ROBUST INCORPORATION OF UNCERTAIN MRI SIDE INFORMATION INTO ECT IMAGE RECONSTRUCTION

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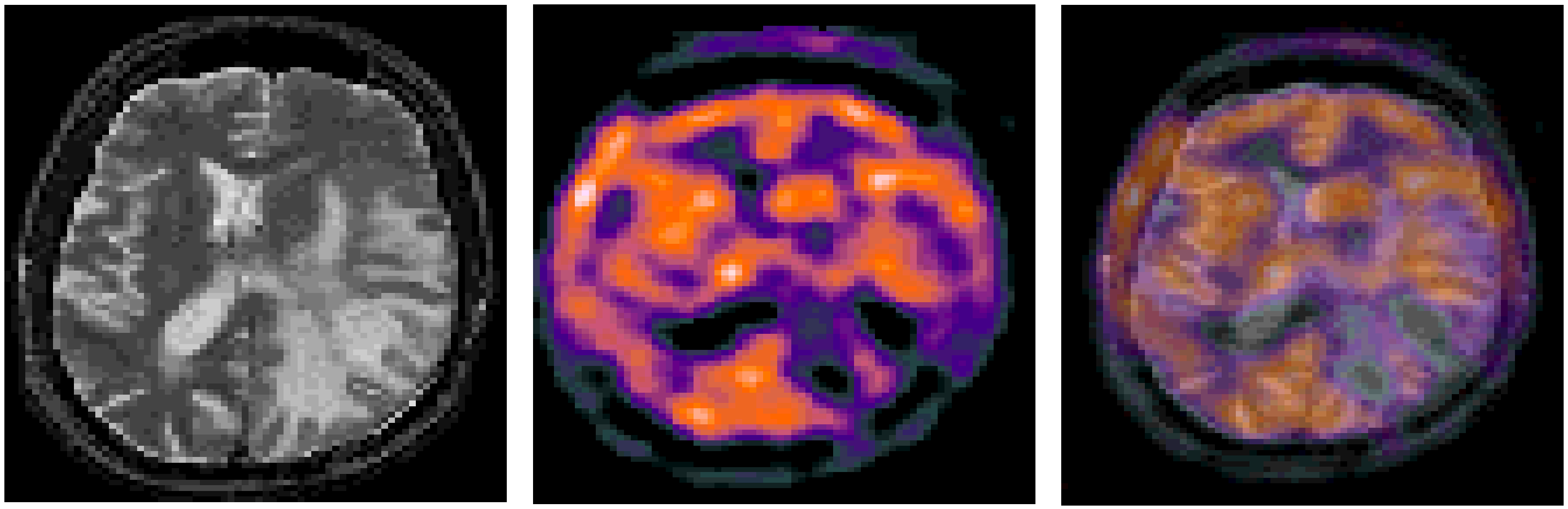
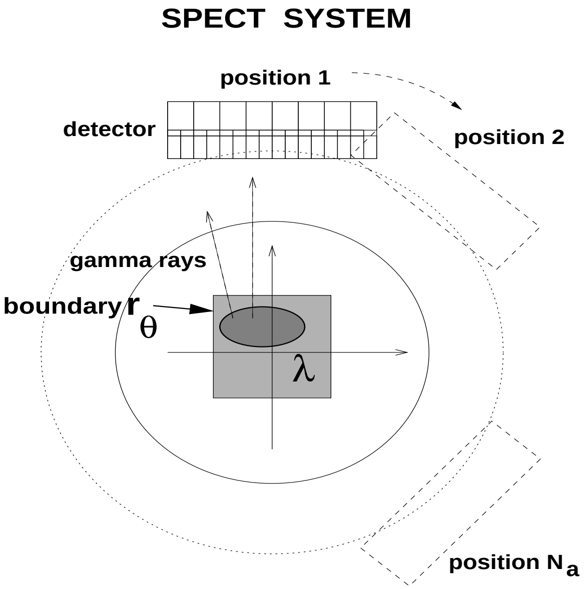
Combining MRI and SPECT For Functional Imaging

- Tracers: oxygen, glucose, iodine, antibodies, etc.
- "Functional" information about physiological processes
- Uptake estimation in a region of interest



Functional Imaging: The ECT Reconstruction Problem

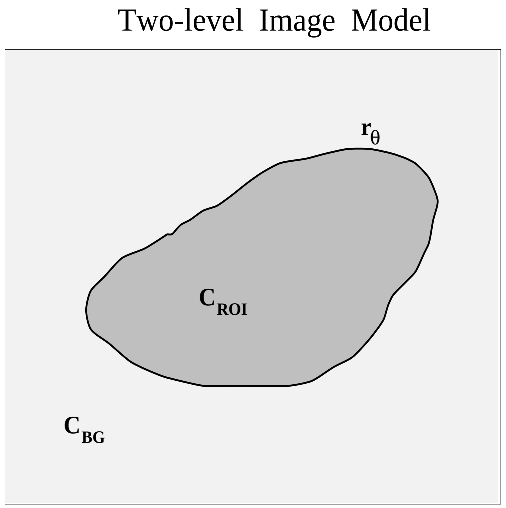


Counts collected approximate line integral values through the 2D function $\boldsymbol{\lambda}$.

$$\begin{aligned}
\boldsymbol{\lambda} &= \text{Unknown emission distribution} \\
\boldsymbol{\theta} &= \text{Parameterization of MRI boundary} \\
A &= \text{System matrix (geometry, etc.)} \\
\boldsymbol{b} &= \text{Background events (scatter, cosmic rays)}
\end{aligned}$$

- **Emission Statistics:** $\boldsymbol{Y}_E \sim \text{Poisson}(A\,\boldsymbol{\lambda} + \boldsymbol{b})$

MRI Boundary Extraction

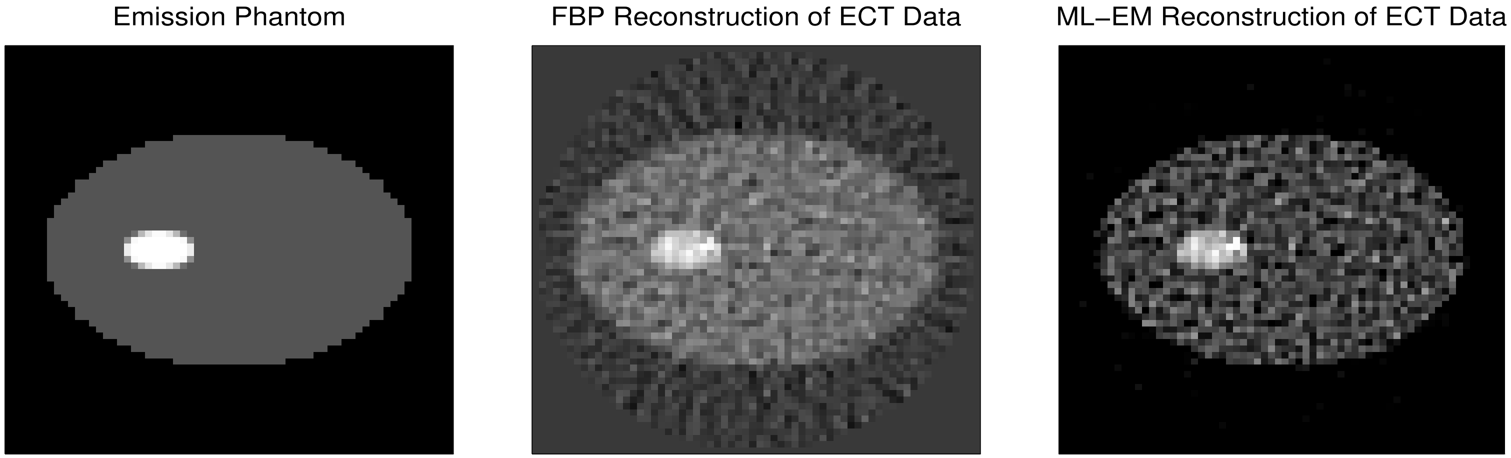


MRI image with organ boundary curve $\boldsymbol{r}_{\boldsymbol{\theta}}$

$$
\begin{aligned}
H &= \text{Symmetric Gaussian PSF} \\
N &= \text{Gaussian noise} \\
\boldsymbol{\theta} &= \text{Parameterization of anatomical boundary}
\end{aligned}
$$

- **Objective:** Estimate anatomical/functional boundary $\boldsymbol{\theta}$
- **MRI Statistics:** $Y_{\text{MRI}} = I_{\boldsymbol{\theta}} ** H + N$

