Bias-Resolution-Variance Tradeoffs for Single Pixel Estimation Tasks using the Uniform Cramér-Rao Bound

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Outline

- Bias Estimator Cramér-Rao (CR) Bound
- Uniform CR Bound
- Interpretations of the UCRB
- Problems with UCRB
- 2nd-moment as measure of resolution
- UCRB with resolution constraint
- Pretty Pictures
- Open Issues

CR-Bound: Scalar Parameter

- Given a Scalar Parameter θ along with the following:
  - Estimator: $\hat{\theta}$
  - Estimator bias: $b(\theta) = E\left[\hat{\theta} - \theta\right]$
  - Log-Likelihood: $l(\theta)$
  - Fisher Information: $F_\theta = E_\theta\left[-\frac{\partial^2}{\partial^2\theta} l(\theta)\right]$

- Biased Estimator CR-Bound

$$\mathrm{var}_\theta\left(\hat{\theta}\right) \geq \left(1 + \frac{\mathrm{db}(\theta)}{\mathrm{d}\theta}\right)^2 F_\theta^{-1}$$

CR-Bound: Scalar Function of a Vector Parameter

- Given a Scalar Function $t_{\underline{\theta}}$ of the vector parameter $\underline{\theta}$ with the following:
  - Estimator: $\hat{t}_{\underline{\theta}}$
  - Estimator mean: $m_{\underline{\theta}} = E_{\underline{\theta}}\left[\hat{t}_{\underline{\theta}}\right]$
  - Estimator bias: $b_{\underline{\theta}} = E_{\underline{\theta}}\left[\hat{t}_{\underline{\theta}} - t_{\underline{\theta}}\right]$
  - Log-Likelihood $l(\underline{\theta})$
  - Fisher Information $F_{\underline{\theta}} = E_{\underline{\theta}}\left[-\nabla^2 l(\underline{\theta})\right]$

Example: Linear Scalar Function

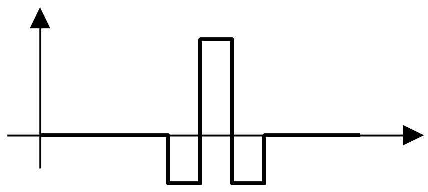
- $t_{\underline{\theta}}$ is a linear combination of the element of $\underline{\theta}$

  $$t_{\underline{\theta}} = \sum h_i \theta_i = \underline{h}^T \underline{\theta}$$

- Estimator gradient has simple form $\nabla t_{\underline{\theta}} = \underline{h}$

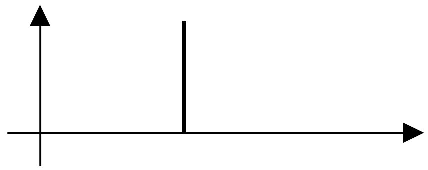
- Examples
  - Contrast Function

    $$\underline{h} = [0, \ldots 0, -1, -1, 2, 2, -1, -1, 0 \ldots, 0]^T$$



  - Single Pixel Estimation

    $$\underline{h} = [0, \ldots 0, 1, 0 \ldots, 0]^T = \underline{e}_p$$



Estimator Mean Response and Bias Functions

- Example: Linear Additive Gaussian Measurements
  - Measurement Equation $\underline{Y} \sim N\left(A\underline{\theta}, \Sigma\right)$
  - Fisher Information Matrix $F_Y = A^T \Sigma^{-1} A$
  - QPML Vector Estimator $\underline{\hat{\theta}}(\underline{Y}) = \left(F_Y + \beta P\right)^{-1} A \Sigma^{-1} \underline{Y}$
  - QPML Pixel Estimator $\hat{t}_{\underline{\theta}}(\underline{Y}) = \underline{e}_p{}^T \underline{\hat{\theta}}$
  - Estimator Mean $m_{\underline{\theta}} = \underline{e}_p{}^T \left(F_Y + \beta P\right)^{-1} F_Y \underline{\theta}$
  - Estimator Bias $b_{\underline{\theta}} = \underline{e}_p{}^T \left[\left(F_Y + \beta P\right)^{-1} F_Y - I\right] \underline{\theta}$

Biased Estimator CR-Bound

- Biased Estimator CR-Bound

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \left[\nabla m_{\underline{\theta}}\right]^T F_{\underline{\theta}}^{-1}\left[\nabla m_{\underline{\theta}}\right]$$

$$\geq \left[\nabla t_{\underline{\theta}} + \nabla b_{\underline{\theta}}\right]^T F_{\underline{\theta}}^{-1}\left[\nabla t_{\underline{\theta}} + \nabla b_{\underline{\theta}}\right]$$

