\log\det\Sigma_Y^2]$$

where $Y = SH$ and Q is an isotropic matrix of size $N \times M$. Let

$$Y_Q = QV\Psi^\dagger H$$

Then $\mathcal{H}(Q\Sigma_Y\Psi_Y^\dagger) = \mathcal{H}(Y_Q)$.

From the proof of the case $T = M = N$, we have $\lim_{\sigma^2\to 0}\mathcal{H}(Y_Q) = \mathcal{H}(QV_0\Psi_0^\dagger H)$. Also,

$$\lim_{\sigma^2\to 0} E[\log\det\Sigma_Y^2] = E[\log\det\Sigma_{Y_0}^2]$$

where $Y_0 = S_0 H$ Therefore, $\liminf_{\sigma^2\to 0}\mathcal{H}(X) \geq \lim_{\sigma^2\to 0}\mathcal{H}(Y) = \mathcal{H}(S_0 H)$.

Now, to show $\lim_{\sigma^2\to 0}\mathcal{H}(X) \leq \mathcal{H}(S_0 H)$. From before

$$\mathcal{H}(X) = \mathcal{H}(Q\Sigma_X\Psi_X^\dagger) + |G(T,N)| + (T-M)E[\log\det\Sigma_X^2]$$

Now $Q\Sigma_X\Psi_X^\dagger$ converges in distribution to $QV_0\Psi_0^\dagger H$. Since the density of $QV_0\Psi_0^\dagger H$ is bounded, from Theorem .12 we have $\limsup_{\sigma^2\to 0}\mathcal{H}(Q\Sigma_X\Psi_X^\dagger) \leq \mathcal{H}(QV_0\Psi_0^\dagger H)$. Also, note that $\lim_{\sigma^2\to 0} E[\log\det\Sigma_X^2] = E[\log\det\Sigma_{Y_0}^2] = \lim_{\sigma^2\to 0} E[\log\det\Sigma_Y^2]$. Which leads to $\limsup_{\sigma^2\to 0}\mathcal{H}(X) \leq \mathcal{H}(S_0 H) = \lim_{\sigma^2\to 0}\mathcal{H}(SH)$.

Therefore, $\lim_{\sigma^2\to 0}\mathcal{H}(X) = \lim_{\sigma^2\to 0}\mathcal{H}(SH)$ and for small σ^2, $\mathcal{H}(X) \approx \mathcal{H}(SH)$.

.11 Proof of Theorem VI.8 in Section 6.4.1

First we note that $\sigma_{\hat{G}}^2 = 1 - \sigma_{\tilde{G}}^2$. This means that

$$\rho_{eff} = \frac{\kappa T\rho + T_c}{(1-r)\kappa T\rho\sigma_{\tilde{G}}^2 + T_c} - 1$$

Therefore, to maximize ρ_{eff} we just need to minimize $\sigma^2_{\tilde{G}}$. Now,

$$\sigma^2_{\tilde{G}} = \frac{1}{NM}\mathrm{tr}\{E[\tilde{\tilde{G}}\tilde{\tilde{G}}^\dagger]\}$$

where

$$E[\tilde{\tilde{G}}\tilde{\tilde{G}}^\dagger] = (I_M + (1-r)\frac{\rho}{M}S_t^\dagger S_t)^{-1} \otimes I_N$$

where $\rho = \frac{M}{\sigma^2}$. Therefore, the problem is the following

$$\min_{S_t:\mathrm{tr}\{S_t^\dagger S_t\} \leq (1-\kappa)TM} \frac{1}{M}\mathrm{tr}\{\left(I_M + (1-r)\frac{\rho}{M}S_t^\dagger S_t\right)^{-1}\}$$

This problem can be restated as

$$\min_{\lambda_1,\ldots,\lambda_M:\sum\lambda_m \leq (1-\kappa)TM} \frac{1}{M}\sum_{m=1}^{M}\frac{1}{1+(1-r)\frac{\rho}{M}\lambda_m}$$

where λ_m, $m = 1,\ldots,M$ are the eigenvalues of $S_t^\dagger S_t$. The solution to the above problem is $\lambda_1 = \ldots = \lambda_M = (1-\kappa)T$. Therefore, the optimum S_t satisfies $S_t^\dagger S_t = (1-\kappa)TI_M$. Which is the same as the optimum signal as derived in [33].