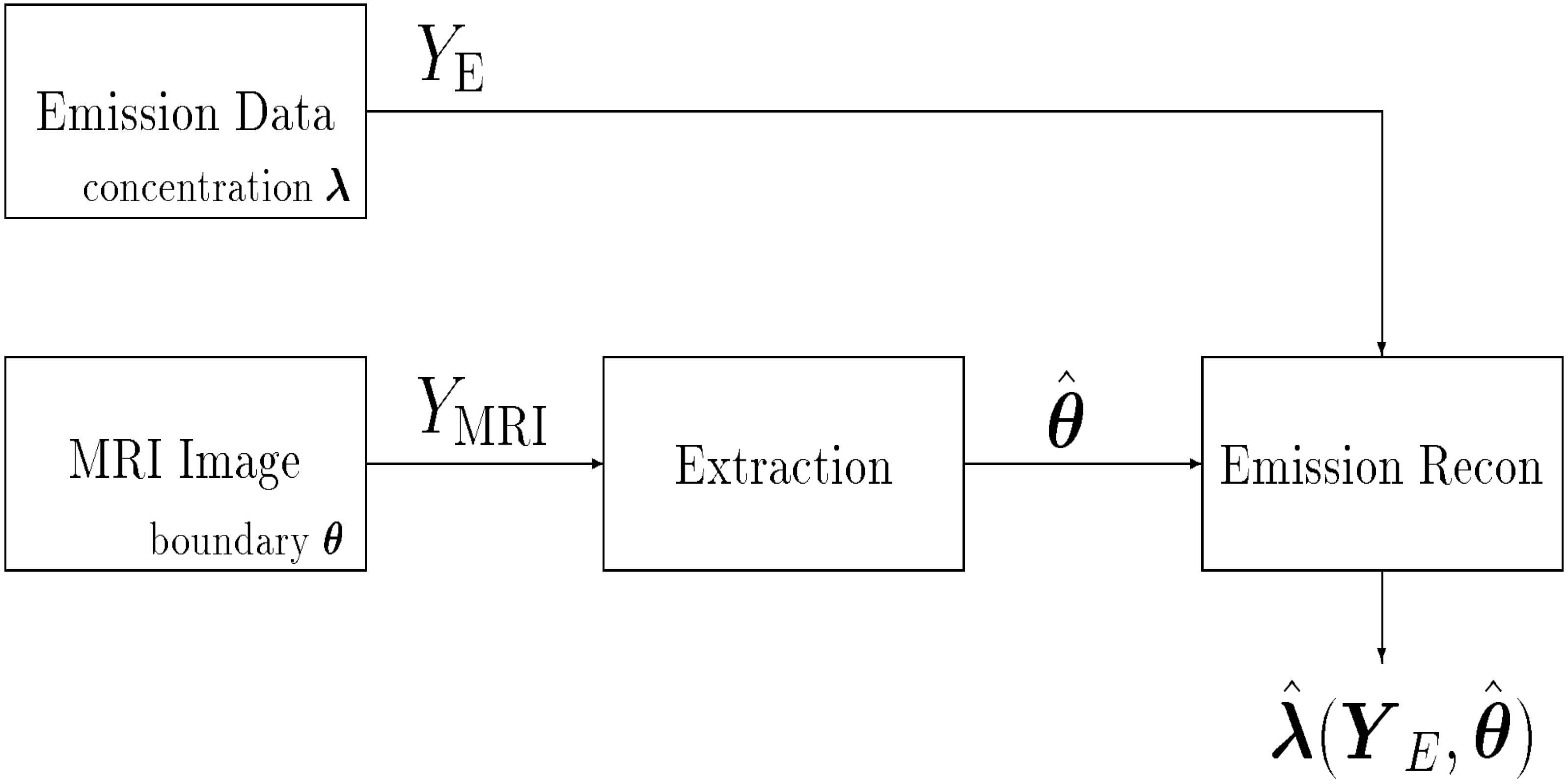
Uptake Estimation



Uptake estimate: $\hat{\alpha} = \underline{1}_{\mathrm{ROI}}^T \hat{\boldsymbol{\lambda}}$

- FBP oversmooths and ignores Poisson statistics or *prior* information
- ML undersmooths and ignores *prior* information

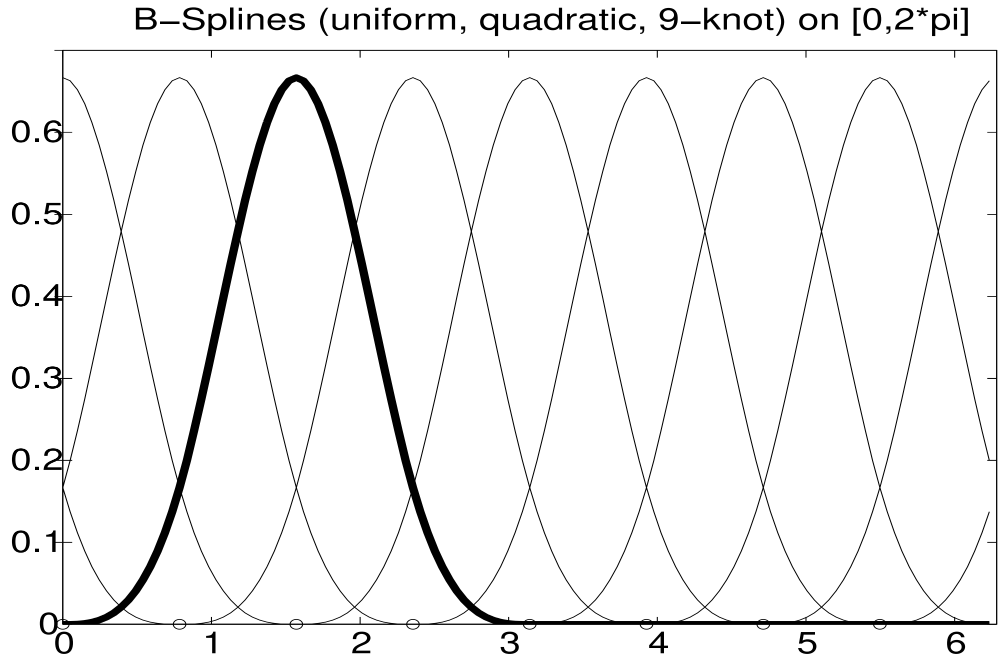
Use of MRI-derived Organ Boundaries



The B-Spline Basis

B-spline parameterization of periodic polar spline:

$$r_{\boldsymbol{\theta}}(\phi) = \sum_{i=1}^{K} \theta_i \, B_i(\phi) \,, \qquad \phi \in [0, 2\pi]$$



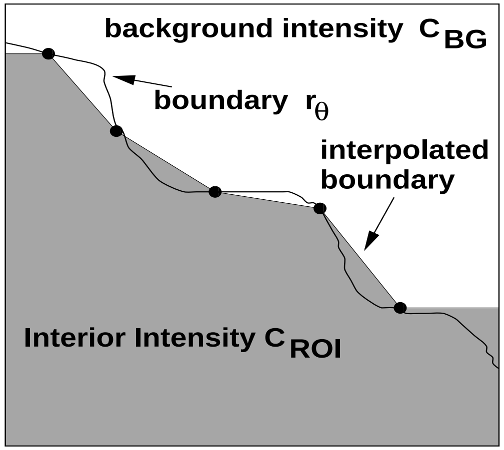
Maximum Likelihood (ML) Boundary Extraction

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}_{\mathrm{ML}} &= \operatorname{argmax}_{\boldsymbol{\theta}} \ln f(Y_{\mathrm{MRI}}; \boldsymbol{\theta}) \\
&= \operatorname{argmin}_{\boldsymbol{\theta}} \|Y_{\mathrm{MRI}} - I_{\boldsymbol{\theta}} ** H\|^2
\end{aligned}
$$

Details:

- ML, LS estimation equivalent
- Nonlinear iterative minimization required

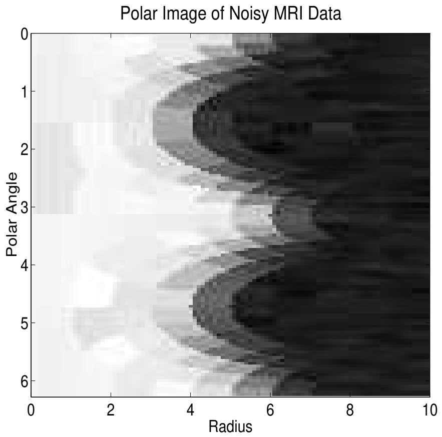
Single Pixel Interpolation

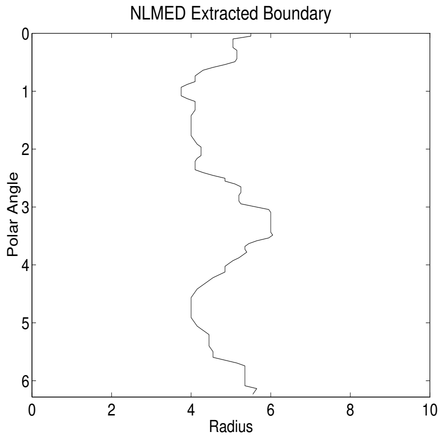


Linear interpolation is used to assign fractional intensities in $[C_{ROI}, C_{BG}]$ to "boundary" pixels of $I_{\boldsymbol{\theta}}$.

Edge Filter (EF) Boundary Extraction

- $Y_{\mathrm{M}RI}$ transformed into polar coordinates
- Edge-finding filter applied to rows of polar image
- Median filter is used to smooths extracted curve

(a)


(b)


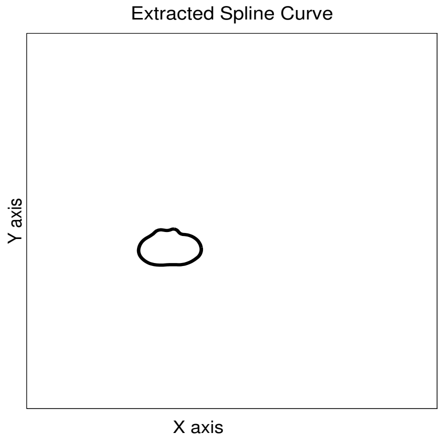
(c)


Figure 1: (a) Polar image of MRI data (b). (b) Polar curve samples extracted using the edge filtering approach. (c) Corresponding boundary curve.

CR Bound on Unbiased Boundary Estimation

Any unbiased estimator $\hat{\boldsymbol{\theta}}(Y_{\mathrm{MRI}})$ of unknown, nonrandom $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_K)$ satisfies

$$\mathrm{cov}(\hat{\boldsymbol{\theta}}) \geq F_{\boldsymbol{\theta}}^{-1} .$$

Matrix $F_{\boldsymbol{\theta}}$ is the Fisher Information matrix

$$\begin{aligned} F_{\boldsymbol{\theta}} &= E\left[\nabla_{\boldsymbol{\theta}}^T \ln f(Y_{\mathrm{MRI}}; \boldsymbol{\theta})\ \nabla_{\boldsymbol{\theta}} \ln f(Y_{\mathrm{MRI}}; \boldsymbol{\theta})\right] \\ &= E\left[-\nabla_{\boldsymbol{\theta}}^2 \ln f(Y_{\mathrm{MRI}}; \boldsymbol{\theta})\right] , \end{aligned}$$

where operator $\nabla_{\boldsymbol{\theta}}$ is the row gradient operator, and the (i, j)th entry of Hessian operator $\nabla_{\boldsymbol{\theta}}^2$ is $\frac{\partial^2}{\partial\theta_i \partial\theta_j}$.