- Biased Pixel Estimator CR-Bound

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]$$

- Note dependence on bias gradient $\nabla b_{\underline{\theta}}$

Mean and Bias Gradient for Point Source Estimation

- Mean Estimator Response to Point Source

$$m_{\underline{\theta}} = \left(F_Y + \beta P\right)^{-1} F_Y \underline{e}_p$$

- Estimator Mean Gradient

$$\nabla m_{\underline{\theta}} = F_Y\left(F_Y + \beta P\right)^{-1} \underline{e}_p$$

- Estimator Bias Gradient

$$\nabla b_{\underline{\theta}} = F_Y\left(F_Y + \beta P\right)^{-1} \underline{e}_p - \underline{e}_p$$

- *Gradients are measures of Point Response Function*

Uniform CR Bound for Point Source Estimation

- Given the Biased Estimator CR-Bound

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]$$

- Find the lower bound among all possible estimator with a given bias gradient *length*

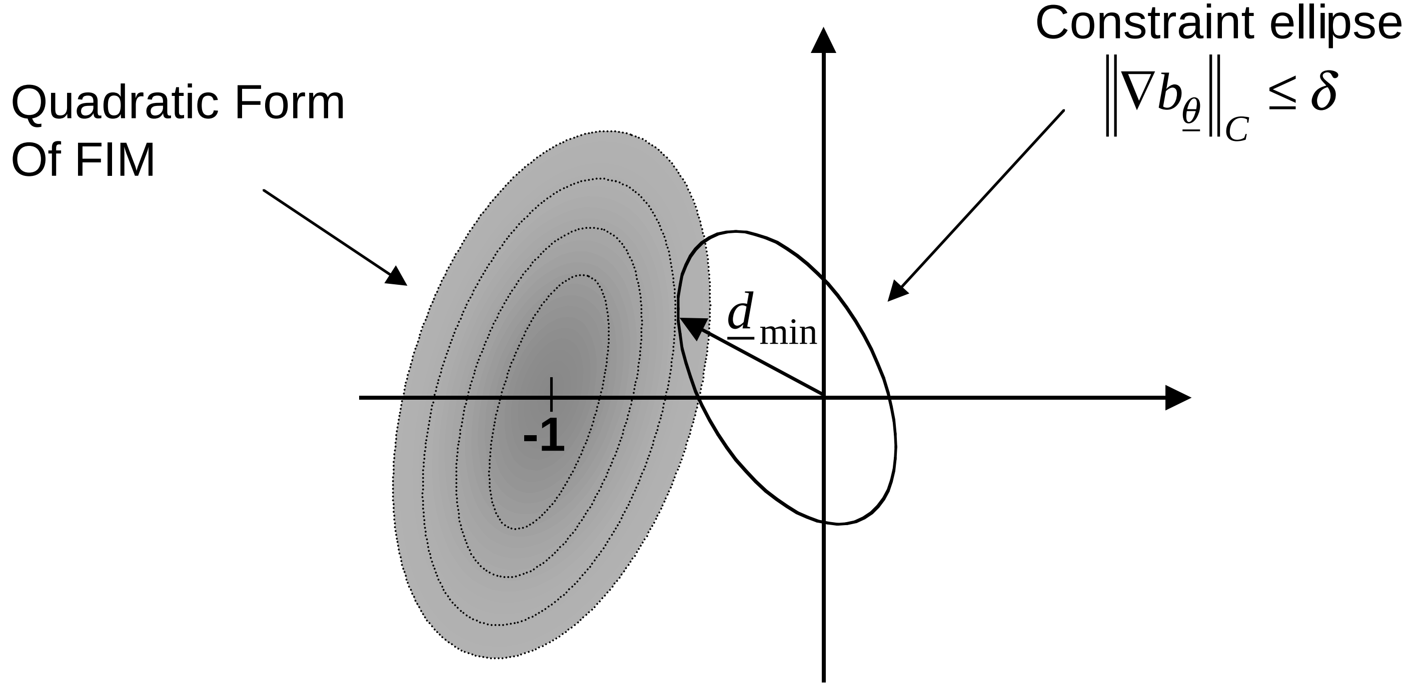
$$\left\|\nabla b_{\underline{\theta}}\right\|_C \leq \delta$$