This gives $\sigma^2_{\tilde{G}} = \frac{1}{1+(1-r)\frac{\rho}{M}(1-\kappa)T}$. Also, for this choice of S_t we obtain the elements of $\hat{G}$ to be zero mean independent with Gaussian distribution. This gives

$$\rho_{eff} = \frac{\kappa T\rho[Mr + \rho(1-\kappa)T]}{T_c(M+\rho(1-\kappa)T) + (1-r)\kappa T\rho M}$$

.12 Proof of Theorem VI.9 in Section 6.4.1

Now

$$\begin{aligned}
\rho_{eff} &= \frac{\kappa T\rho[Mr + \rho(1-\kappa)T]}{T_c(M+\rho(1-\kappa)T) + (1-r)\kappa T\rho M} \\
&= \frac{T\rho}{T_c - (1-r)M}\frac{(1-\kappa)\kappa + \kappa\frac{rM}{T\rho}}{\frac{MT_c + T\rho T_c}{T\rho[T_c-(1-r)M]} - \kappa} \qquad T_c \neq (1-r)M \\
&= \frac{T^2\rho^2}{T_c(M+T\rho)}[(1-\kappa)\kappa + \kappa\frac{rM}{T\rho}] \qquad T_c = (1-r)M
\end{aligned}$$

Consider the following three cases for the maximization of ρ_{eff} over $0 \leq \kappa \leq 1$.

Case 1. $T_c = (1-r)M$:

We need to maximize $(1-\kappa)\kappa + \kappa\frac{rM}{T\rho}$ over $0 \leq \kappa < 1$. The maximum occurs at $\kappa = \min\{\frac{1}{2} + \frac{rM}{2T\rho}, 1\}$ which states that for sufficient small ρ no training is required. This result agrees with the result in Section 6.3.3 where the rician fading channel basically behaves like an AWGN channel. In this case

$$\frac{T^2\rho^2}{(1-r)M(M+T\rho)}[\kappa\frac{rM}{T\rho} + \kappa(1-\kappa)]$$

with $\kappa = \min\{\frac{1}{2} + \frac{rM}{2T\rho}, 1\}$.

Case 2. $T_c > (1-r)M$:

In this case,

$$\rho_{eff} = \frac{T\rho}{T_c - (1-r)M}\frac{(1-\kappa)\kappa + \kappa\eta}{\gamma - \kappa}$$

where $\eta = \frac{rM}{T\rho}$ and $\gamma = \frac{MT_c + T\rho T_c}{T\rho[T_c - (1-r)M]} > 1$. We need to maximize $\frac{(1-\kappa)\kappa+\kappa\eta}{\gamma-\kappa}$ over $0 \leq \kappa \leq 1$ which occurs at $\kappa = \gamma - \sqrt{\gamma^2 - \gamma - \eta\gamma}$. Therefore,

$$\rho_{eff} = \frac{T\rho}{T_c - (1-r)M}(\sqrt{\gamma} - \sqrt{\gamma - 1 - \eta})^2$$

When $\kappa = 1$ we obtain $T_c = T$. Substituting $\kappa = 1$ in the expression for ρ_{eff}

$$\rho_{eff} = \frac{\kappa T\rho[Mr + \rho(1-\kappa)T]}{T_c(M + \rho(1-\kappa)T) + (1-r)\kappa T\rho M}$$

we obtain $\rho_{eff} = \frac{rT\rho}{T+(1-r)T\rho}$.

Case 3. $T_c < (1-r)M$:

In this case,

$$\rho_{eff} = \frac{T\rho}{(1-r)M - T_c}\frac{(1-\kappa)\kappa + \kappa\eta}{\kappa - \gamma}$$

where $\gamma = \frac{MT_c + T\rho T_c}{T\rho[T_c - (1-r)M]} < 0$. Maximizing $\frac{(1-\kappa)\kappa+\kappa\eta}{\gamma-\kappa}$ over $0 \leq \kappa \leq 1$ we obtain $\kappa = \gamma + \sqrt{\gamma^2 - \gamma - \gamma\eta}$. Therefore,

$$\rho_{eff} = \frac{T\rho}{T_c - (1-r)M}(\sqrt{-\gamma} - \sqrt{-\gamma + 1 + \eta})^2$$

Similar to the case $T_c < (1-r)M$ when $\kappa = 1$ we obtain $T_c = T$ and $\rho_{eff} = \frac{rT\rho}{T+(1-r)T\rho}$.