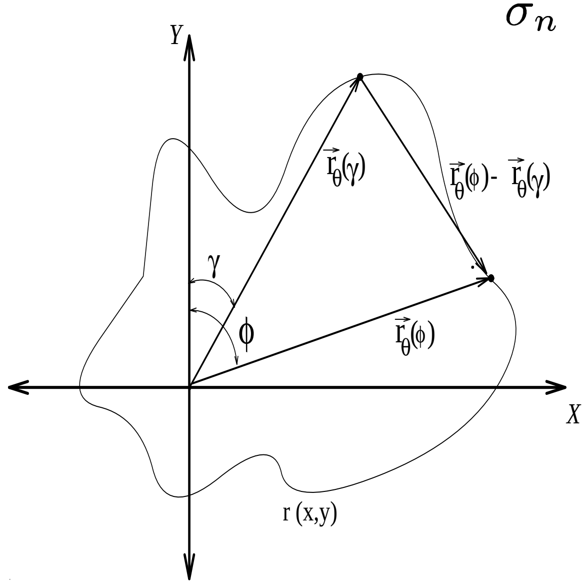
Continuous Data:

$$Y_{\mathrm{MRI}}(x, y) = (I_{\boldsymbol{\theta}} ** H)(x, y) + N(x, y)$$

$F_{\boldsymbol{\theta}}$ for Continuous Data $Y_{\mathrm{MRI}}(x,y)$

$$[F_{\boldsymbol{\theta}}]_{i,j} = C_{\mathrm{CN}} \iint_0^{2\pi} \exp\left[-\frac{\|\vec{r}_{\boldsymbol{\theta}}(\phi) - \vec{r}_{\boldsymbol{\theta}}(\gamma)\|^2}{4\sigma_s^2}\right] r_{\boldsymbol{\theta}}(\phi)\, r_{\boldsymbol{\theta}}(\gamma)\, B_i(\phi)\, B_j(\gamma)\ \ d\phi\, d\gamma$$

$$\begin{aligned} \text{where} \quad C_{\mathrm{CN}} &= \frac{(C_{ROI} - C_{BG})^2}{4\pi\sigma_s^2\sigma_n^2} \\ \sigma_s &= \text{MRI PSF standard deviation} \\ \sigma_n &= \text{MRI noise standard deviation} \end{aligned}$$



Geometric interpretation of $\|\vec{r}_{\boldsymbol{\theta}}(\phi) - \vec{r}_{\boldsymbol{\theta}}(\gamma)\|^2$ part of Fisher information.

Fisher information $F_{\mathbf{r}}$ for radial parameterization

Express

$$r_{\boldsymbol{\theta}}(\phi) \quad = \quad \mathbf{B}^T(\phi)\boldsymbol{\theta}$$

$$\mathbf{B}(\phi) \quad = \quad [B_1(\phi), \ldots, B_K(\phi)]^T$$

Fisher information for boundary radii: $\{\vec{r}_{\boldsymbol{\theta}}(\phi)\}_{\phi \in [0,2\pi]}$

$$[F_{\mathbf{r}}]_{\xi,\psi} \quad = \quad \mathbf{B}^T(\xi)\; F_{\boldsymbol{\theta}}\; \mathbf{B}(\psi)$$

$$= C_{\mathrm{CN}} \iint_0^{2\pi} \exp\left[-\frac{\|\vec{r}_{\boldsymbol{\theta}}(\phi) - \vec{r}_{\boldsymbol{\theta}}(\gamma)\|^2}{4\sigma_s^2}\right] r_{\boldsymbol{\theta}}(\phi)\, r_{\boldsymbol{\theta}}(\gamma)\, \mathbf{B}^T(\xi)\mathbf{B}(\phi)\, \mathbf{B}^T(\gamma)\mathbf{B}(\psi) \;\; d\phi\, d\gamma$$

Geometric implications

- Since B-spline basis is complete in L^2

$$\lim_{K\to\infty} [F\mathbf{r}]_{\xi,\psi} = C_{\mathrm{CN}}\ \exp\left(-\frac{\|\vec{r}_{\boldsymbol{\theta}}(\xi) - \vec{r}_{\boldsymbol{\theta}}(\psi)\|^2}{4\sigma_s^2}\right) r_{\boldsymbol{\theta}}(\xi)\, r_{\boldsymbol{\theta}}(\psi)$$

- Define:
  - mod. of cty. per unit angle: $\rho = \inf_{\phi,\gamma} \frac{\|\vec{r}_{\boldsymbol{\theta}}(\phi)-\vec{r}_{\boldsymbol{\theta}}(\gamma)\|}{|\phi-\gamma|}$
  - shape/resolution ratio: $\alpha = \rho/\sigma_s^2$

Then :

$$\lim_{\alpha,K\to\infty} [F\mathbf{r}]_{\xi,\psi} = \rightarrow \begin{cases} C_{\mathrm{CN}} \cdot r^2_{\boldsymbol{\theta}}(\psi), & \xi = \psi \\ \\ 0, & o.w. \end{cases}$$

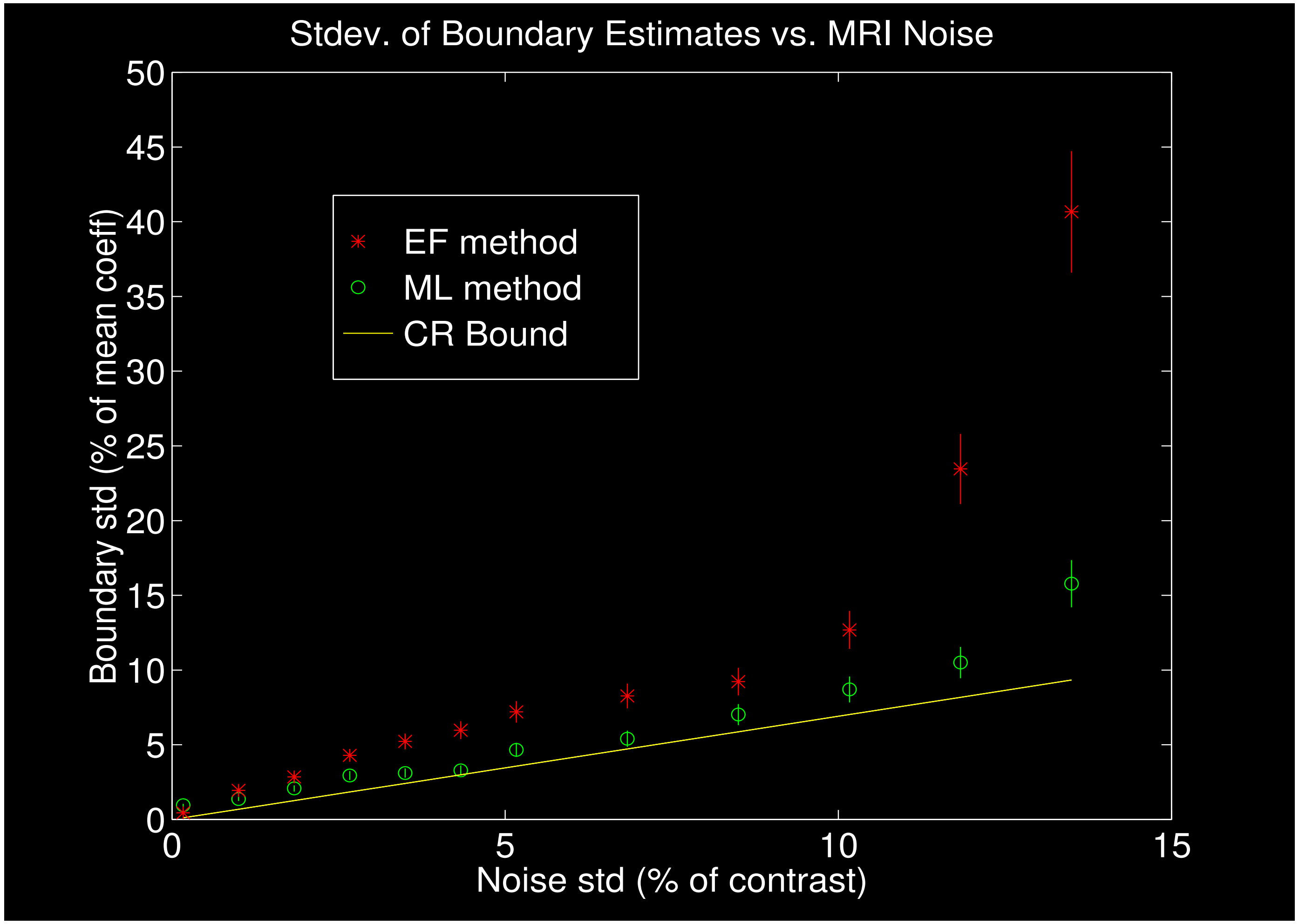
- In this case, average estimator variance satisfies

$$\overline{\sigma^2}_{geom}/K \quad \rightarrow \quad \frac{1}{C_{\mathrm{CN}} \cdot \int_0^{2\pi} r^2_{\boldsymbol{\theta}}(\phi) d\phi} \quad = \quad \frac{1}{2C_{\mathrm{CN}} \cdot \mathrm{area}}$$

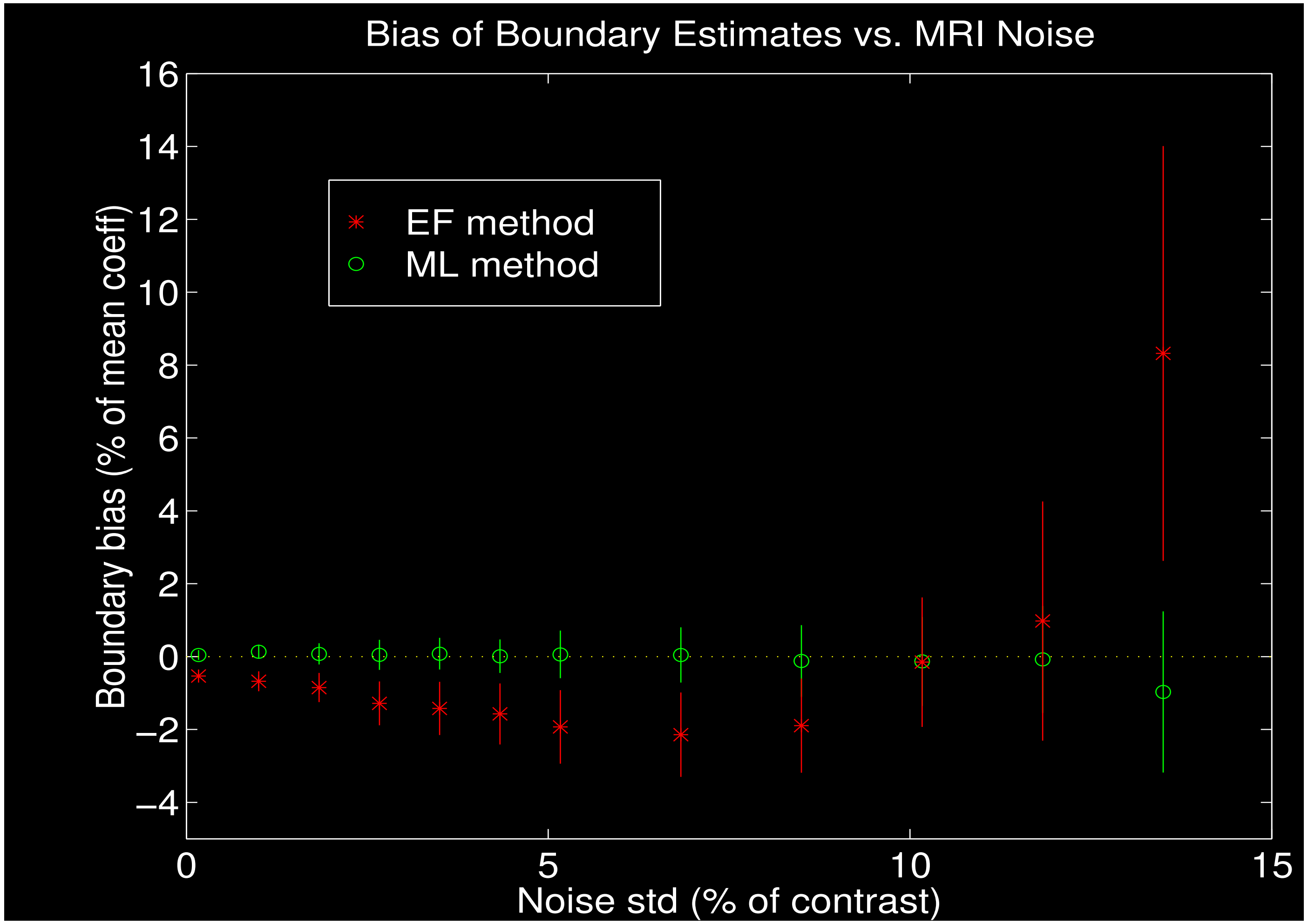
$$\overline{\sigma^2}_{arith}/K \quad \rightarrow \quad \frac{1}{C_{\mathrm{CN}}} \cdot \int_0^{2\pi} 1/r^2_{\boldsymbol{\theta}}(\phi) d\phi \quad = \quad \frac{1}{2C_{\mathrm{CN}}} \cdot \int_0^{2\pi} \left(\frac{d\phi}{dA}\right) d\phi$$

