Cut-off Rate and Signal Design for the Rayleigh Fading Space-Time Channel

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Outline

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2. Random coding exponent and cutoff rate
3. Integral representation for cutoff rate
4. Case of discrete K-dimensional constellations
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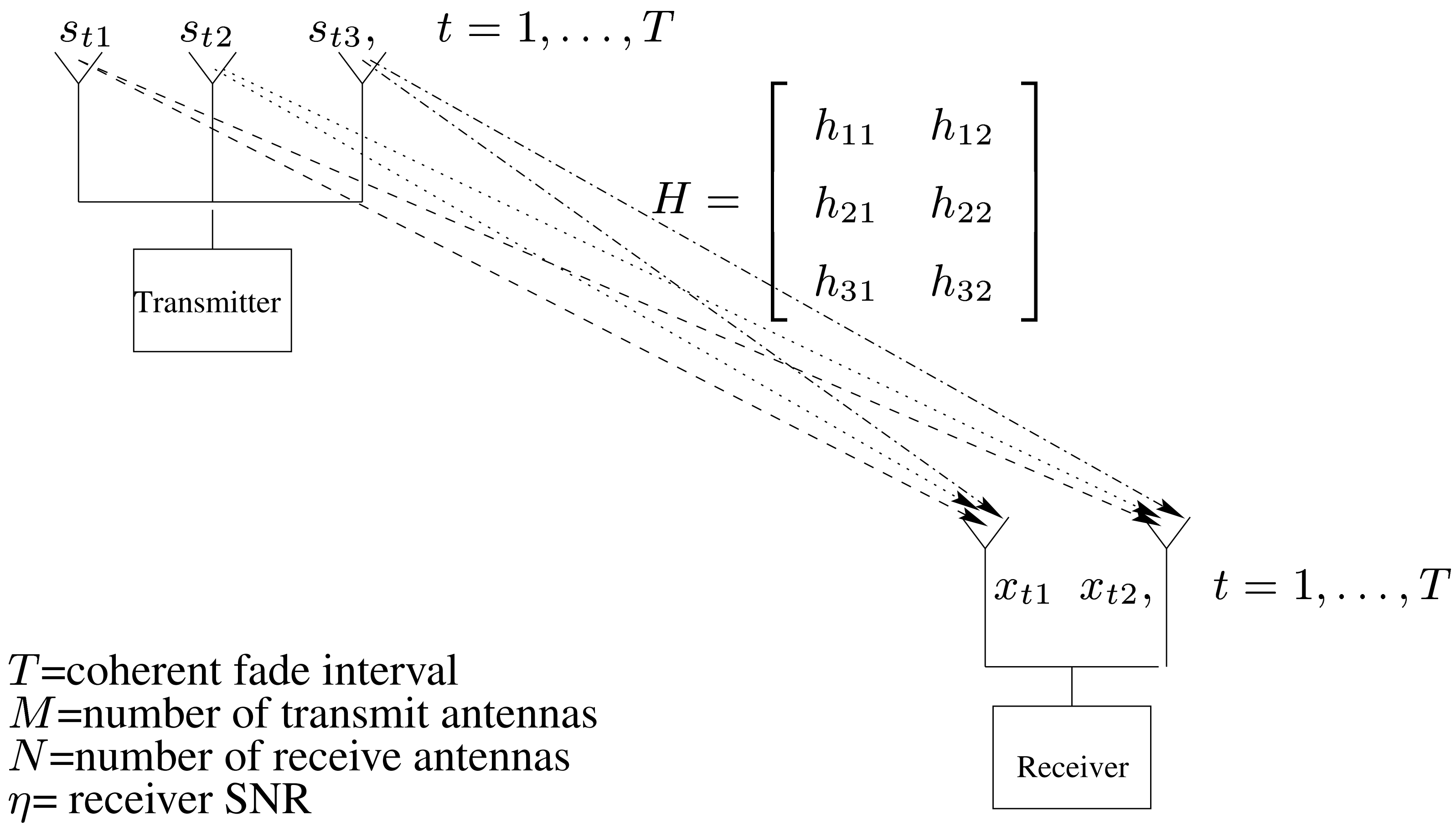

