Analysis of the Sequential Partial Update LMS Algorithm

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Abstract

Partial updating of LMS filter coefficients is an effective method for reducing the computational load and the power consumption in adaptive filter implementations. The Sequential Partial Update LMS algorithm is one popular algorithm in this category. In [5] a first order stability analysis of this algorithm was performed on wide sense stationary signals under the restrictive assumption of small step size parameter μ. The necessary and sufficient condition derived on μ for convergence in the mean was identical to the one for guaranteeing stability in the mean of LMS. In [9] first order sufficient conditions were derived for stability without the aforementioned small μ assumption. The sufficient region of convergence derived was smaller than that of regular LMS. In this paper, we establish that for stationary signals the sequential algorithm converges in mean for the same values of the step size parameter μ for which the regular LMS does. In other words, we show that the conclusion drawn in [5] holds without the restrictive assumption of small μ. We also derive sufficient conditions for stability on μ for cyclo-stationary signals.

Keywords: partial update LMS algorithms, random updates, sequential algorithm, periodic algorithm.

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1 Introduction

The least mean-squares (LMS) algorithm is a popular algorithm for adaptation of weights in adaptive beamformers using antenna arrays and for channel equalization to combat intersymbol interference. Many other application areas of LMS include interference cancellation, echo cancellation, space time modulation and coding, signal copy in surveillance and wireless communications. Although there exist algorithms with faster convergence rates like RLS, LMS is popular because of its ease of implementation and low computational costs.

The least mean-squares (LMS) algorithm is an approximation of the steepest descent algorithm used to arrive at the Weiner-Hopf solution for computing the weights (filter coefficients) of a finite impulse response (FIR) filter. The filter coefficients are computed so as to produce the closest approximation in terms of mean squared error to a *desired output*, which is stochastic in nature from the input to the filter, which is also stochastic in nature. The Weiner-Hopf solution involves an inversion of the input signal correlation matrix. The steepest descent algorithm avoids this inversion by recursively computing the filter coefficients using the gradient computed using the input signal correlation matrix. The LMS algorithm differs from the steepest algorithm in that it uses a "stochastic gradient" as opposed to the exact gradient. Knowledge of the exact input signal correlation matrix is not required for the algorithm to function. The reduction in complexity of the algorithm comes at an expense of greater instability and degraded performance in terms of final mean squared error. Therefore, the issues with the LMS algorithm are "filter stability", "final misadjustment" and "convergence rate" [11, 13, 15].

Partial updating of the LMS adaptive filter has been proposed to reduce computational costs and power consumption [6, 7, 14] which is quite attractive in the area of of mobile computing and communications. Partial update algorithms have application in many fields including adaptive beamforming, channel equalization in communications and space-time modulation/coding. Sequential Partial Update LMS algorithm is one such algorithm. However, for this algorithm theoretical performance predictions on convergence rate and steady state tracking error are more difficult to derive than for standard full update LMS. Accurate theoretical predictions are important as it has been observed that for the non-stationary signal scenario the standard LMS conditions on the step size parameter fail to ensure convergence of S-LMS.

One of the variants of LMS is the Partial Update LMS (PU-LMS) Algorithm. Many applications in wireless communications like channel equalization and echo cancellation require the adaptive filter to have a very large number of coefficients. Updating the entire coefficient is constly in terms of power, memory, and computation and is sometimes impractical for mobile units. Therefore, partial updating of the LMS adaptive filter has been proposed to reduce these costs [6, 7, 14]. In this era of mobile computing and communications, such implementations are especially attractive for reducing power consumption. However, theoretical performance predictions on convergence rate and steady state tracking error are more difficult than for standard full update LMS. Accurate theoretical predictions are important as it has been observed that the standard LMS conditions on the step size parameter fail to ensure convergence of the partial update algorithm [7].

Two types of partial update LMS algorithms are prevalent in the literature and have been described in [5]. They are referred to as the "Periodic LMS algorithm" and the "Sequential LMS algorithm". To reduce computation by a factor of P, the Periodic LMS algorithm (P-LMS) updates all the filter coefficients every P^{th} iteration instead of every iteration. The Sequential LMS (S-LMS) algorithm updates only a fraction of coefficients every iteration. Another variant referred to as "Max Partial Update LMS algorithm" (Max PU-LMS) has been proposed in [3, 4] and [1]. In this algorithm, the subset of coefficients to be updated is dependent on the input signal. The subset is chosen so as to minimize the increase in the mean squared error due to partial as opposed to full updating. The input signals multiplying each coefficient are ordered according to their magnitude and the coefficients corresponding to the largest $\frac{1}{P}$ of input signals are chosen for update in an iteration. Some analysis of this algorithm has been performed in [4] for the special case of $P = 1$ but, analysis for more general cases still needs to be completed.