- Pose problem as minimization of a quadratic form

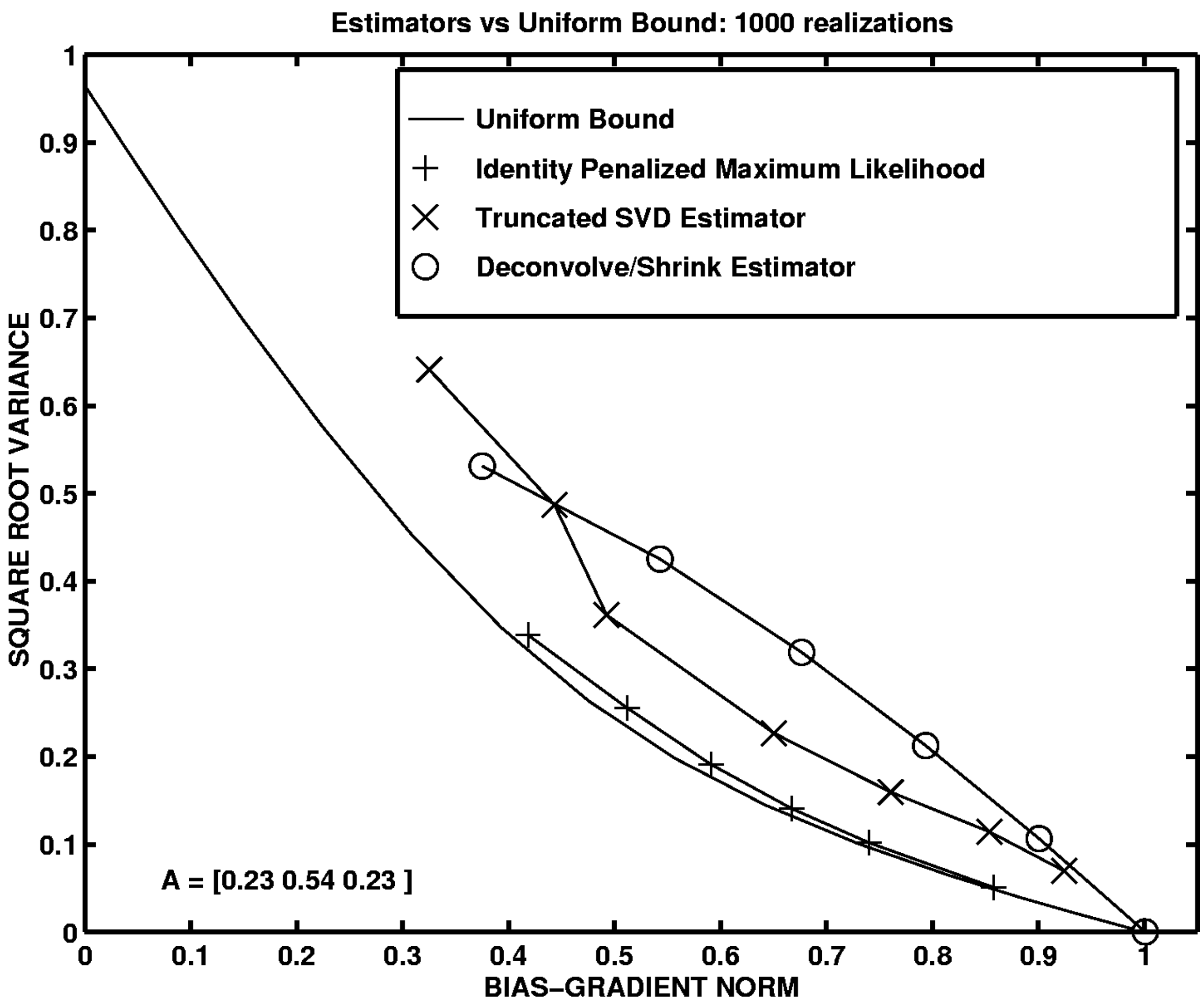
$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \underset{\underline{d}\,:\,\|\underline{d}\|_C \leq \delta}{\text{arg - min}} \left[\underline{e}_p + \underline{d}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \underline{d}\right]$$

Graphical Interpretation of UCRB for Point Source Estimation

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \underset{\underline{d}\,:\,\|\underline{d}\|_C \leq \delta}{\text{arg - min}} \left[\underline{e}_p + \underline{d}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \underline{d}\right]$$

