PENALIZED LIKELIHOOD ECT RECONSTRUCTION WITH UNCERTAIN MRI SIDE INFORMATION VIA ASYMPTOTIC MARGINALIZATION (DRAFT)

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ABSTRACT

In [1] a methodology for incorporating extracted MRI anatomical boundary information into penalized likelihood (PL) ECT image reconstructions and tracer uptake estimation was proposed. This methodology used quadratic penalty based on Gibbs weights which enforced smoothness constraints everywhere in the image except across the MRI-extracted boundary of the ROI. When high quality estimates of the anatomical boundary are available and MRI and ECT images are perfectly registered, the performance of this method was shown to be very close to that attainable using ideal side information, i.e. noiseless anatomical boundary estimates. However when the variance of the MRI-extracted boundary estimates becomes significant this penalty function method performs poorly. We give a modified Gibbs penalty function implemented with a set of averaged Gibbs weights, where the averaging is performed with respect to a limiting form of the posterior distribution of the MRI boundary parameters.

1. INTRODUCTION

Radio-tracer uptake estimation is an essential tool in medicine and biological sciences for evaluating metabolic function of living systems. Emission computed tomography (ECT) is very useful in this regard due to its ability to image in three dimensions. Critical to uptake estimation accuracy is a reliable estimate of the anatomical region of interest (ROI), e.g. a target organ within the body. While it is possible to estimate the ROI directly from the acquired radio-isotope image, better estimates can be obtained from higher resolution imaging systems, such as X-ray computed tomgraphy (CT) or magnetic resonance images (MRI), which are specifically adapted to imaging anatomy. Many researchers have suggested ways to incorporate such MRI or CT side information into ECT image reconstructions and tracer uptake estimates [2, 3, 4, 5, 6, 7]. Here we present a method for using noisy MRI side information in ECT which is based on asymptotic marginalization of the penalty likelihood method proposed in [8, 1].

The method of [9, 8, 1] performs ECT image reconstrion using a penalized (Poisson) likelihood function approach. A quadratic penalty based on Gibbs weights is introduced which enforces smoothness constraints everywhere in the image except across the MRI-extracted boundary of the ROI. When high quality estimates of the anatomical boundary are available and MRI and ECT images are perfectly registered, the performance of this method was shown [9] to be very close to that attainable using ideal side information, i.e. noiseless anatomical boundary estimates. However when the variance of the MRI-extracted boundary estimates becomes significant this penalty function method performs poorly. This poor performance is due in large part to the fact that the Gibbs weights do not account for estimation errors in the anatomical boundary estimates. In [10] a technique called "variance corrected weighting" was proposed which relied on a local Taylor series expansion of the Gibbs weights about the estimated spline parameters. This scheme applies only to weight assignments that are smooth functions of the spline parameters for which Taylor series can be applied. Furthermore, even for the smooth weight assignments presented in [10] the method is accurate only when the uncertainty in the extracted boundary estimates are less than a single MRI pixel width.

Here we derive a variance corrected weighting which applies globally to non-smooth weights. The method is based on marginalization [11, Sec 3.2.2] which averages the penalized likelihood over an asymptotic normal approximation to the posterior distribution of the side information. Jensen's inequality provides a lower bound on the resultant marginal which gives a simple penalty function requiring averaging of the weight maps. This averaging is performed with respect to the asymptotic distribution of the boundary estimates which is a multivariate Gaussian density centered at the boundary estimate $\{\hat{r}_\phi\}_{\phi\in[0,2\pi]}$ and with covariance matrix equal to the inverse (observed) Fisher information.