Figure 1. Flat Rayleigh fading model for $M = 3$, $N = 2$

Received signal in l-th block of T secs. ($t = 1, \ldots, T$)

$$[x_{t1}^l, \ldots, x_{tn}^l] = \sqrt{\eta}[s_{t1}^l, \ldots, s_{tm}^l] \begin{bmatrix} h_{11}^l & \cdots & h_{1n}^l \\ \vdots & \vdots & \vdots \\ h_{m1}^l & \cdots & h_{mn}^l \end{bmatrix} + [w_{t1}^l, \ldots, w_{tn}^l],$$

or, equivalently

$$X^l = \sqrt{\eta} S^l H^l + W^l$$

- X^l: $T \times N$ received signal matrices
- S^l: $T \times M$ transmitted signal matrices
- H^l: i.i.d. $M \times N$ propagation matrices $\sim \mathcal{CN}(0, I_M \bigotimes I_N)$
- W^l: i.i.d. $T \times N$ noise matrices $\sim \mathcal{CN}(0, I_T \bigotimes I_N)$
- Transmitted signals satisfy *peak power constraint*

$$\|S\|^2 = \mathrm{tr}\{S^H S\} \leq TM$$

- Objective: Find optimal signal set $\{S_i\}$ from which to choose S^l's.
- Optimality criterion: rate optimal signal set $\{S_i\}$

1 Random Coding Error Exponent

The minimum error probability of any decoder of a block code over L channel utilizations satisfies

$$\min P_e \leq e^{-LE_U(R)}, \quad R < C$$

where

- $E_U(R)$: error exponent
- R: symbol rate (nats/symbol)
- C: channel capacity (nats/symbol)

$\Rightarrow E_U(R)$ studied by Abou-Faycal & Hochwald [2]– (avg power)

$\Rightarrow C$ studied by Hochwald & Marzetta (99') [3] – (avg power)