- For fixed perimeter, average estimator variance minimized for disk shaped objects.
- Good boundaries are circular
- Bad boundaries have sharp edges

Boundary Estimate Stdev. vs. MRI Noise



Boundary Estimate Bias vs. MRI Noise



ECT Reconstruction via Spatially Variant Gibbs Model: $\boldsymbol{\theta}$ known

- Assume conditional Gibbs distribution for λ

$$f(\boldsymbol{\lambda}|\boldsymbol{\theta}) = C\ \exp\left(-\beta R(\boldsymbol{\lambda}, \boldsymbol{\theta})\right)$$

$$R(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \sum_j \sum_k \omega_{jk}(\boldsymbol{\theta})\, (\lambda_j - \lambda_k)^2$$

- **Min P_e criterion**: design $\hat{\boldsymbol{\lambda}}$ to minimize

$$\min_{\hat{\boldsymbol{\lambda}}} P_{\boldsymbol{\theta}} \left(\|\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\| > \epsilon \right), \qquad \epsilon \text{ small}$$

Solution: Penalized Maximum Likelihood Objective

- $\boldsymbol{\lambda}$ maximizes posterior likelihood

$$\ln f(\boldsymbol{\lambda}|Y_E, Y_M; \theta) = \ln f(\boldsymbol{Y}_E; \boldsymbol{\lambda}) - \beta\, R(\boldsymbol{\lambda}, \boldsymbol{\theta})$$

- "Smoothing parameter" β controls bias-variance tradeoff.
- Side-information dependent Gibbs weights $\{\omega_{jk}(\boldsymbol{\theta})\}$.

$$R(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \sum_j \sum_k \omega_{jk}(\boldsymbol{\theta})\, (\lambda_j - \lambda_k)^2$$

Gibbs Weight Mapping $\omega_{jk}(\boldsymbol{\theta})$

- **Non-Negativity**

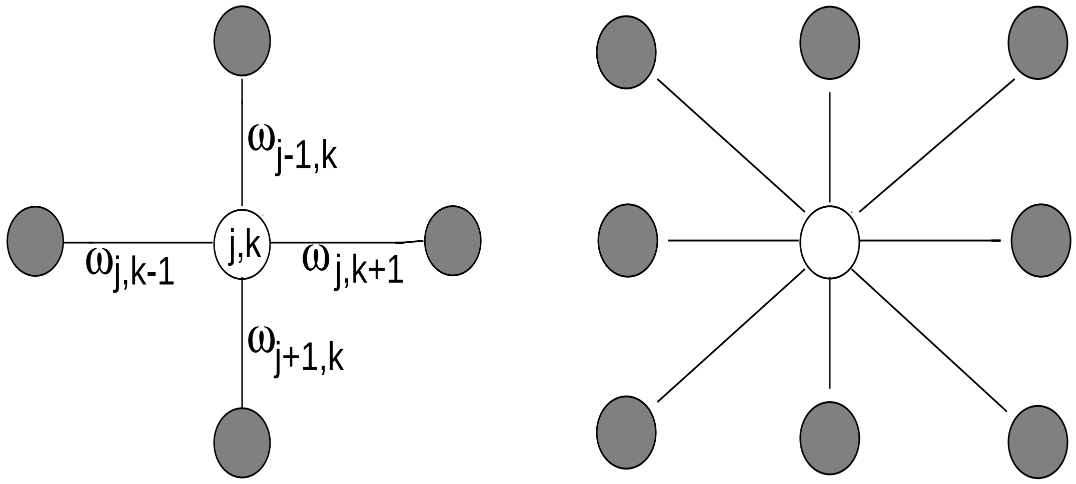
  $\omega_{jk} \geq 0, \quad \forall\, j, k$

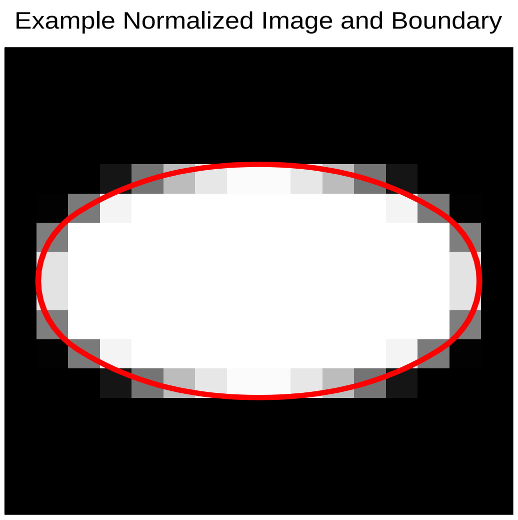
- **Symmetry**

  $\omega_{jk} = \omega_{kj}, \quad \forall\, j, k$

- **Locality**

  $\omega_{jk} = 0 \;\; \text{(non-neighbors)}$





- **Boundary relaxed weight mapping**

  Normalized "set membership" image $J(\boldsymbol{\theta})$ used to prevent smoothing across boundary.

Weight Matrices in Image Domain: θ Known

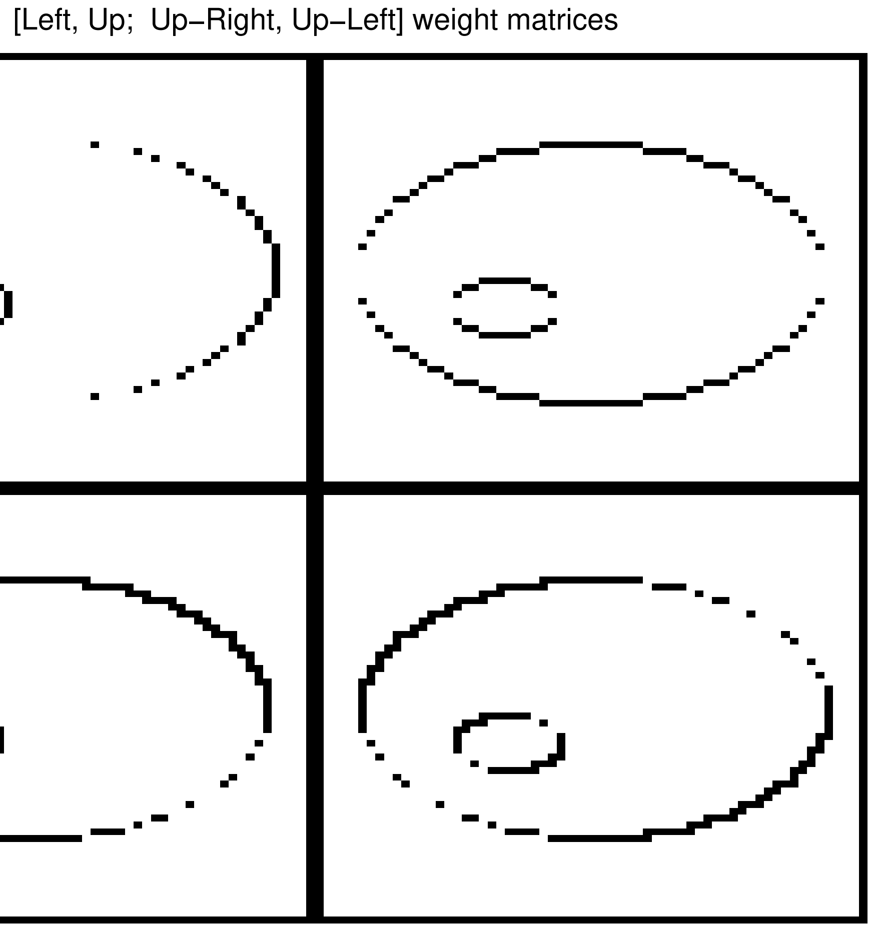


Figure 2: Ideal weights: a) left, b) up, c) up-right, d) up-left.