In [5] convergence conditions were derived under the assumption of small step-size parameter (μ) which turned out to be the same as those for the standard LMS algorithm. We have been interested in investigating the Sequential Partial Update LMS Algorithm (S-LMS) under more general condition on μ. In [9] we derived a less restrictive sufficient condition on μ for stationary signals. However, we were unable to find an example of a stationary signal for which regular LMS was stable but S-LMS unstable. We also point out that the motivating example in Section 2 of [9] is incorrect.

This has led us to look for more accurate bounds on μ which hold for stationary signals and arbitrary fixed sequence of partial updates. Here, we show that for stationary signals first order stability of LMS implies first order stability of S-LMS.. We also extend the analysis in [9] to cyclo-stationary signals. We show that for the cyclo-stationary case there exists some μ such that the use of it in S-LMS could lead to divergence even if full update LMS converges for this μ.

The main contributions of this paper can be summarized as follows:

- For stationary signals and arbitrary sequence of updates, we have conclusively shown, without the restrictive assumption of small μ [5], that convergence in the mean of LMS implies convergence in the mean of S-LMS (Theorem 2).
- For a special class of cyclo-stationary signals and even-odd sequence of updates, we have derived sufficient conditions for convergence in the mean of S-LMS (Theorem 3).

The organization of the paper is as follows. First in Section 2, a brief description of the sequential partial update algorithm is given. The algorithm with arbitrary sequence of updates is analyzed for the case of stationary signals in Section 3. This is followed by the analysis of algorithm with the special case of alternate even and odd coefficient updates for cyclo-stationary signals in Section 4. In Section 5 an example is given to illustrate the usefulness of the bounds on step-size derived in Section 4. Finally, conclusions and directions for future work are indicated in Section 6.

2 Algorithm Description

The block diagram of S-LMS for a N-tap LMS filter with alternating even and odd coefficient updates is shown in Figure 1. We refer to this algorithm as even-odd S-LMS.

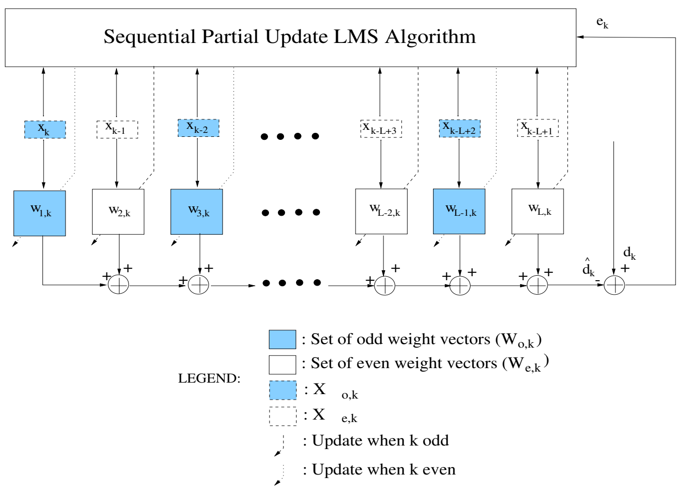

Figure 1: Block diagram of S-LMS for the special case of alternating even/odd coefficient update

It is assumed that the LMS filter is a standard FIR filter of even length, N. For convenience, we start with some definitions. Let $\{x_{i,k}\}$ be the input sequence and let $\{w_{i,k}\}$ denote the coefficients of the adaptive filter. Define

$$
\begin{aligned}
W_k &= [w_{1,k}\ w_{2,k}\ \ldots\ w_{N,k}]^T \\
X_k &= [x_{1,k}\ x_{2,k}\ x_{3,k}\ \ldots\ x_{N,k}]^T
\end{aligned}
$$

where the terms defined above are for the instant k and T denotes the transpose operator. In addition, Let d_k denote the desired response. In typical applications d_k is a known training signal which is transmitted over a noisy channel with unknown FIR transfer function.

