Cut-off Rate and Signal Design for the Rayleigh Fading Space-Time Channel[1]

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March 10, 2000

Abstract

We consider the computational cut-off rate and its implications on signal design for the complex Rayleigh flat fading spatio-temporal channel under a peak power constraint. When the analysis is restricted to finite dimensional sets of signals interesting characterizations of the optimal rate-achieving constellation can be obtained. For arbitrary finite dimension, the optimal constellation must admit an equalizer distribution, i.e., a positive set of signal probabilities which equalizes the conditional decoding error probabilities. When the number of signal matrices in the constellation is less than the ratio of the number of time samples to the number of transmit antennas, the optimal cut-off rate attaining constellation is a set of equiprobable mutually-orthogonal unitary matrices. When the SNR is below a specified threshold the matrices in the constellation are rank one and the cutoff rate is achieved by applying all transmit power to a single antenna and using orthogonal signaling. Finally, we derive recursive necessary and sufficient conditions for a constellation to lie in the feasible set.

Keywords: space-time coding, spatial diversity, optimal signal design, constellation of unitary matrices, peak power constraint.

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This work was performed, in part, while the first author was visiting the Mathematical Sciences Research Center.

1 Introduction

In this paper we investigate the single-user cut-off rate for the Rayleigh flat fading spatio-temporal channel model studied by Marzetta and Hochwald [13] under a maximum peak transmitted power constraint. The signal set for this channel consists of complex matrices whose rows are indexed over temporal coordinates that span the interval over which the Rayleigh fading coefficients are constant, and whose columns are indexed over spatial coordinates that span the locations of the transmitter antennas. Fading is accounted for by a propagation matrix of Rayleigh distributed coefficients which are random variables unknown to both transmitter and receiver.

The channel cut-off rate R_o is a lower bound on the Shannon channel capacity C. R_o also specifies an upper bound on the error rate of an optimal decoder operating at a given symbol rate which can be used to majorize the minimum probability of decoding error. Cut-off rate analysis has frequently been adopted to establish practical coding limits [22, 8] as the cut-off rate specifies the highest information rate beyond which sequential decoding becomes impractical [20, 23] and as it is frequently simpler to calculate than channel capacity. Cut-off rate analysis has also been used to evaluate relative merits between different coding and modulation schemes [15], signal design for optical communications [21], and establishing achievable rate regions for multiple access channels [17, 18].

The following are some of the principal results obtained.

1. An integral representation for the cut-off rate is obtained (Lemma 1) which depends on a pairwise dissimilarity measure over the set of signal matrices. This dissimilarity measure is a decreasing function of the spatial correlation between pairs of signal matrices.

2. For low SNR the dissimilarity measure reduces to a distance metric equal to the trace norm of pairwise differences between outerproducts of the signal matrices (Lemma 8).

3. A lower bound is given on the largest possible minimum distance for arbitrary sets of signal matrices of fixed finite dimension (Proposition 2). This is also a lower bound on the maximum distance of signals in the optimal cut-off rate attaining signal set (Proposition 3).

4. A necessary and sufficient condition for a signal probability distribution to attain cut-off rate is that it equalize the decoder error rate over all possible signal matrices (Lemma 2). We call this the equalization condition and it plays a central role in this work.

5. The K dimensional cut-off rate, defined as the cut-off rate for constellations whose dimension does not exceed K, is the appropriate limiting factor for signal sets generated with finite numerical

precision. A finite dimensional analog to the aforementioned equalization condition on the signal probabilities must be satisfied by the rate achieving constellation (Proposition 4).

6. The determination of the K dimensional cut-off rate reduces to maximization of a quadratic form over the set of feasible constellations, defined as those constellations which satisfy both the peak power constraint and the finite dimensional equalization condition (Proposition 5). This quadratic form is similar to that arising in the Capon/MVDR method for adaptive beamforming arrays. If the feasible set of K dimensional constellations is empty then the optimal constellation is necessarily of dimension less than K.

7. For low symbol-rate the optimal constellation is a set of scaled mutually orthogonal unitary matrices in $\mathbb{C}^{T\times M}$ (Proposition 9). This constellation also maximizes minimum distance over all constellations of the same dimension (Corollary 2). When SNR is low the rank of the signal matrices in the constellation is one and cut-off rate is achieved by transmitting all power from a single antenna element. As the SNR increases the rank of the signal matrices increases and more and more antenna elements are utilized. Interestingly, the number of receive antennas N plays no role whatsoever in determining how many transmit antennas should be used.

8. When K exceeds T/M but is finite, the K dimensional cut-off rate is not closed form and must be determined by numerical optimization of the cut-off rate objective function over the feasible set of constellations. We derive simple necessary and sufficient conditions for a constellation to be in the feasible set (Lemma 4). The necessary conditions characterize properties of the optimal cut-off rate achieving constellation while the sufficient conditions can be used to generate feasible signal sets. To satisfy these conditions, among other attributes, the signal matrices should have pairwise distances of low variability. As these conditions can be used to check if a candidate set of constellations is in the feasible set or not, they provide a potentially useful signal design tool, especially when dealing with large constellations of high dimensional signal matrices for which computing the globally optimal signal set may be impractical.