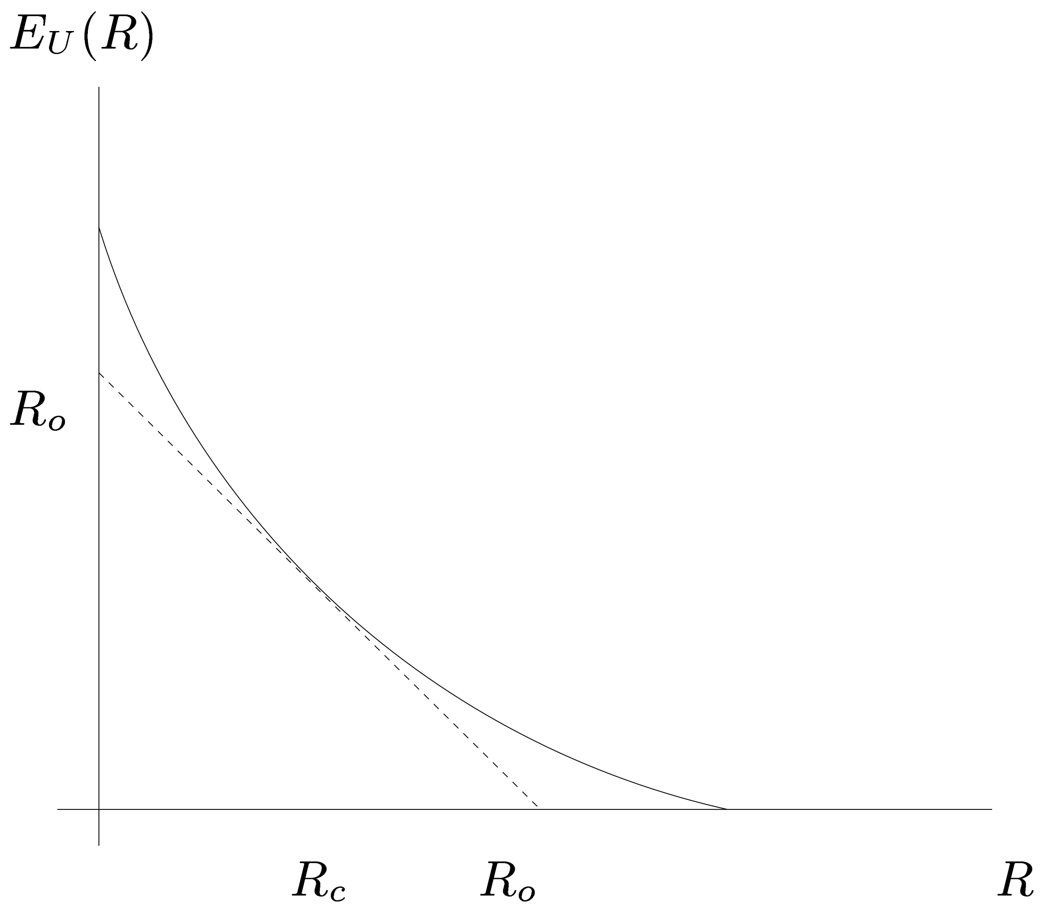

Figure 2. Error exponent $E_U(R)$ and cut-off rate bound $y(R) = R_o - R$.

R_o ***computational cut-off rate*** **lower bound**:

$$E_U(R) \geq R_o - R, \quad R \leq R_o$$

$$R_o = \max_{P \in \mathcal{P}} -\ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} \sqrt{p(X|S)} dP(S) \right]^2 dX, \quad \text{nats/symbol}$$

where $\mathcal{P}$ are suitably constrained distributions over $\mathbb{C}^{T \times M}$

FACTS:

- $R_o \leq C$
- $E_U(R) \approx R_o - R$ when $R \approx R_c$, the critical rate
- R_o is highest practical rate for sequential decoders
- R_o specifies upper bound on optimal decoder error

$$P_e \leq e^{-L(R_o - R)}, \quad R \leq R_o$$

2 Integral Representation for R_o

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{S_1 \in \mathcal{S}} dP(S_1) \int_{S_2 \in \mathcal{S}} dP(S_2)\ e^{-ND(S_1 \| S_2)}.$$

where

$$D(S_1 \| S_2) \quad \stackrel{\text{def}}{=} \frac{1}{2} \ln \frac{\left| I_T + \frac{\eta}{2}(S_1 S_1^H + S_2 S_2^H) \right|^2}{\left| I_T + \eta S_1 S_1^H \right| \left| I_T + \eta S_2 S_2^H \right|}.$$

Low SNR approximation:

$D(S_1 \| S_2) = \eta^2 / 8 \| S_1 S_1^H - S_2 S_2^H \|^2 + o(\eta^2)$

The following parallels Theorems 1 and 2 of [3]

Proposition 1 *Assume that the transmitted signal S is constrained to satisfy the peak power constraint $\|S\|^2 \leq MT$. There is no advantage to using $M > T$ transmit antennas. Furthermore, for $M \leq T$ the signal matrices achieving R_o can be expressed as*

$$S = \begin{bmatrix} \\ V \\ \\ \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix}$$

Where

- $V^H V = I_M$
- $\Lambda = M \times M$ *non-negative diagonal matrix.*

3 Case of Discrete K-dimensional Constellations

Specialize $\mathcal{P}$ to the discrete distributions over $\mathbb{C}^{T\times M}$

Then $R_o = \tilde{R}_o(K)$ is given by

$$\max_{\{P_i,S_i\}_{i=1}^K} -\ln \sum_{i=1}^K P_i \sum_{j=1}^K P_j \; e^{-ND(S_i\|S_j)} = -\ln \min_{\{P_i,S_i\}_{i=1}^K} \underline{P}^T E_K \underline{P}$$

where

- $E_k = ((D(S_i|S_j))_{i,j=1}^K$: dissimilarity (distance) matrix
- $\underline{P} = [P_1, \ldots, P_K]^T$