ECT Reconstruction: $\boldsymbol{\theta}$ unknown

- Estimate-and-plug [Titus&etal:1995]: $\boldsymbol{\theta} \longrightarrow \hat{\boldsymbol{\theta}}(Y_M)$ (MLE, EF, etc)
- Min-max P_e criterion:
  - Design $\hat{\boldsymbol{\lambda}}$ to minimize error over range of $\hat{\boldsymbol{\theta}}$

$$\min_{\hat{\boldsymbol{\lambda}}} \max_{\boldsymbol{\theta}} P_{\boldsymbol{\theta}} \left(\|\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\| > \epsilon \right),$$

  - equivalent to:

$$\min_{\hat{\boldsymbol{\lambda}}} \int_{\Theta} P_{\boldsymbol{\theta}} \left(\|\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}\| > \epsilon \right) \, \tilde{f}(\boldsymbol{\theta}) \, d\boldsymbol{\theta}$$

  - $\tilde{f}(\boldsymbol{\theta})$ is the "equalizer density" which makes $\min_{\hat{\boldsymbol{\lambda}}} P_{\hat{\boldsymbol{\theta}}}$ constant

Equivalent form for Min-Max Estimator

- Use **Factorization Identity**

$$f(Y_E, \boldsymbol{\lambda}, \boldsymbol{\theta}, Y_M) = f(Y_E|\boldsymbol{\lambda}) f(\boldsymbol{\lambda}|\boldsymbol{\theta}) \tilde{f}(\boldsymbol{\theta}|Y_M) \tilde{f}(Y_M)$$

- to show that min-max $\hat{\boldsymbol{\lambda}}$ maximizes

$$\ln \tilde{f}(\boldsymbol{\lambda}|Y_E, Y_M) = \ln f(Y_E|\boldsymbol{\lambda}) + \ln \int_{\Theta} \exp(-\beta R(\boldsymbol{\lambda}; \boldsymbol{\theta})) \tilde{f}(\boldsymbol{\theta}|Y_M) d\boldsymbol{\theta} + c$$

Note:

- This is a modified PML function for estimating $\boldsymbol{\lambda}$
- Penalty term remains convex in $\boldsymbol{\lambda}$
- Penalty term is non-linear in weights ω_{ij}

Simplification of modified PML via Jensen's Inequality

$$\ln \tilde{f}(\boldsymbol{\lambda}|Y_E, Y_M) \geq \ln f(Y_E|\boldsymbol{\lambda}) - \beta \sum_{ij} \tilde{w}_{ij}(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j)^2 + c$$

where $\tilde{\omega}_{ij}$ are averaged weights

$$\tilde{w}_{ij} = \int \omega_{ij}(\boldsymbol{\theta})\tilde{f}(\boldsymbol{\theta}|Y_M)d\boldsymbol{\theta}$$

Remaining issue:

- How to compute $\tilde{f}(\boldsymbol{\theta}|Y_M)$?

Asymptotic Marginalization

Theorem: Assume that $\tilde{f}(Y_M|\boldsymbol{\theta})$ is a smooth function of $\boldsymbol{\theta}$. Then the maximum likelihood estimator $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(Y_M)$ is consistent and

$$f(\boldsymbol{\theta}|Y_M) = \frac{|F_{\hat{\boldsymbol{\theta}}}|^{\frac{1}{2}}}{(\sqrt{2\pi})^p} \cdot \exp\left\{-\tfrac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \hat{F}_{\boldsymbol{\theta}}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\right\} \left(1 + O(\Delta^2)\right)$$

where $\hat{F}_{\boldsymbol{\theta}}$ is the *observed* Fisher information matrix

$$\hat{F}_{\hat{\boldsymbol{\theta}}} = -\nabla^2_{\hat{\boldsymbol{\theta}}} \ln f(Y_M|\hat{\boldsymbol{\theta}}),$$

and $O(\Delta^2)$ decays to zero faster than $\Delta^2 = \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2$.

Implementation

Define n-vector $\mathbf{r}$ as sampled version of $\{r_{\boldsymbol{\theta}}(\phi)\}_\phi$

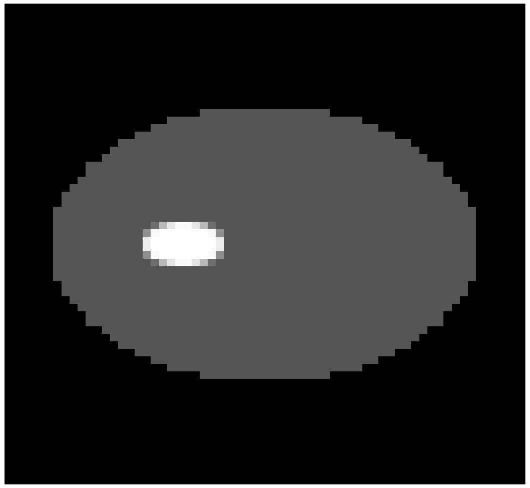
Then using Fisher matrix for $\mathbf{r}$, smoothed weights take form:

$$
\begin{aligned}
\tilde{w}_{ij}(\hat{\mathbf{r}}) &= \frac{|\hat{F}_{\hat{\mathbf{r}}}|^{\frac{1}{2}}}{\sqrt{2\pi}^n} \cdot \sum_{\mathbf{r}} w_{ij}(\mathbf{r}) \exp\left\{-\tfrac{1}{2}(\mathbf{r}-\hat{\mathbf{r}})^T \hat{F}_{\hat{\mathbf{r}}}^{\dagger}(\mathbf{r}-\hat{\mathbf{r}})\right\} \Delta\mathbf{r} \\
&= w_{ij}(\hat{\mathbf{r}}) \, * \, h(\hat{\mathbf{r}})
\end{aligned}
$$

- smoothed weights can be computed offline
- weight averaging operation is an n-D convolution
- convolution can be approximated by sequence of 2D FFT's due to 2nd order neighborhood property

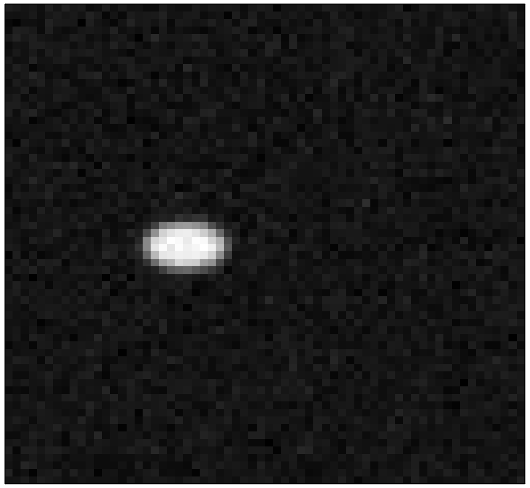
Numerical Experiments

Emission Phantom



- **Phantom:** 64×64 ($C_{ROI}=9,\ C_{BG}=3$)
- **Spline $\boldsymbol{\theta}$:** 16-knot, 128 ϕ samples
- **Realizations:** 400
- **Background:** 15%
- **Projections:** 64 bins, 60 angles
- **Reconstruction:** 60 iters SAGE3

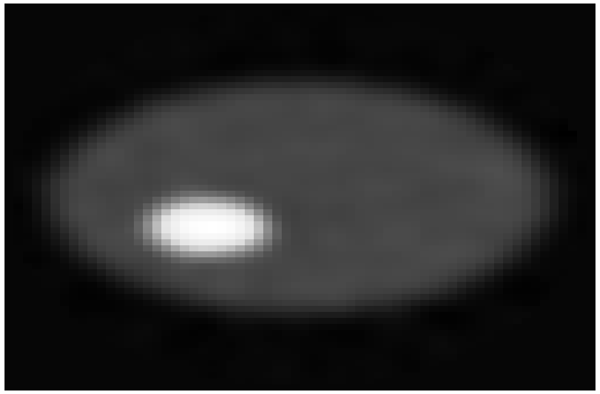
Realization of Noisy MRI Data



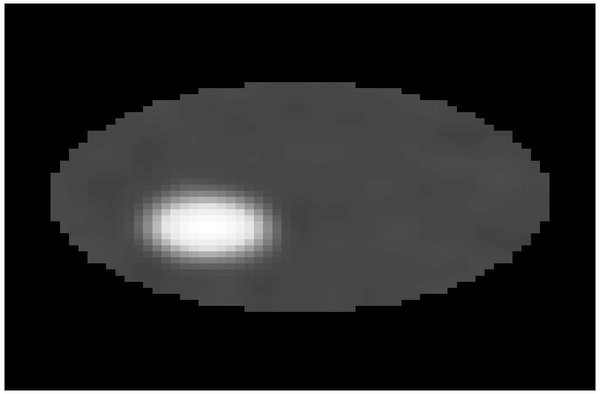
- **MRI:** 64×64 ($C_{ROI}=6,\ C_{BG}=0$)
- $\boldsymbol{\sigma_s}$: .7 pixel (19% mean coeff)
- $\boldsymbol{\sigma_n}$: $.18 - .5$ ($3\% - 10\%$ of contrast)

- **ROI:** ideal, raw extracted weights, smoothed weights

Reconstruction: no side info

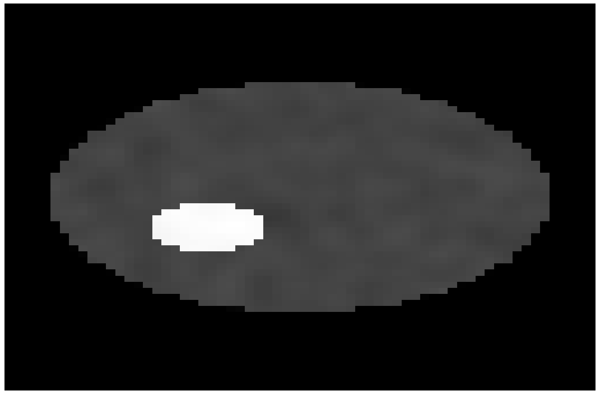


(a) FBP - Hamming

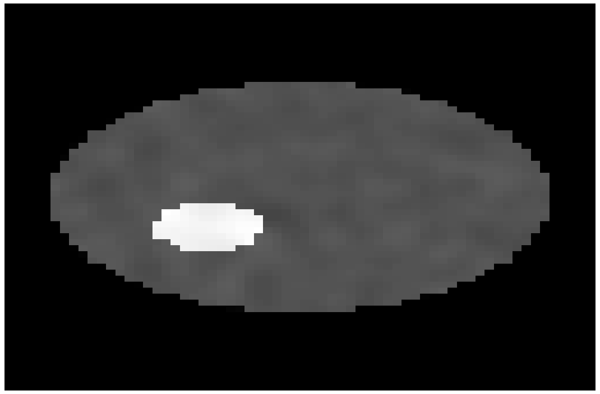


(b) SAGE - uniform

Reconstruction: unsmoothed side info

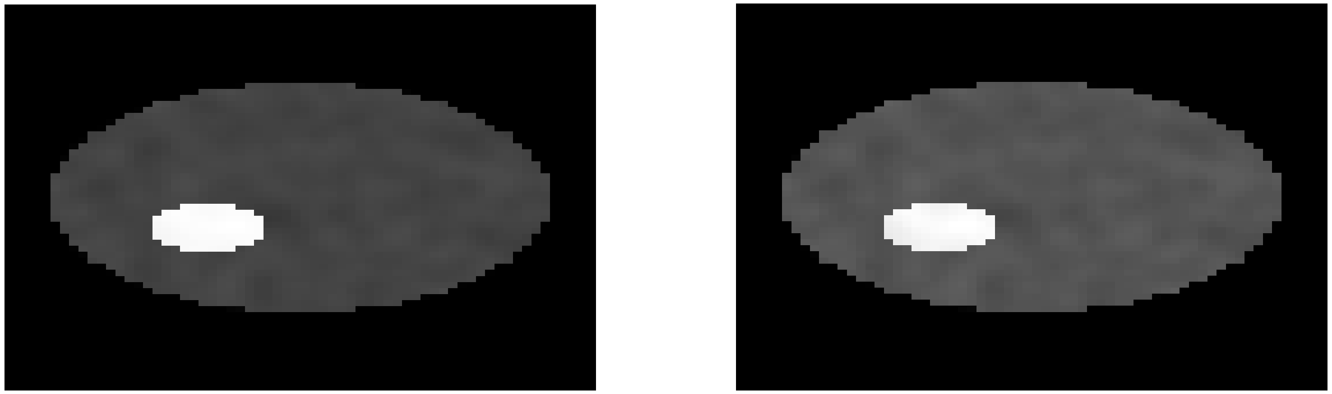


(c) SAGE - ideal



(d) SAGE - extracted

Reconstruction: smoothed side info



(e) SAGE - ideal (f) SAGE - corrected

Figure 3: Reconstructed images a single realization of MRI and PET data with MRI noise $\sigma_n = 0.36$ (6% of contrast). The PML-SAGE3 method with $\beta = 1$ was used for (c-f).