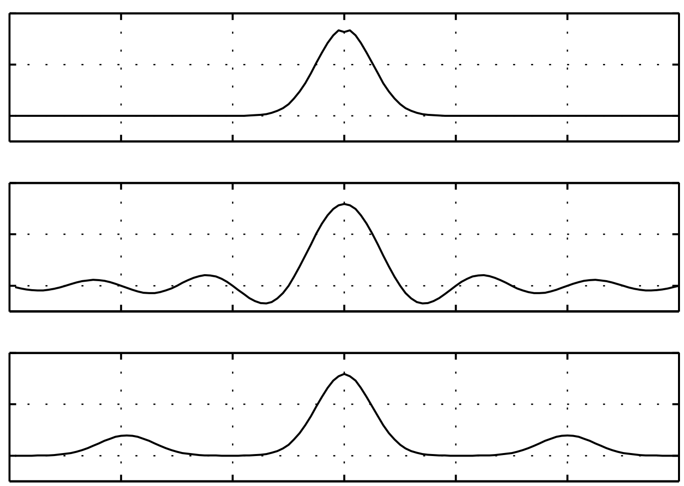


Example UCRB Calculation



Problem with UCRB

- Multiple different estimator point response functions can have *identical* bias gradient length, but with widely different resolution properties



Extend the UCRB: Add a Resolution Measure

- Estimator mean gradient is related to the point response function
- Pose a resolution criteria on $\nabla m_{\underline{\theta}}$
  - *2nd-Moment relative to pth-pixel*

$$\gamma^2 = \frac{\sum_i (p-i)^2 \left(\nabla m_{\underline{\theta}}\right)_i^2}{\sum_i \left(\nabla m_{\underline{\theta}}\right)_i^2}$$

$$\gamma^2 = \frac{\nabla m_{\underline{\theta}}^T M_p \nabla m_{\underline{\theta}}}{\nabla m_{\underline{\theta}}^T \nabla m_{\underline{\theta}}} = \frac{\left\| \underline{e}_p + \nabla b_{\underline{\theta}} \right\|_{M_p}^2}{\left\| \underline{e}_p + \nabla b_{\underline{\theta}} \right\|_2^2}$$