In this section we assume that d_k itself obeys an FIR model given by $d_k = W_{opt}^H X_k + n_k$ [5] where W_{opt} are the coefficients of an FIR model given by $W_{opt} = [w_{1,opt}\ \ldots\ w_{N,opt}]^T$ and H denotes the hermitian operator. Here $\{n_k\}$ is assumed to be a zero mean i.i.d sequence that is independent of the input sequence X_k.

For description purposes we will assume that the filter coefficients can be divided into P mutually exclusive subsets of equal size, i.e. the filter length N is a multiple of P. For convenience, define the index set $S = \{1, 2, \ldots, N\}$. Partition S into P mutually exclusive subsets of equal size, $S_1,\ S_2, \ldots, S_P$. Define I_i by zeroing out the j^{th} row of the identity matrix I if $j \notin S_i$. In that case, $I_i X_k$ will have precisely $\frac{N}{P}$ non-zero entries. Let the sentence "choosing S_i at iteration k" stand to mean "choosing the weights with their indices in S_i for update at iteration k".

The S-LMS algorithm is described as follows. At a given iteration, k, one of the sets S_i, $i = 1, \ldots, P$, is chosen in a pre-determined fashion and the update is performed.

$$
w_{k+1,j} = \begin{cases} w_{k,j} + \mu e_k^* x_{k,j} & \text{if } j \in S_i \\ w_{k,j} & \text{otherwise} \end{cases}
$$

where $e_k = d_k - W_k^H X_k$. The above update equation can be written in a more compact form in the following manner

$$
W_{k+1} = W_k + \mu e_k^* I_i X_k \tag{1}
$$

In the special case of even and odd updates, $P = 2$ and S_1 consists of all even indices and S_2 of all odd indices as shown in Figure 1.

We also define the coefficient error vector as

$$
V_k = W_k - W_{opt}
$$

which leads to the following coefficient error vector update for S-LMS when k is odd

$$\begin{aligned} V_{k+2} &= (I-\mu I_2 X_{k+1}X_{k+1}^H)(I-\mu I_1 X_k X_k^H)V_k + \\ &\quad \mu(I-\mu I_2 X_{k+1}X_{k+1}^H)n_k I_1 X_k + \mu n_{k+1} I_2 X_{k+1}, \end{aligned}$$

and the following when k is even

$$\begin{aligned} V_{k+2} &= (I-\mu I_1 X_{k+1}X_{k+1}^H)(I-\mu I_2 X_k X_k^H)V_k + \\ &\quad \mu(I-\mu I_1 X_{k+1}X_{k+1}^H)n_k I_2 X_k + \mu n_{k+1} I_1 X_{k+1}. \end{aligned}$$

3 Analysis: Stationary Signals

Assuming that d_k and X_k are jointly WSS random sequences, we analyze the convergence of the mean coefficient error vector $E[V_k]$. We make the standard assumptions that V_k and X_k are mutually uncorrelated and that X_k is independent of X_{k-1} [2]. For regular full update LMS algorithm the recursion for $E[V_k]$ is given by

$$E[V_{k+1}] = (I-\mu R)E[V_k] \tag{2}$$

where I is the N-dimensional identity matrix and $R = E\left[X_k X_k^H\right]$ is the input signal correlation matrix. The well known necessary and sufficient condition for $E[V_k]$ to converge is given by [11]

$$\rho(I-\mu R) < 1$$

where $\rho(B)$ denotes the spectral radius of B ($\rho(B) = \max|\lambda_i(B)|$). This leads to

$$0 < \mu < 2/\lambda_{max}(R) \tag{3}$$

where $\lambda_{max}(R)$ is the maximum eigen-value of the input signal correlation matrix R.

Taking expectations under the same assumptions as above, using the independence assumption on the sequences X_k, n_k, the mutual independence assumption on X_k and V_k, and simplifying we obtain for even-odd S-LMS when k is odd

$$E[V_{k+2}] = (I-\mu I_2 R)(I-\mu I_1 R)E[V_k] \tag{4}$$

and when k is even

$$E[V_{k+2}] = (I-\mu I_1 R)(I-\mu I_2 R)E[V_k]. \tag{5}$$

It can be shown that under the above assumptions on X_k, V_k and d_k, the convergence conditions for odd ($\rho((I-\mu I_2 R)(I-\mu I_1 R)) < 1$) and even update equations ($\rho((I-\mu I_1 R)(I-\mu I_2 R)) < 1$) are identical. We therefore focus on (4). We will show that if $\rho(I-\mu R) < 1$ then $\rho((I-\mu I_2 R)(I-\mu I_1 R)) < 1$.