Ideal Weight Matrices

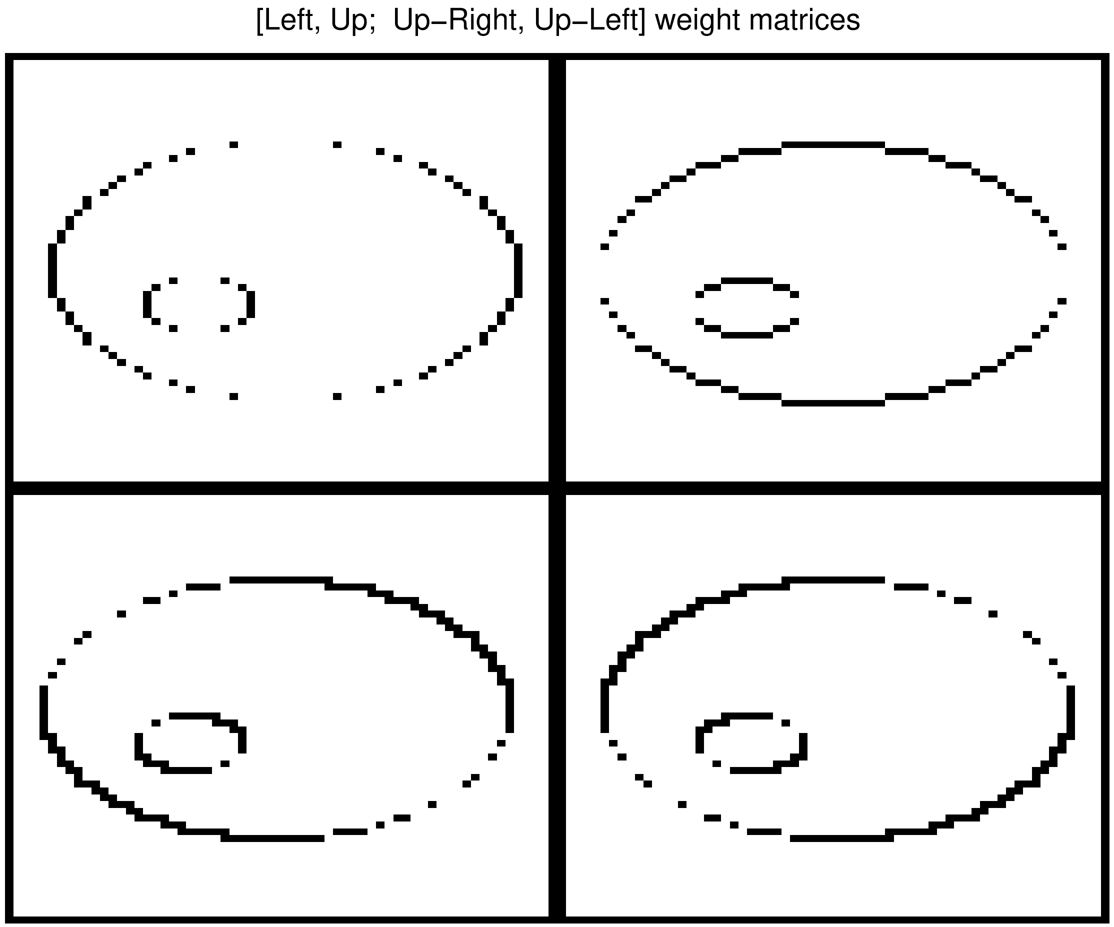


Figure 4: Ideal weights: a) left, b) up, c) up-right, d) up-left.

Extracted Weight Matrices (6% noise)

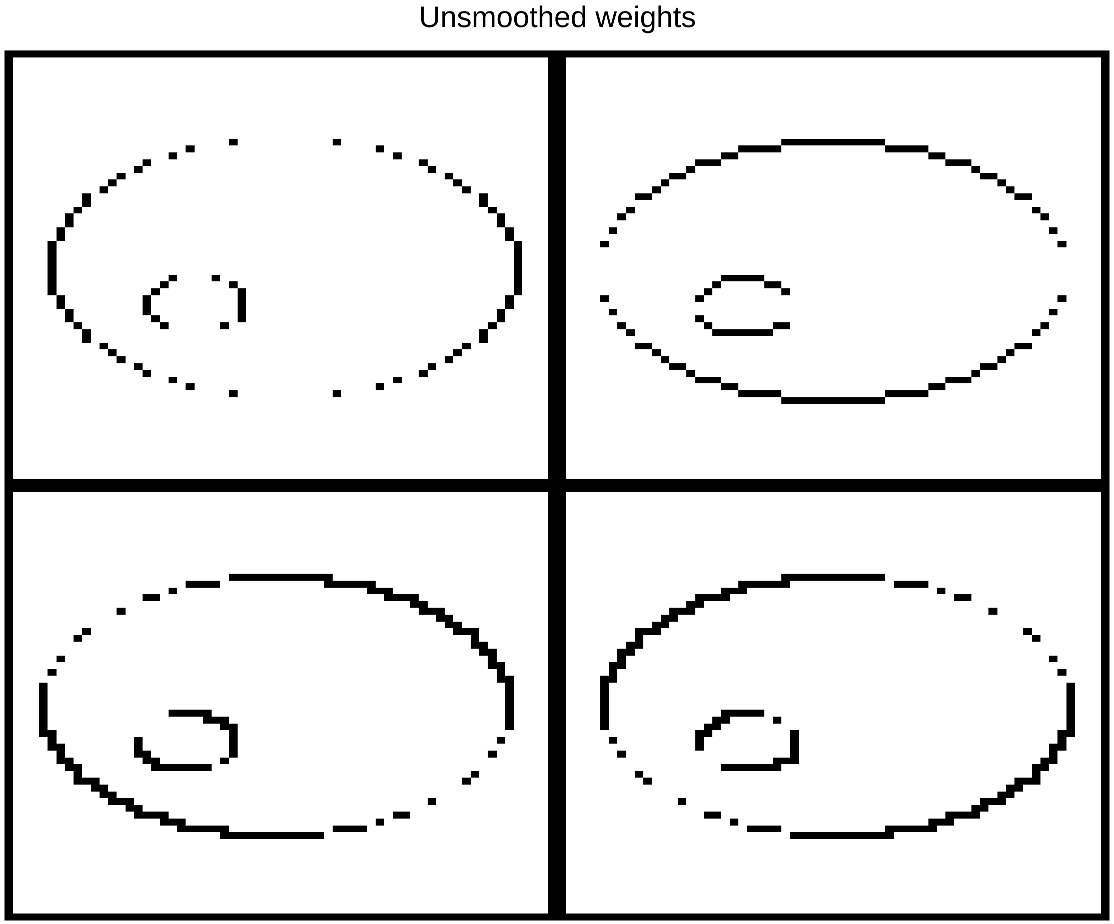


Figure 5: Extracted weights: a) left, b) up, c) up-right, d) up-left.

Smoothed Extracted Weight Matrices (6% noise)