.13 Proof of Theorem VI.10 in Section 6.4.1

Note that optimization over T_c makes sense only when $\kappa < 1$. First, we examine the case $T_c > (1-r)M$. The other two cases are similar. Let $Q = \min\{M, N\}$ and let λ_i denote the i^{th}, $i = 1, \ldots, Q$ non-zero eigenvalue of $\frac{H_1 H_1^\dagger}{M}$. Then we have

$$C_t \geq \sum_{i=1}^{Q} \frac{T_c}{T} E \log(1 + \rho_{eff}\lambda_i)$$

Let C_l denote the lower bound on the RHS in the expression above. The idea is to maximize C_l as a function of T_c. We have

$$\frac{dC_l}{dT_c} = \sum_{i=1}^{Q} \left\{ \frac{1}{T} E \log(1 + \rho_{eff}\lambda_i) + \frac{T_c}{T} \frac{d\rho_{eff}}{dT_c} E \left[\frac{\lambda_i}{1 + \rho_{eff}\lambda_i} \right] \right\}$$

Now, ρ_{eff} for $T_c > (1-r)M$ is given by

$$\rho_{eff} = \frac{T\rho}{T_c - (1-r)M} (\sqrt{\gamma} - \sqrt{\gamma - 1 - \eta})^2$$

where $\gamma = \frac{MT_c + T\rho T_c}{T\rho[T_c - (1-r)M]}$ and $\eta = \frac{rM}{T\rho}$. It can be easily verified that

$$\frac{d\rho_{eff}}{dT_c} = \frac{T\rho(\sqrt{\gamma} - \sqrt{\gamma - 1 - \eta})^2}{[T_c - (1-r)M]^2} \left[\sqrt{\frac{(1-r)M(M + T\rho)}{T_c(T_c + T\rho + rM)}} - 1 \right]$$

Therefore,

$$\frac{dC_l}{dT_c} = \frac{1}{T} \sum_{i=1}^{Q} E \left[\log(1 + \rho_{eff}\lambda_i) - \frac{\rho_{eff}\lambda_i}{1 + \rho_{eff}\lambda_i} \frac{T_c}{T_c - (1-r)M} \left[1 - \sqrt{\frac{(1-r)M(M + T\rho)}{T_c(T_c + T\rho + rM)}} \right] \right]$$

Since, $\frac{T_c}{T_c - (1-r)M} \left[1 - \sqrt{\frac{(1-r)M(M+T\rho)}{T_c(T_c+T\rho+rM)}} \right] < 1$ and $\log(1+x) - x/(1+x) \geq 0$ for all $x \geq 0$ we have $\frac{dC_l}{dT_c} > 0$. Therefore, we need to increase T_c as much as possible to maximize C_l.

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ABSTRACT

Antenna Arrays in Wireless Communications

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Multiple-antenna wireless systems have been gaining in popularity due to demand for higher data rates and the limited bandwidth available for communication. Multiple antennas have been used both at the transmitter and the receiver traditionally in a beamformer (array processing). However, it was shown recently that in the presence of fading the optimal solution in terms of capacity (upper limit on data rates with arbitrarily small error probabilities) involves other structures that are not beamformers.

We investigate both aspects of multiple-antenna wireless communication systems in this thesis. We investigate variants of the popular adaptive LMS algorithm known as the Partial Update LMS Algorithms for the purposes of beamforming in a wireless communication scenario. Partial update LMS algorithms involve partial updating of the filter coefficients as opposed to the full update in a regular LMS algorithm. For example, the coefficients can be divided into even and odd coefficient sets and

the sets could be updated alternatively. We investigate the existing partial update algorithms and also a propose a new variant which exhibits none of the disadvantages of the existing algorithms.

The second aspect of the multiple-antenna wireless communcation systems is the shannon capacity. Existing work so far has concentrated on deriving capacity expressions for the case of rayleigh fading channels. In this thesis, we investigate the maximum possible data rate affordable by communication systems in the presence of rician fading. We derive capacity expressions, capacity achieving signal structures as well as upper and lower bounds on capacity for two new models (proposed for tractable analysis) for rician fading. We also derive asymptotic capacity expressions, capacity achieving signal structures and upper and lower bounds on capacity for the standard rician fading channel model.