Now, if instead of just two partitions of even and odd coefficients ($P = 2$) we have any number of arbitrary partitions ($P \geq 2$) then the update equations can be similarly written as above with $P > 2$. Namely,

$$E[V_{k+P}] = \prod_{i=1}^{P}(I - \mu I_{(i+k)\%P} R) E[V_k]$$

where $(i+k)\%P$ stands for $(i+k)$ modulo P. I_i, $i = 1, \ldots, P$ is obtained from I, the identity matrix of dimension $N \times N$, by zeroing out some rows in I such that $\sum_{i=1}^{P} I_i$ is positive definite.

We will show that for any arbitrary partition of any size ($P \geq 2$); S-LMS converges in the mean if LMS converges in the mean(Theorem 2). The case $P = 2$ follows as a special case. The intuitive reason behind this fact is that both the algorithms try to minimize the mean squared error $V_k^H R V_k$. This error term is a quadratic bowl in the V_k co-ordinate system. Note that LMS moves in the direction of the negative gradient $-RV_k$ by retaining all the components of this gradient in the V_k co-ordinate system whereas S-LMS discards some of the components at every iteration. The resulting gradient vector (the direction in which S-LMS updates its weights) obtained from the remaining components still points towards the bottom of the quadratic bowl and hence if LMS reduces the mean squared error then so does S-LMS.

We will show that if R is a positive definite matrix of dimension $N \times N$ with eigenvalues lying in the open interval $(0, 2)$ then $\prod_{i=1}^{P}(I - I_i R)$ has eigenvalues inside the unit circle.

The following theorem is used in proving the main result in Theorem 2.

Theorem 1 *[12, Prob. 16, page 410] Let B be an arbitrary $N \times N$ matrix. Then $\rho(B) < 1$ if and only if there exists some positive definite $N \times N$ matrix A such that $A - B^H A B$ is positive definite. $\rho(B)$ denotes the spectral radius of B ($\rho(B) = \max_{1,\ldots,N} |\lambda_i(B)|$).*

Theorem 2 *Let R be a positive definite matrix of dimension $N \times N$ with $\rho(R) = \lambda_{max}(R) < 2$ then $\rho(\prod_{i=1}^{P}(I - I_i R)) < 1$ where I_i, $i = 1, \ldots, P$ are obtained by zeroing out some rows in the identity matrix I such that $\sum_{i=1}^{P} I_i$ is positive definite. Thus if X_k and d_k are jointly W.S.S. then S-LMS converges in the mean if LMS converges in the mean.*

Proof: Let $\mathbf{x}_0 \in \mathbb{C}^N$ be an arbitrary non-zero vector of length N. Let $\mathbf{x}_i = (I - I_i R)\mathbf{x}_{i-1}$. Also, let

$\mathbf{P} = \prod_{i=1}^{P}(I - I_i R)$.

First we will show that $\mathbf{x}_i^H R\mathbf{x}_i \leq \mathbf{x}_{i-1}^H R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1}$, where $\alpha = \frac{1}{2}(2 - \lambda_{max}(R)) > 0$.

$$\begin{aligned}
\mathbf{x}_i^H R\mathbf{x}_i &= \mathbf{x}_{i-1}^H (I - RI_i)R(I - I_iR)\mathbf{x}_{i-1} \\
&= \mathbf{x}_{i-1}^H R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} - \\
&\quad \beta\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} + \mathbf{x}_{i-1}^H RI_iRI_iR\mathbf{x}_{i-1}
\end{aligned}$$

where $\beta = 2 - \alpha$. If we can show $\beta RI_iR - RI_iRI_iR$ is positive semi-definite then we are done. Now

$$\beta RI_iR - RI_iRI_iR = \beta RI_i(I - \frac{1}{\beta}R)I_iR.$$

Since $\beta = (1 + \lambda_{max}(R)/2) > \lambda_{max}(R)$ it is easy to see that $I - \frac{1}{\beta}R$ is positive definite. Therefore, $\beta RI_1R - RI_1RI_1R$ is positive semi-definite and

$$\mathbf{x}_i^H R\mathbf{x}_i \leq \mathbf{x}_{i-1}^H R\mathbf{x}_{i-1} - \alpha\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1}.$$

Combining the above inequality for $i = 1, \ldots, P$, we note that $\mathbf{x}_P^H R\mathbf{x}_P < \mathbf{x}_0^H R\mathbf{x}_0$ if $\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} > 0$ for at least one i, $i = 1, \ldots, P$. We will show by contradiction that is indeed the case.