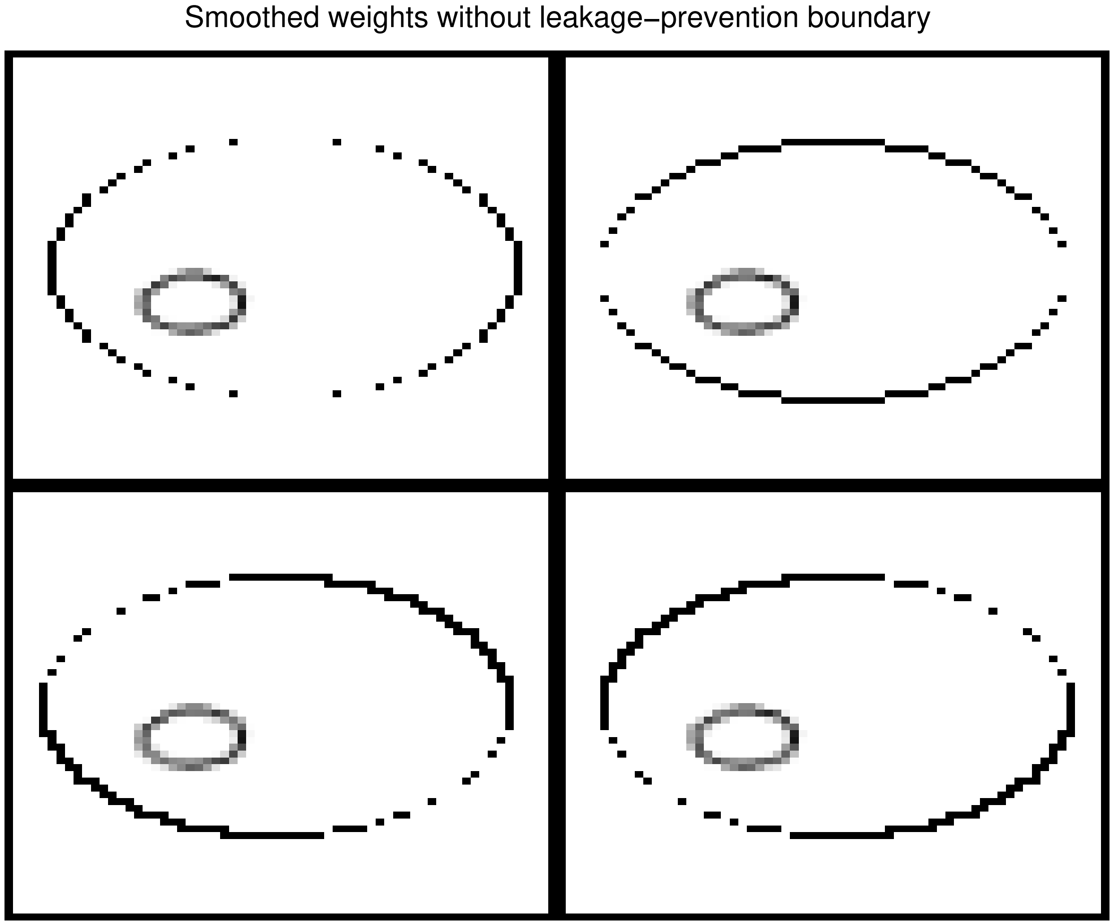
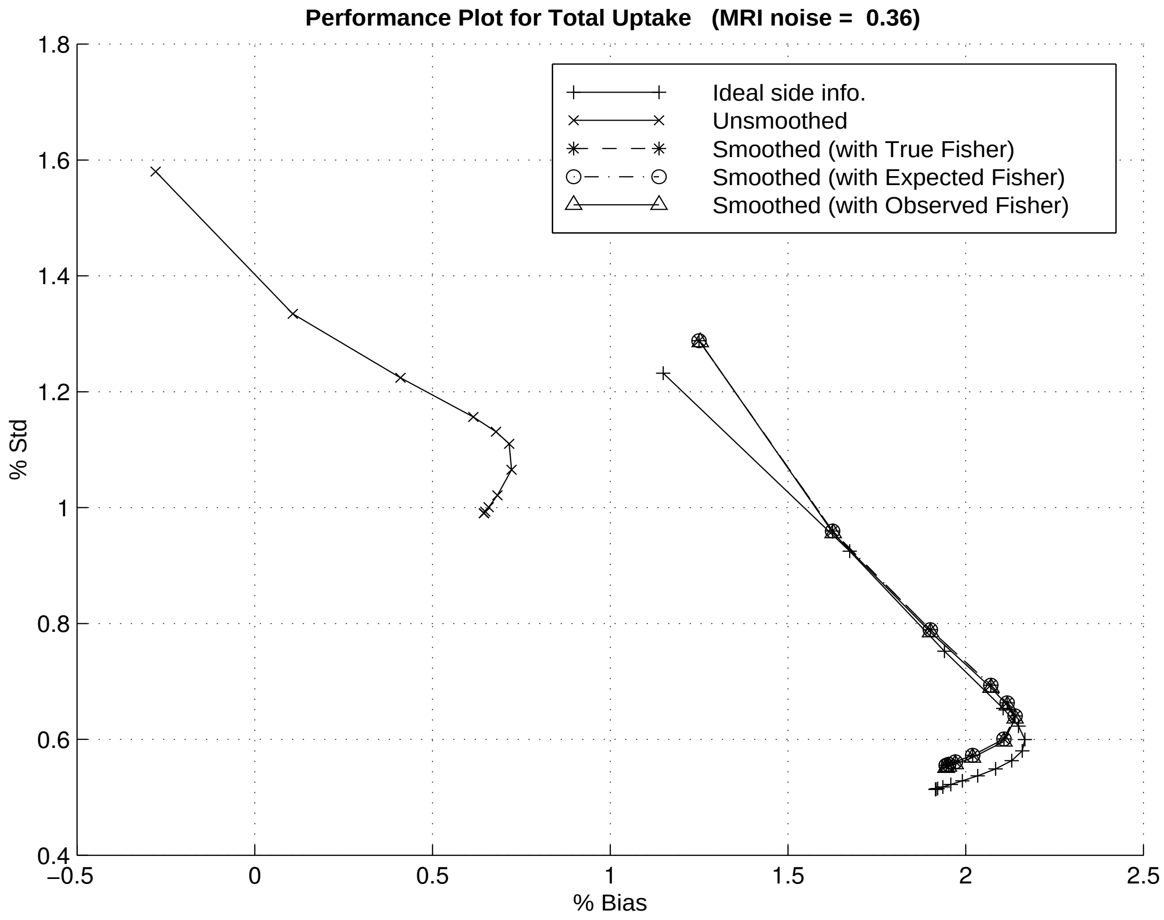
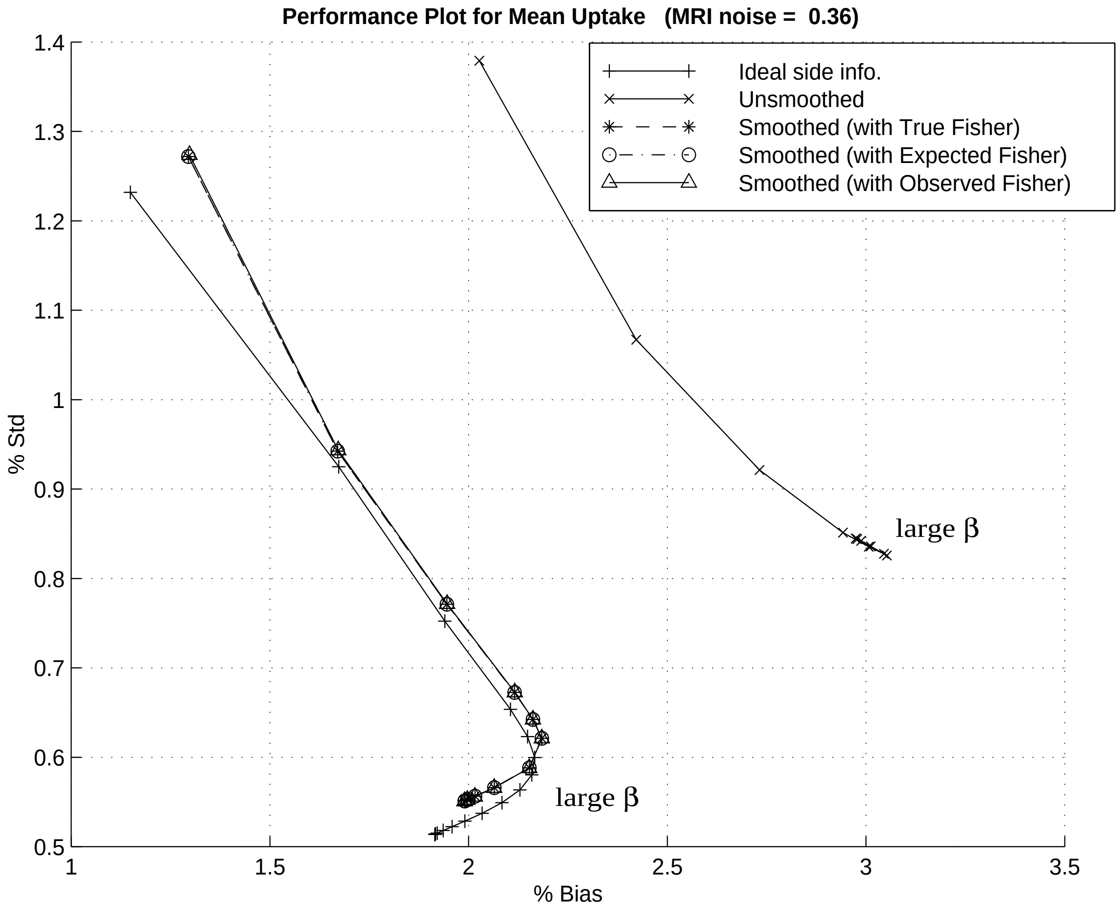


Figure 6: Smoothed weights: a) left, b) up, c) up-right, d) up-left.