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2. BACKGROUND

Let Y_E and Y_M be two random measurements corresponding to (Poisson) ECT projections and (Gaussian) NMR spin density images, respectively. In [10] two different methodologies were proposed for extracting anatomical boundary estimates from noisy MRI images for the purpose of incorporation into ECT. One was based on non-linear maximum likelihood estimation of coefficients θ in a periodic B-spline model for the closed boundary. The other was based on direct estimation of the boundary defined by the radial distance function $r(\phi), \quad \phi \in [0, 2\pi]$ which specifies the (polar) coordinates (r, ϕ) of the boundary relative to an origin defined within the ROI. This latter method performed a polar to rectangular coordinate conversion, Canny or Marr-Hildreth edge detection, followed by smoothing of the extracted edge by a median filter. It was observed through simulations that both boundary extraction methdologies yielded boundary estimates which were approximately unbiased over a wide range of SNR. The Fisher information matrix was derived and it was observed that the methods also came close to the CR lower bound on attainable estimator variance for low to moderate additive noise levels (σ_n less than 15% of edge contrast).

The boundary estimates, which are denoted by common notation $\hat{\theta}$ for spline coefficient parameterization or for radial distance parameterization, are incorporated into ECT image reconstruction via maximizing the penalized likelihood (PL) objective over the ECT intensity distribution λ

$$J(\lambda) = \ln f(Y_E|\lambda, \hat{\theta}) - \beta R(\lambda; \hat{\theta}) \tag{1}$$

where $f(Y_E|\lambda)$ is the ECT likelihood function in which the penalty function $R(\lambda; \hat{\theta})$ depends on the MRI measurements and is specified to enforce smoothness within the estimated boundary of the ROI. The penalty is of the form of a Gibbs potential function

$$R(\lambda; \hat{\theta}) = -\beta \sum_{i,j} w_{ij}(\hat{\theta})(\lambda_i - \lambda_j)^2$$

where w_{ij} are Gibbs weights which depend on $\hat{\theta}$ through one of many possible functional assignments [10, Ch. 4].

3. A BAYESIAN SETTING

Assume that λ and θ are random parameters (ECT intensity and MRI boundary estimates). Under the assumption that Y_M and Y_E are conditionally independent given λ, and that λ is a function of θ, the joint density density function factors in the following manner:

$$f(\lambda, Y_E, \theta, Y_M) = f(Y_E|\lambda) f(\lambda|\theta) f(\theta|Y_M) f(Y_M). \tag{2}$$

The first three factors on the right hand side of the above equation can be recognized as the likelihood function of λ without side information, the conditional density linking θ to λ, and the posterior density of θ given Y_M.

When θ is known exactly, the posterior mode (MAP) estimate $\hat{\lambda}$ of λ maximizes the posterior loglikelihood $\ln f(\lambda|Y_E, \theta)$ which is equivalent to maximizing the objective

$$J_\theta(\lambda) = \ln f(Y_E|\lambda, \hat{\theta}) + \ln f(\lambda|\theta).$$

Comparing this to (1) we can identify $f(\lambda|\theta)$ as a Gaussian "Gibbs-type" distribution of λ:

$$f(\lambda|\theta) = \alpha \exp\{-\beta R(\lambda; \theta)\} \tag{3}$$

where α is a normalizing constant.

When θ is not known exactly there are several ways to approach the estimation of λ. Many researchers [2, 3, 4, 5] have investigated joint estimation of λ and θ under various models for the prior $f(\theta)$ and $f(\lambda|\theta)$. While this is an important task for applications where both λ and θ are of intrinsic interest, this approach does not generally yield Bayes optimal estimates of λ which are given by the posterior mode estimator of λ obtained by marginalizing $f(\lambda, \theta|Y_E, Y_M)$ by integrating out θ.

For random unknown θ the posterior mode estimate of λ maximizes the posterior $f(\lambda|Y_E, Y_M)$ which is equivalent to maximizing the following joint density function over λ:

$$\begin{aligned} f(\lambda, Y_E, Y_M) &= \int f(\lambda, Y_E, \theta, Y_M) d\theta \\ &= f(Y_E|\lambda) \cdot f(Y_M) \cdot \int f(\lambda|\theta) f(\theta|Y_M) d\theta \\ &= \ln f(Y_E|\lambda) + \ln \int f(\lambda|\theta) f(\theta|Y_M) d\theta \end{aligned}$$