Suppose not, then $\mathbf{x}_{i-1}^H RI_iR\mathbf{x}_{i-1} = 0$ for all i, $i = 1, \ldots, P$. Since, $\mathbf{x}_0^H RI_1R\mathbf{x}_0 = 0$ this implies $I_1R\mathbf{x}_0 = \mathbf{0}$. Therefore, $\mathbf{x}_1 = (I - I_1R)\mathbf{x}_0 = \mathbf{x}_0$. Similarly, $\mathbf{x}_i = \mathbf{x}_0$ for all i, $i = 1, \ldots, P$. This in turn implies that $\mathbf{x}_0^H RI_iR\mathbf{x}_0 = 0$ for all i, $i = 1, \ldots, P$ which is a contradiction since $R(\sum_{i=1}^{P} I_i)R$ is a positive-definite matrix and $0 = \sum_{i=1}^{P} \mathbf{x}_0^H RI_iR\mathbf{x}_0 = \mathbf{x}_0^H R(\sum_{i=1}^{P} I_i)R\mathbf{x}_0 \neq 0$.

Finally, we conclude that

$$\begin{aligned}
\mathbf{x}_0^H \mathbf{P}^H R\mathbf{P}\mathbf{x}_0 &= \mathbf{x}_P^H R\mathbf{x}_P \\
&< \mathbf{x}_0^H R\mathbf{x}_0.
\end{aligned}$$

Since $\mathbf{x}_0$ is arbitrary we have $R - \mathbf{P}^H R\mathbf{P}$ to be positive definite so that applying Theorem 1 we conclude that $\rho(\mathbf{P}) < 1$.

Finally, if LMS converges in the mean we have $\rho(I - \mu R) < 1$ or $\lambda_{max}(\mu R) < 2$. Which from the above proof is sufficient for concluding that $\rho(\prod_{i=1}^{P}(I - \mu I_iR)) < 1$. Therefore, S-LMS also converges in the mean.

□

4 Analysis: Cyclo-Stationary Signals

Next, we consider the case when X_k and d_k are jointly cyclo-stationary with covariance matrix R_k. We limit our attention to even-odd S-LMS as shown in Figure 1. Let X_k be a cyclo-stationary signal with period L. i.e, $R_{i+L} = R_i$. For simplicity, we will assume L is even. For the regular LMS algorithm we have the following L update equations

$$E\left[V_{k+L}\right] = \prod_{i=0}^{L-1}(I - \mu R_{i+d})E\left[V_k\right]$$

for $d = 1,\ 2,\ \ldots, L$, in which case we would obtain the following sufficient condition for convergence

$$0 < \mu < \min_i \{2/\lambda_{max}(R_i)\}$$

where $\lambda_{max}(R_i)$ is the largest eigenvalue of the matrix R_i.

Define $A_k = (I - \mu I_1 R_k)$ and $B_k = (I - \mu I_2 R_k)$ then for the partial update algorithm the $2L$ valid update equations are

$$E\left[V_{k+L}\right] = \left(\prod_{i=0}^{\frac{L-1}{2}} B_{2*i+1+d} A_{2*i+d}\right) E\left[V_k\right] \tag{6}$$

for $d = 1,\ 2, \ldots, L$ and odd k and

$$E\left[V_{k+L}\right] = \left(\prod_{i=0}^{\frac{L-1}{2}} A_{2*i+1+d} B_{2*i+d}\right) E\left[V_k\right] \tag{7}$$

for $d = 1,\ 2, \ldots, L$ and even k.