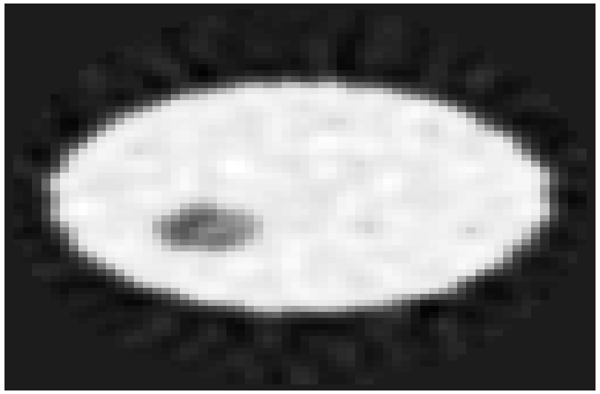


(a) Total Uptake Bias Variance Tradeoff

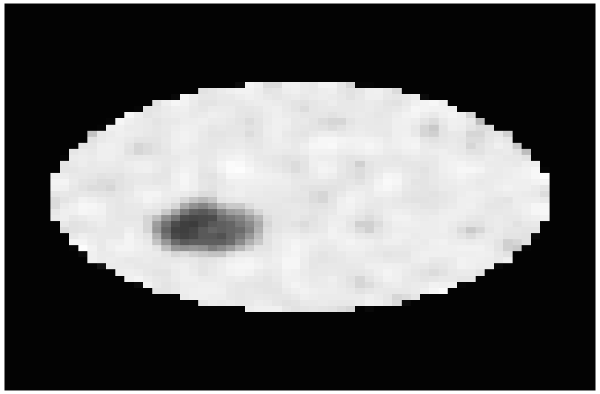


(b) Mean Uptake Bias Variance Tradeoff

Cold Spot Reconstruction: no side info

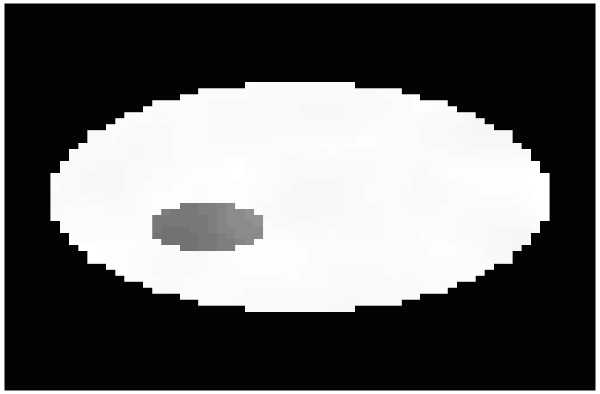


(c) FBP - Hamming

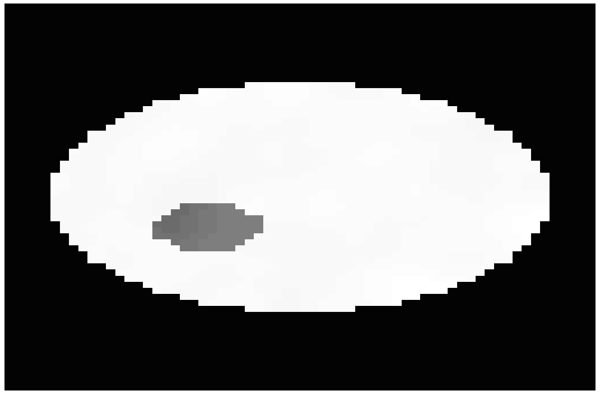


(d) SAGE - uniform

Cold Spot Reconstruction: unsmoothed side info

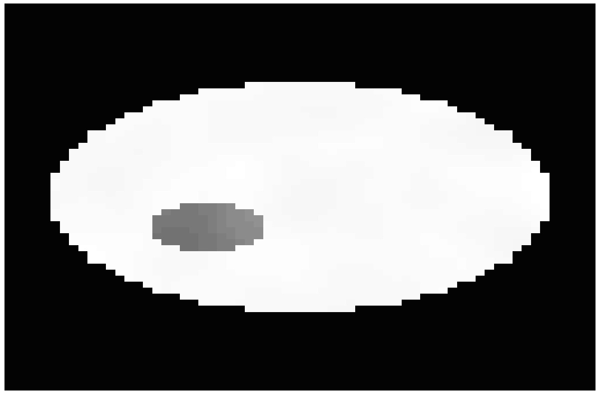


(e) SAGE - ideal

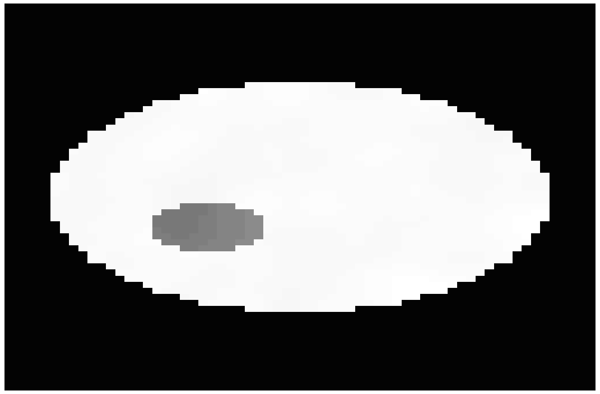


(f) SAGE - extracted

Cold Spot Reconstruction: smoothed side info

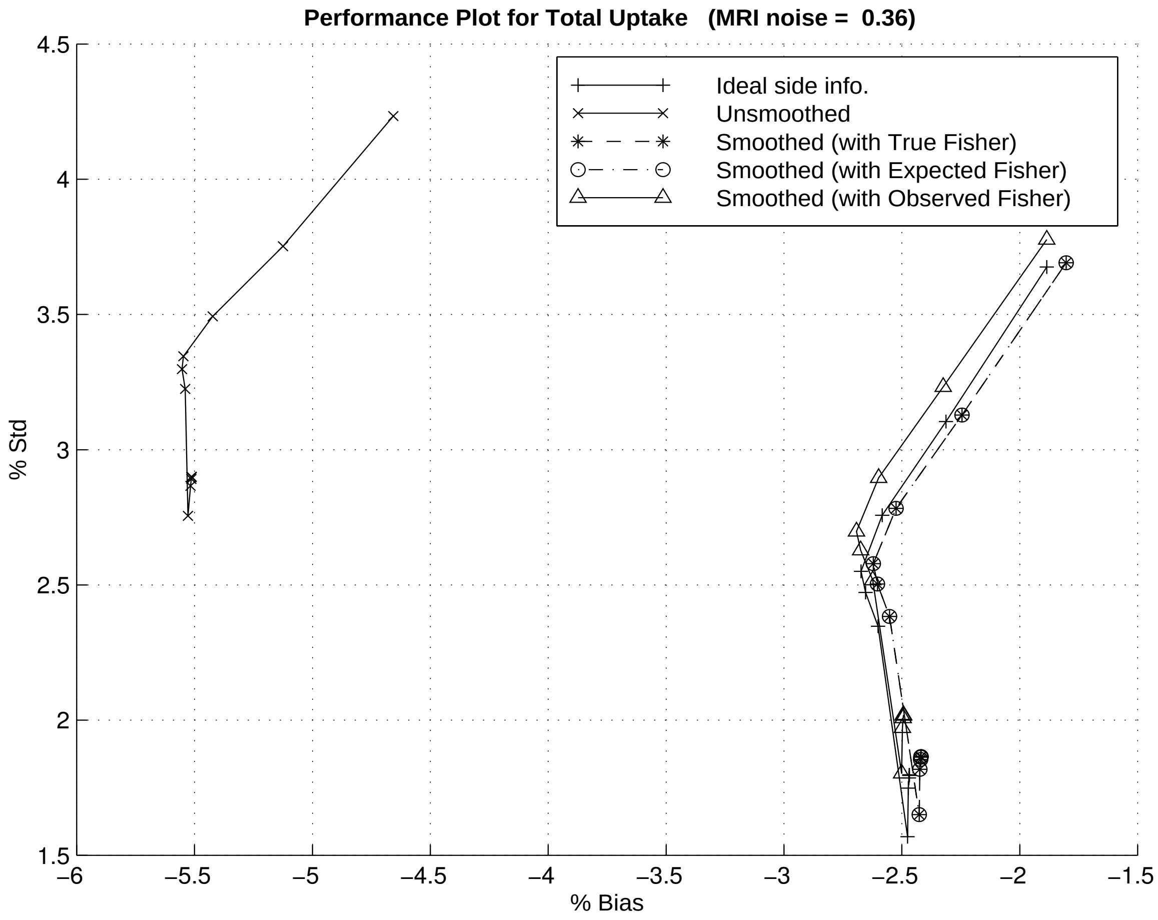


(g) SAGE - ideal

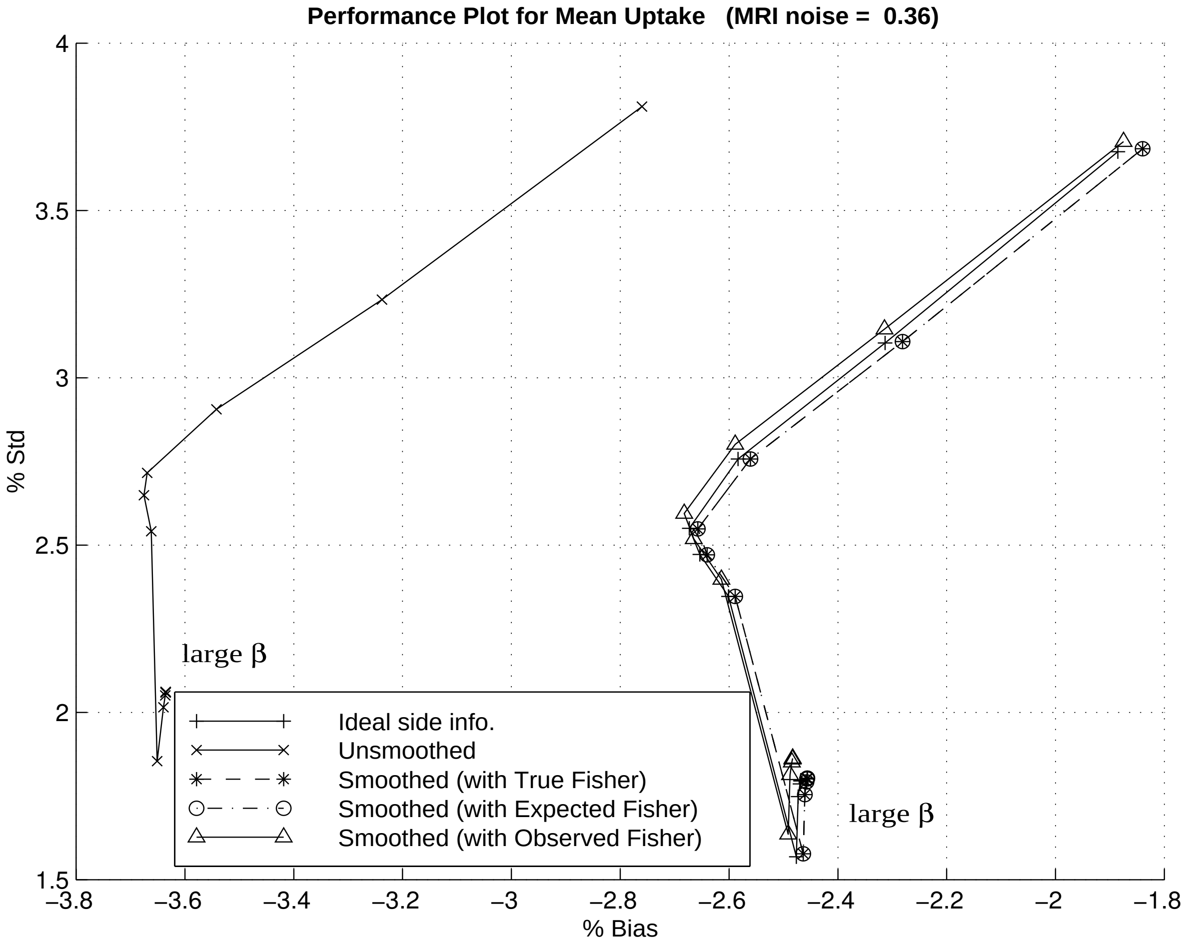


(h) SAGE - corrected

Figure 7: Reconstructed images a single realization of MRI and PET data with MRI noise $\sigma_n = 0.36$ (6% of contrast). The PML-SAGE3 method with $\beta = 1$ was used for (c-f).



(a) Total Uptake Bias Variance Tradeoff



(b) Mean Uptake Bias Variance Tradeoff

Conclusions and future work

- Iterative incorporation of extracted side information via Gibbs penalty can greatly improve uptake estimation
- Asymptotic marginalization plus Gibbs property leads to fast implementation.
- Expected Fisher information appears to be a somewhat better smoother as compared to Observed Fisher information.
- Extension to non-star shaped boundaries?
- Extension to 3D objects?
- Comparison to joint estimation approaches?