Using the Gibbs conditional distribution (3) for $f(\lambda|\theta)$ this becomes

$$\hat{\lambda} = \text{argmax}_\lambda \left\{ \ln f(Y_E|\lambda) + \ln \int \exp\{-\beta R(\lambda; \theta)\} f(\theta|Y_M) d\theta \right\} \tag{4}$$

This is of the form of a penalized likelihood estimator (1) where the penalty is obtained by taking the log of the $f(\theta|Y_M)$-averaged exponentiated Gibbs penalty function. This penalty is convex in λ and for piecewise

constant $(0,1)$ weights w_{ij} can be expressed as:

$$\ln \int \exp\{-\beta R(\lambda;\theta)\}\, f(\theta|Y_M)d\theta = \ln \sum_l \exp\{-\beta R(\lambda;\theta_l)\} \int_{\Theta_l} f(\theta|Y_M)d\theta \quad (5)$$

where $\{\Theta_l\}_l$ is a partition of $\mathbb{R}^p$ into sets over which $R(\lambda;\theta)$ is constant and θ_l is a point within the set Θ_l.

4. SIMPLIFICATION BY JENSENS INEQUALITY

Because of the exponential form of the average in 4) or (5) the copmputation of the penalty function is numerically unstable for even moderate vaules of teh smoothing constant β. A simplification to (4) can be made by applying Jensen's inequality to lower bound this penalty term

$$\begin{aligned} & \ln \int \exp\{-\beta R(\lambda;\theta)\}\, f(\theta|Y_M)d\theta \\ \geq\ & -\beta \int R(\lambda;\theta) f(\theta|Y_M)d\theta \\ =\ & -\beta E_{\theta|Y_M}[R(\lambda;\theta)] \\ =\ & -\beta \sum_{ij} \tilde{w}_{ij}(\lambda_i - \lambda_j)^2 \end{aligned}$$

where $E_{\theta|Y_M}$ denotes expectation with respect to the posterior density $f(\theta|Y_M)$ and $\tilde{w}_{ij} = E_{\theta|Y_M}[w_{ij}(\theta)]$ are average weights. Therefore, use of the simpler penalty $-\beta E_{\theta|Y_M}[R(\lambda;\theta)]$ instead of $\ln \int \exp\{-\beta R(\lambda;\theta)\}\, f(\theta|Y_M)d\theta$ entails an overall reduction in the influence of the penalty factor for any given β. However, the underpenalization can be be compensated by increasing the value of β.

The computation of the penalty requires averaging over a posterior density $f(\theta|Y_M)$ which requires knowing the prior distribution $f(\theta)$. When multiple realizations of $f(Y_M, \theta)$ are available empirical Bayes techniques can be used to accurately estimate the posterior density. Empirical Bayes techniques were formalized by Robbins [12] as a methodology for dealing with uncertain parameters. Empirical Bayes results in the substitution of an empirically determined distribution for the unknown prior distribution required for Bayes parameter estimation, e.g. the conditional mean or the posterior mode (maximum a posteriori) estimators. Another technique that eliminates the need for an explicit prior $f(\theta)$ are use of asymptotic approximations to $f(\theta|Y_M)$.

5. ASYMPTOTIC MARGINALIZATION

Let $\hat{\theta} = \text{argmax}_\theta f(Y_M|\theta)$ be the maximum likelihood estimate of θ based on Y_M. We now state the following limit theorem which can be shown using techniques of [13], and [14, Secs 12.4 and 17.7].

Theorem 1 *Assume that $f(Y_M|\theta)$ is a smooth function of θ in the sense of satisfying the regularity conditions [13, p. 131] and that $f(\theta)$ is a smooth function in the neighborhood of $\hat{\theta}$. Then $\hat{\theta}$ is an asymptotically consistent estimator of $\theta \in \mathbb{R}^p$ and*

$$f(\theta|Y_M) = \frac{|F_{\hat{\theta}}|^{\frac{1}{2}}}{(\sqrt{2\pi})^p} \cdot \exp\left\{-\tfrac{1}{2}(\theta-\hat{\theta})^T F_{\hat{\theta}}(\theta-\hat{\theta})\right\}(1+O(\Delta^2))$$

where $F_{\hat{\theta}}$ is the observed Fisher information matrix

$$F_{\hat{\theta}} = -\nabla^2_{\hat{\theta}} \ln f(Y_M|\hat{\theta})$$

and $O(\Delta^2)$ decays to zero faster than $\Delta^2 = \|\theta - \hat{\theta}\|^2$.