Under peak power constraint, $\|S_i\| \leq TM$,

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K} \left(\min_{\{P_i\}_{i=1}^K} \underline{P}^T E_K \underline{P} \right)$$

Inner maximization:

$$\min_{\underline{P}>0:\underline{1}_K^T \underline{P}=1} \{\underline{P}^T E_K \underline{P}\}$$

Lagrangian

$$J(\underline{P}) = \underline{P}^T E_K \underline{P} - 2c(\underline{1}_K^T \underline{P} - 1)$$

minimized for *equalizer probability* $\underline{P} = \underline{P}^*$

$$E_K \underline{P}^* = c\underline{1}_K \quad \Rightarrow \quad \sum_{j=1}^{K} P_j \; e^{-ND(S_i \| S_j)} = c$$

if $E_K^{-1}\underline{1}_K \geq 0$ then

$$\underline{P}^* = cE_K^{-1}\underline{1}_K, \quad \text{and} \quad c = \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K}$$

and

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K} \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K} = \max_{\{S_i\}_{i=1}^K} \ln \underline{1}_K^T E_K^{-1} \underline{1}_K$$

4 Bound on minimum distance

To $o(\eta^2)$ we have bounds

$$D^{**}_{\min} = \max_{\{S_i\}_{i=1}^{K}} \min_{i \neq j} D(S_i \| S_j) \geq \frac{\eta^2}{8} \frac{(TM)^2}{(2^p-1)^2} > \frac{\eta^2 (TM)^2}{128} K^{-2/T}$$

(a) (b)

Figure 3. *Constellations of signal matrix singular values*

Implication: for rate optimal constellation $\{S_i^*\}$

$$\overline{D(S_i^* \| S_j^*)} \geq D_{\min}^{**} > \frac{\eta^2 (TM)^2}{128} K^{-2/T} + o(\eta^2)$$

5 Low Dimensional Constellations

For given η, T and M define the integer M_o

$$M_o = \text{argmax}_{m \in \{1,\ldots,M\}} \left\{ m \ln \frac{(1+\eta TM/(2m))^2}{1+\eta TM/m} \right\}.$$

Proposition 2 *Let $2M \leq T$. Then*

$$D_{\max} \stackrel{\text{def}}{=} \max_{S_1, S_2 \in \mathcal{S}^K_{\text{peak}}} D(S_1 \| S_2) = M_o \ln \frac{(1+\eta TM/(2M_o))^2}{1+\eta TM/M_o}. \tag{1}$$

Furthermore, the optimal signal matrices which attain $D_{\max}$ can be taken as scaled rank M_o mutually orthogonal unitary $T \times M$ matrices of the form

$$S_1 = \eta \text{TM}\ \Phi_1, \qquad S_2 = \eta \text{TM}\ \Phi_2$$

where, for $j = 1, 2$,

$$\Phi_j^H \Phi_j = I_{M_o}, \quad \text{and} \quad \Phi_i^H \Phi_j = 0, \;\; i \neq j.$$

Proof is based on alternative representation for $D(S_1\|S_2)$

$$D(S_1\|S_2) = \quad \tfrac{1}{2}\ln \frac{\left|I_M+\frac{\eta}{2}S_1^H S_1\right|^2\left|I_M+\frac{\eta}{2}S_2^H S_2\right|^2}{\left|I_M+\eta S_1^H S_1\right|\left|I_M+\eta S_2^H S_2\right|}\left|I_M-\kappa^H\kappa\right|^2,$$

