THE EM ALGORITHM AND BEYOND

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Outline

1. **Some difficult ML and PML inverse problems**
   EM algorithm: the basics
   Focus applications
   Convergence properties
   Acceleration of EM: SAGE

Concluding remarks

1 Some difficult ML and PML inverse problems

1. **Gaussian inverse problem:**

– **Estimating amplitudes of superimposed signals**

2. **Poisson inverse problem:**

– **PET image reconstruction**

3. Filtered Poisson inverse problem: estimating shot noise parameters

– estimating shot noise parameters

1.1 Gaussian inverse problem

Measure a linear comb. of n samples of p signals

$1, \cdots, p$

$$\underline{Y} = \underline{a}_1 \theta_1 + \ldots + \underline{a}_p \theta_p +_y$$

$$= \quad \mathbf{A}\underline{\theta} + {}_y$$

* $\mathbf{A} = [{}_1, \ldots, {}_p]$: **known signal vectors**
* ${}_y \sim \mathcal{N}_n(0,\ \mathbf{R}_y)$: **Gaussian noise**
* $\underline{\theta}$: **vector of unknown amplitudes**

This is linearized version of general superimposed signal model

$$\underline{Y} \quad = \quad \underline{a}_1(\theta_1) + \ldots + \underline{a}_p(\theta_p) + {}_y$$

Penalized Log-Likelihood function:

$$
\begin{aligned}
l(\underline{\theta}) &= \ln f(\underline{Y};\underline{\theta}) - \underline{\theta}^T \mathbf{P} \underline{\theta} = \\
&= \underbrace{-\tfrac{1}{2}[\underline{Y} - \mathbf{A}\underline{\theta}]^T \mathbf{R}_y^{-1} [\underline{Y} - \mathbf{A}\underline{\theta}] - \tfrac{1}{2}\ln(2\pi|\mathbf{R}_y|)}_{=\ln f} - \eta \underline{\theta}^T \mathbf{P} \underline{\theta}
\end{aligned}
$$

where

*** $\mathbf{P} > 0$ is a penalty (smoothing) matrix**

*** $\eta > 0$ is a regularization (smoothing) parameter**

Penalized Maximum Likelihood (PML) estimator

$$
\begin{aligned}
&= \operatorname{argmax}_{\underline{\theta}}\, l(\underline{\theta}) \\
&= [\mathbf{A}^T \mathbf{R}_y^{-1} \mathbf{A} + \eta \mathbf{P}]^{-1} \mathbf{A}^T \mathbf{R}_y^{-1} \underline{Y}
\end{aligned}
$$

Online computational complexity: $O(np)$

1.2 Poisson inverse problem

Measure a set $\underline{Y}$ **of** n **projections of** p **pixels** $\underline{\theta}$ **onto** n **detectors**

$$\underline{Y} = \mathbf{A}\underline{\theta} + \underline{n}_y$$

*** A: known matrix of (projection) transition probabilities**

*** $\underline{n}_y$: Poisson noise with covariance**

$$\mathrm{cov}_{\underline{\theta}}\left(\mathrm{diag}[\mathbf{A}\underline{\theta} + \underline{r}]\right)$$

*** $\underline{\theta}$: unknown vector (image)**

Penalized Log-Likelihood function:

$$l(\underline{\theta}) = \sum_{d=1}^{n} \left[-\mu_d(\underline{\theta}) + y_k \log \mu_k(\underline{\theta})\right] - \eta \underline{\theta}^T \mathbf{P} \underline{\theta}$$

where $\mu_d(\underline{\theta}) = E_{\underline{\theta}}[y_d]$

$$\mu_d(\underline{\theta}) = \sum_{i=1}^{p} a_{di}\theta_i + r_d$$

ML and PML are not of analytic form

1.3 Filtered Poisson process

Measure a waveform $y(t)$ consisting of shot noise and Gaussian noise

$$y(t) = \sum_{i=1}^{N_t} a_i g(t - t_i) + w_y(t), \quad t \in [0, T]$$

* $g(t)$**: known function of time**
* $a_i \sim \mathcal{N}(\mu_a, \sigma_a^2)$**: i.i.d. Gaussian amplitudes**
* $\{N_t : t \in [0, T]\}$**: inhomogeneous Poisson process**

with intensity $\lambda(t;\underline{\theta})$

*** $t_1, \ldots, t_n$ arrival times of points**

*** $\underline{\theta}$: unknown parameters to be estimated**

Likelihood function is not closed form:

$$
\begin{aligned}
L(\underline{\theta}) &= \sum_{n=0}^{\infty} e^{-\Lambda - \frac{n\mu_a^2}{2\sigma_a^2}} \\
&= \int_{t_1,\ldots,t_n} \phi(t_1, \ldots, t_n)\, dt_1 \ldots\, dt_n \quad .
\end{aligned}
$$

where

$$\phi(t_1,\ldots,t_n) = \lambda(t_i,\underline{\theta}) \exp\left[\frac{\mu_a^2}{2\sigma_a^2}[\alpha\underline{X}_f + \underline{1}]^T[I + R_p\alpha]^{-1}[\alpha\underline{X}_f + \underline{1}]\right]$$

and $\mathbf{R}_p = ((< g(t-t_i), g(t-t_j) >))_{i,j=1}^n$

2 EM algorithm: the basics

Standard iterative algorithms one might consider:

*** Exhaustive search: impractical in many cases**