This theorem gives an asymptotic Gaussian form for the posterior $f(Y_M|\theta)$ which depends on the estimate $\hat{\theta}$ and the observed Fisher information at $\hat{\theta}$ but is independent of the explicit form of the prior density $f(\theta)$. This form is identical to the profile posterior approximation proposed in [15] when the joint distribution is specialized to the factorization (2). In some cases it is reasonable to use the (expected) Fisher information in place of F. In particular $F(\hat{\theta})$ can be approximated by the Fisher information evaluated at $\hat{\theta}$ [16]:

$$I_{\hat{\theta}} = -\int f(Y_M|\theta)\nabla^2_\theta \ln f(Y_M|\theta) dY_M|_{\theta=\hat{\theta}}.$$

6. BACK TO PENALIZED LIKELIHOOD

Application of the above results gives the following form for the variance compensated penalized likelihood estimator

$$\hat{\lambda} = \text{argmax}_\lambda \left\{ \ln f(Y_E|\lambda) - \beta \sum_{ij} \tilde{w}_{ij}(\hat{\theta})(\lambda_i - \lambda_j)^2 \right\}$$

where

$$\tilde{w}_{ij}(\hat{\theta}) = |F_{\hat{\theta}}|^{\frac{1}{2}} \int w_{ij}(\theta)\phi(F_{\hat{\theta}}^{-\frac{1}{2}}[\theta - \hat{\theta}])d\theta$$

Note that the weight averaging corresponds to considering θ as a p-variate Gaussian random vector with mean $\hat{\theta}$ and covariance matrix $F_{\hat{\theta}}^{-1}$ and $\phi(x_1, \ldots, x_p)$ is the standard p-dimensional Gaussian density function with zero mean and identity covariance. This average is over the the posterior uncertainty region of θ.

For the binary weight scheme the computation of $\tilde{w}_{ij}$ can be performed directly in the pixel domain by averaging the weight map over all perturbations of θ

which produce changes in the boundary by at least a single pixel. Since this average may be difficult, it may be better to use the reparametrization $w_{ij}(\hat{\theta}) = w_{ij}(\hat{r}, \nu)$, where $\hat{r}, \nu$ index the locations of the boundary in polar coordinates, and do the averaging in the pixel domain

$$\tilde{w}_{ij}(\hat{\theta}) = \frac{|F_{\hat{r}}|^{\frac{1}{2}}}{(\sqrt{2\pi})^p} \cdot \sum_r w_{ij}(\hat{r}, \nu) \exp\left\{-\tfrac{1}{2}(r-\hat{r})^T F_{\hat{r}}(r-\hat{r})\right\}$$

where the sum is over the variation of radii r_ϕ which give rise to different weight maps and $F_{\hat{r}}^+$ denotes the pseudo-inverse $F_{\hat{r}}$:

$$F_{\hat{r}} = B F_{\hat{\theta}} B^T.$$

As a practical approximation, the sum can be truncated by extracting p principle components $\xi_1, \ldots, \xi_p$ (eigenvalues ordered in increasing rank $\lambda_1 \leq \lambda_2 \leq \ldots$) from the eigendecomposition of $F_{\hat{r}}$ and summing over the variations in $(r-\hat{r})^T \xi_i$ which: i) produce changes in the weight maps; ii) have magnitude changes over range: $|(r-\hat{r})^T \xi_i|^2 \leq \frac{3}{\lambda_i}$, $i = 1, \ldots, p$. This will cover a range of changes in r which will account for the majority of the mass in the averaging distribution (plus or minus 2 standard deviations from the mean).

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