Let $\|A\|$ denote the spectral norm $\lambda_{max}(AA^H)^{1/2}$ of the matrix A. Since $\rho(A) \leq \|A\|$ and $\|\prod A_i\| \leq \prod \|A_i\|$, for ensuring the convergence of the iteration (6) and (7) a sufficient condition is

$$\|B_{i+1}A_i\| < 1 \quad \text{and} \quad \|A_{i+1}B_i\| < 1 \quad \text{for } i = 1,\ 2,\ \ldots, L.$$

Since we can write $B_{i+1}A_i$ as

$$B_{i+1}A_i = (I - \mu R_i) + \mu I_2 (R_i - R_{i+1}) + \mu^2 I_2 R_{i+1} I_1 R_i$$

and $A_{i+1}B_i$ as

$$A_{i+1}B_i = (I - \mu R_i) + \mu I_1 (R_i - R_{i+1}) + \mu^2 I_1 R_{i+1} I_2 R_i$$

we have the the following expression which upper bounds both $\|B_{i+1}A_i\|$ and $\|A_{i+1}B_i\|$

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\|.$$

This tells us that the sufficient condition to ensure convergence of both (6) and (7) is

$$\|I - \mu R_i\| + \mu\|R_{i+1} - R_i\| + \mu^2\|R_{i+1}\|\|R_i\| < 1 \tag{8}$$

for $i = 1, \ldots, L$.

If we make the assumption that

$$\mu < \min_i\{\frac{2}{\lambda_{max}(R_i) + \lambda_{min}(R_i)}\} \tag{9}$$

and

$$\delta_i = \|R_{i+1} - R_i\| < \max\{\lambda_{min}(R_i), \lambda_{min}(R_{i+1})\} = \eta_i \tag{10}$$

for $i = 1, 2, \ldots, L$ then (8) translates to

$$1 - \mu\eta_i + \mu\delta_i + \mu^2\lambda_{max}(R_i)\lambda_{max}(R_{i+1}) < 1$$

which gives

$$0 < \mu < \min_{i=1}^{L}\{\frac{\eta_i - \delta_i}{\lambda_{max}(R_i)\lambda_{max}(R_{i+1})}\}. \tag{11}$$

Equation (11) is the sufficient condition for convergence of even-odd S-LMS with cyclostationary signals.

Therefore, we have the following theorem.

Theorem 3 *Let X_k and d_k be jointly cyclostationary. Let R_i, $i = 1, \ldots, L$ denote the L covariance matrices corresponding to the period L of cyclo-stationarity. If we assume X_k is slowly varying in the sense given by (10) and μ is small enough given by (9) then the sufficient condition on μ for the convergence of iterations (6) and (7) is given by (11)*

5 Example

The usefulness of the bound on step-size for the cyclo-stationary case can be gauged from the following example. Consider a 2-tap filter with $X_k = [x_k \ x_{k-1}]^T$ such that $\{x_k\}$ is a cyclo-stationary signal with period 2 having the following auto-correlation matrices

$$\begin{aligned} R_1 &= \begin{bmatrix} 5.1354 & -0.5733 - 0.6381i \\ -0.5733 + 0.6381i & 3.8022 \end{bmatrix} \\ R_2 &= \begin{bmatrix} 3.8022 & 1.3533 + 0.3280i \\ 1.3533 - 0.3280i & 5.1354 \end{bmatrix} \end{aligned}$$

For this choice of R_1 and R_2, η_1 and η_2 turn out to be 3.38 and we have $\|R_1 - R_2\| = 2.5343 < 3.38$. Therefore, R_1 and R_2 satisfy the assumption made for analysis. Now, $\mu = 0.33$ satisfies the condition for the regular LMS algorithm but, the eigenvalues of B_2A_1 for this value of μ have magnitudes 1.0481 and 0.4605. Since one of the eigenvalues lies outside the unit circle (6) is unstable for this choice of μ. On the other hand (11) gives $\mu = 0.0254$. For this choice of μ the eigenvalues of B_2A_1 turn out to have magnitudes 0.8620 and 0.8773. Hence (6) is stable.

We have plotted the evolution trajectory of the 2-tap filter with input signal satisfying the above properties. We chose $W_{opt} = [0.4\ 0.5]$ in Figures 2 and 3. For Figure 2 μ was chosen according to be 0.33 and for Figure 3 μ was chosen to be 0.0254. For simulation purposes we set $d_k = W_{opt}^H S_k + n_k$ where $S_k = [s_k\ s_{k-1}]^T$ is a vector composed of the cyclo-stationary process $\{s_k\}$ with correlation matrices given as above, and $\{n_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$. We set $\{x_k\} = \{s_k\} + \{v_k\}$ where $\{v_k\}$ is a white sequence, with variance equal to 0.01, independent of $\{s_k\}$.