where κ is a $M \times M$ multiple signal correlation matrix

$$\kappa = \tilde{S}_2^H \tilde{S}_1$$

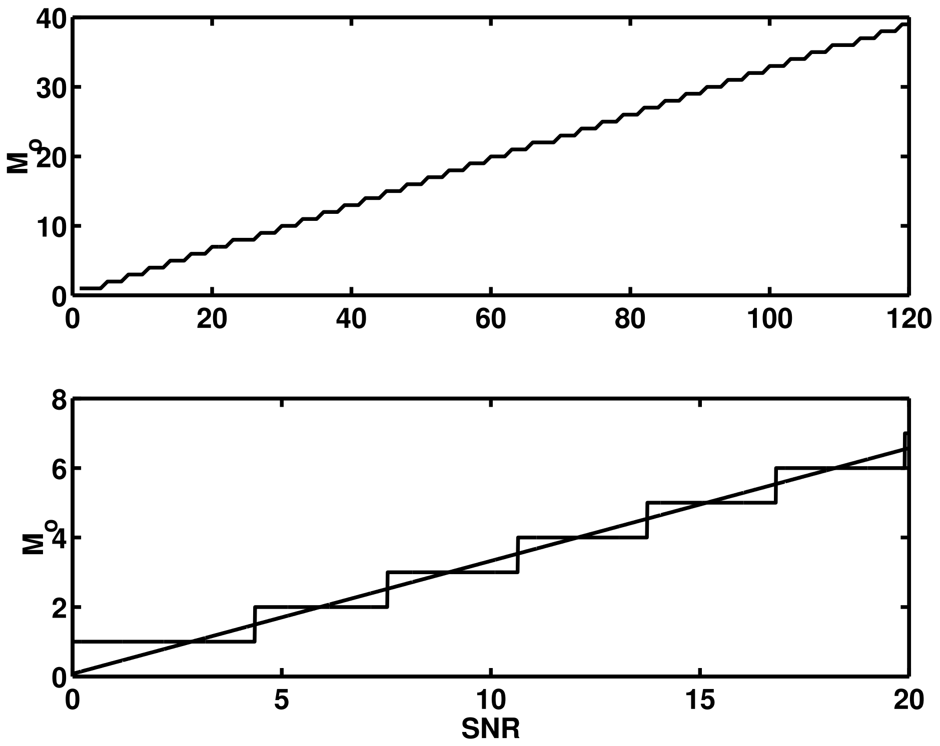

Figure 4. Top panel: M_o as a function of the SNR parameter $\eta T M$. Bottom panel: blow up of first panel over a reduced range of SNR.

Proposition 3 *Let* $2M \leq T$ *and let* M_o *be as defined in (1). Suppose that* $M_o \leq \min\{M, T/K\}$. *Then the peak constrained* K *dimensional cut-off rate is*

$$\tilde{R}_o(K) = \ln\left(\frac{K}{1 + (K-1)e^{-ND_{\max}}}\right)$$

and $D_{\max}$ *is given by (1). Furthermore, the optimal constellation attaining* $\tilde{R}_o(K)$ *is the set of* K *rank* M_o *mutually orthogonal unitary matrices and the optimal probability assignment is uniform:* $P_i^* = 1/K$, $i = 1, \ldots, K$.

Example constellations for $T \times M = 4 \times 2$

- $M_o = 1$, $K = 4$:

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- $M_o = 2$, $K = 2$:

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

6 Conclusions

- Peak power contrained cut-off rate invlolves minimizing Q-form
- optimal constellation equalizes the decoder error rates
- Average distance for optimal K-dim constellation decreases at most by $K^{-2/T}$
- Optimal low rate constellation is a set of scaled mutually orthogonal unitary matrices.
- Rank of the unitary signal matrices decreases in SNR
- For very low SNR, transmitter should apply all power to a single antenna element at a time.

References

[1] A. O. Hero and T. L. Marzetta, "On computational cut-off rate for space time coding," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 2000.

[2] I. Abou-Faycal and B. M. Hochwald, "Coding requirements for multiple-antenna channels with unknown Rayleigh fading," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 1999.

[3] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh fading," *IEEE Trans. on Inform. Theory*, vol. IT-45, pp. 139–158, Jan. 1999.