Solving for Extended UCRB

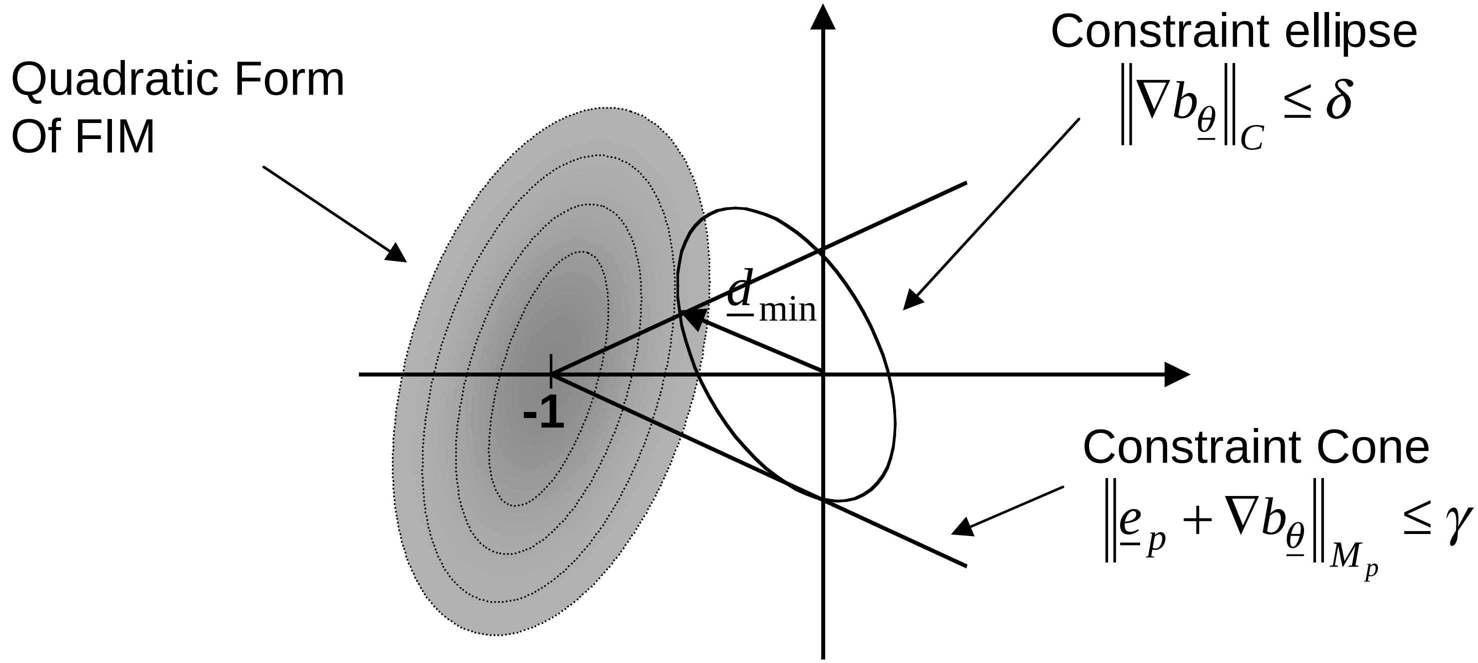
- Perform same minimization as before

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \underset{\underline{d}}{\text{arg-min}} \; \left[\underline{e}_p + \underline{d}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \underline{d}\right]$$

- Subject to the following two constraints:
  - 1) Bias Gradient Constraint $\left\|\nabla b_{\underline{\theta}}\right\|_C \leq \delta$
  - 2) Resolution Constraint $\left\|\underline{e}_p + \nabla b_{\underline{\theta}}\right\|_{M_p} \leq \gamma$
- Calculate resulting Bias-Resolution-Variance surface

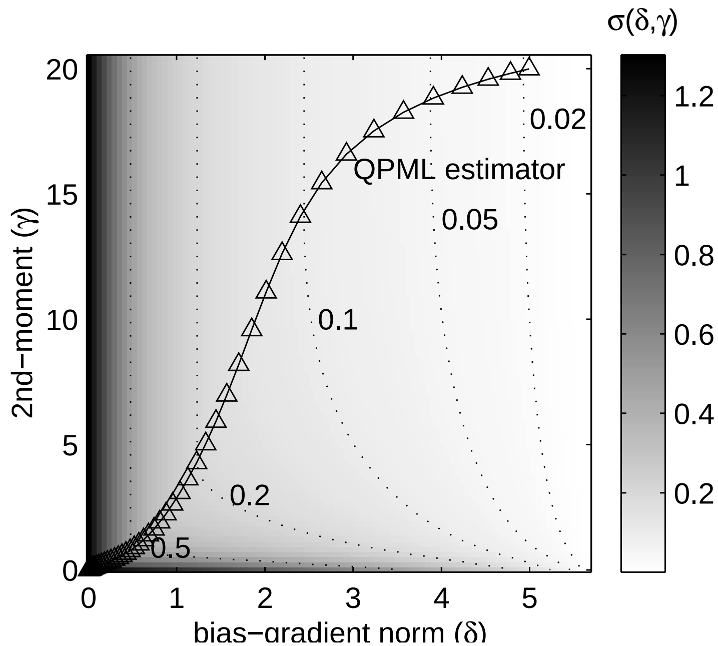
Graphical Interpretation of UCRB with Resolution Constraint

$$\mathrm{var}_{\underline{\theta}}\left(\hat{t}_{\underline{\theta}}\right) \geq \underset{\underline{d}}{\text{arg-min}} \left[\underline{e}_p + \underline{d}\right]^T F_{\underline{\theta}}^{-1}\left[\underline{e}_p + \underline{d}\right]$$



Example UCRB Calculation with Resolution Constraint

- Linear Additive Gaussian Inverse Problem
- Single Pixel Estimation Task
- QPML Estimator overlaid on top



Achievability of UCRB

For $\underline{\theta} =$ source intensity vector $\underline{\lambda}$,

UCRB is nearly attainable by the

Penalized Maximum Likelihood estimator

$$\hat{\underline{\theta}} = \arg\max_{\hat{\underline{\theta}} \geq 0} \left\{ l(\underline{\theta}) + \beta \underline{\theta}^T P \underline{\theta} \right\}$$

- Issue: This is for UCRB.
  - How does Resolution constraint kick in?
  - For Linear Additive Gaussian problem, the QPML estimator seems to have min variance / max resolution