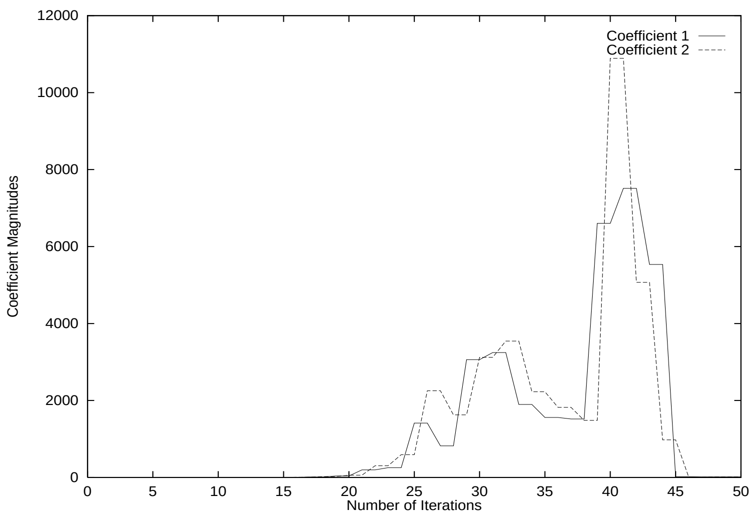

Figure 2: Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.33$ for a 2-tap adaptive filter in a cyclo-stationary signal environment described in Section 5

This section shows a cyclo-stationary signal example for which there exists a region of μ in which S-LMS converges. In fact, there exist signals for which S-LMS does not converge irrespective of the value of μ. Such examples have been described in [8].

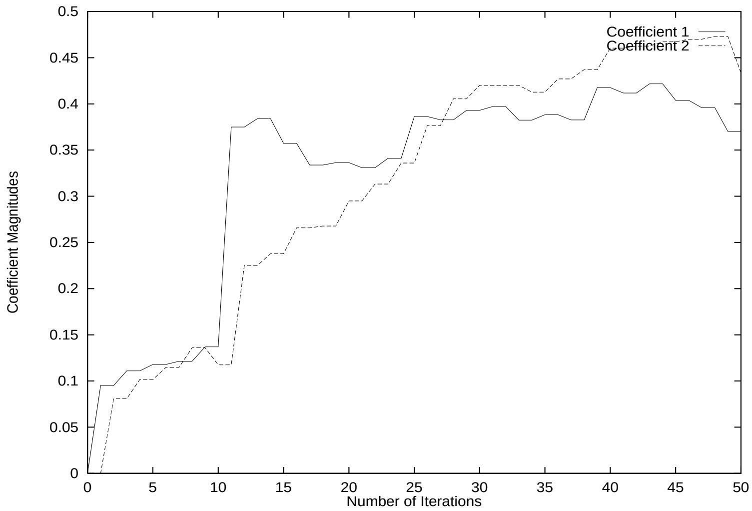

Figure 3: Trajectory of $w_{1,k}$ and $w_{2,k}$ for $\mu = 0.0254$ for the same adaptive system as shown in Figure 2

6 Conclusion

We have analyzed the alternating odd/even partial update LMS algorithm and we have derived stability bounds on step-size parameter μ for wide sense stationary and cyclo-stationary signals based on extremal properties of the matrix 2-norm. For the case of wide sense stationary signals we have shown that if the regular LMS algorithm converges in mean then so does the sequential LMS algorithm for the general case of arbitrary but fixed ordering of the sequence of partial coefficient updates. For cyclo-stationary signals the bounds derived may not be the weakest possible bounds but they do provide the user with a useful sufficient condition on μ which ensures convergence in the mean. We believe the analysis undertaken in this paper is the first step towards deriving concrete bounds on step-size without making small μ assumptions. The analysis also leads directly to an estimate of mean convergence rate.

In the future, it would be useful to analyze partial update algorithm, without the assumption of independent snapshots and also, if possible, perform a second order analysis (mean square convergence). Furthermore, as S-LMS exhibits poor convergence in non-stationary signal scenarios [8] it is of interest to develop new partial update algorithms with better convergence properties. One such algorithm based on randomized partial updating of filter coefficients is described in [8].

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