9. The number of signals in the optimal constellation increases as a function of the number of antenna receivers (Proposition 6).

The outline of this paper is as follows. In Section 2 the Rayleigh fading measurement model is introduced. In Section 3 we briefly review the definition of computational cut-off rate and its relation to the error exponent. In Section 4 we specialize to the finite dimensional cut-off rate which reduces to maximization of a quadratic form over signal dissimilarity matrices. In Section 5 we consider quadratic programming for computing the cut-off rate. In Section 6 general characterizations for the feasible set are given in terms of necessary and sufficient conditions on the signal pairwise distances. In Section 7 we specialize to the case of finite constellations with dimension smaller than T/M, for which the finite dimensional rate is closed form.

2 Rayleigh Flat Fading Spatio-Temporal Channel

We use substantially the same notation as in the papers of Hochwald and Marzetta [13, 10]. Let there be M transmitter antennas and N receiver antennas and let the MN channel fading coefficients be constant over an interval of length T time periods. T will be called the coherent fade sampling interval. A transmitted signal S is a $T \times M$ matrix having complex valued entries. Let $\mathcal{S} = \mathbb{C}^{T\times M}$ denote the set of all possible signal matrices. The norm of a matrix $S \in \mathcal{S}$ is defined as

$$\|S\| = \sqrt{\mathrm{tr}\{SS^H\}} = \sqrt{\sum_{i=1}^{T} |\lambda_i|^2},$$

where S^H denotes the Hermitian transpose of S and $\{\lambda_i\}_{i=1}^T$ are the singular values of S. Note that if $T > M$ only M of these singular values will be non-zero.

For N receiver antennas and an observation time interval of LT seconds the received signal is a sequence $\{X_i\}_{i=1}^L$ of L complex valued $T \times N$ matrices $X_i \in \mathcal{X} = \mathbb{C}^{T\times N}$ which has the representation [13]:

$$X_i = \sqrt{\eta} S_i H_i + W_i, \quad i = 1, \ldots, L, \tag{1}$$

where $S_i \in \mathcal{S}$ is the i-th transmitted signal, $\eta = \rho/M$ is the normalized signal-to-noise ratio (SNR) with $\rho > 0$ the expected SNR at each receiver per transmit antenna, H_i is an $M \times N$ matrix of complex channel coefficients, and W_i is a $T \times N$ matrix of complex noises. The piecewise constant Rayleigh flat fading model corresponds to taking the $LN(T+M)$ elements of the matrices $\{H_i\}_{i=1}^L$ and $\{W_i\}_{i=1}^L$ to be i.i.d. complex zero mean (circularly symmetric) Gaussian random variables with unit variance. Therefore the joint conditional probability density of the observations factors into a product of marginals

$$p(X_1, \ldots, X_L | S_1, \ldots, S_L) = \prod_{i=1}^{L} p(X_i|S_i),$$

where

$$p(X_i|S_i) = \frac{\exp\left(-\mathrm{tr}\{[I_T + \eta S_i S_i^H]^{-1} X_i X_i^H\right)}{\pi^{TN} \left|I_T + \eta S_i S_i^H\right|^N}, \tag{2}$$

I_T is the $T \times T$ identity matrix, and $|A| = |\det(A)|$ denotes the magnitude determinant of square matrix A.

3 Error Exponents and the Channel Cutoff Rate

It is well known [23] that at any transmission rate R (nats) below capacity C the minimum attainable decoding error probability P_e of a code of block length L satisfies

$$P_e \leq e^{-LE_U(R)}, \tag{3}$$

where $E_U(R) > 0$ is a reliability function controlling the error rate of the optimal decoder. This paper is concerned with a lower bound on the quantity $E_U(R)$, called the random coding error exponent, which is defined as [6]

$$E_U(R) = \max_{\mu \in [0,1]} \max_{P \in \mathcal{P}} \left\{ -\mu R - \ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} (p(X|S))^{1/(1+\mu)} \, dP(S) \right]^{1+\mu} dX \right\}, \quad \text{nats/symbol}, \tag{4}$$

where the inner maximization is performed over a suitably constrained set $\mathcal{P}$ of probability distributions P defined over the set of signal matrices $\mathcal{S}$. Additional constraints on P are determined by factors such as total power budget or envelope constraints and are left implicit in (4). The distribution $P = P^*$ which attains the maximum in $E_U(R)$ gives an optimal signal distribution for which there exists a decoder achieving minimum probability of decoding error for sufficiently large blocklengths. Generally, P^* is not discrete and an optimal set of signal matrices can only be constructed by a random coding procedure. The function $E_U(R)$ was studied for the spatio-temporal Rayleigh flat fading model by Abou-Faycal and Hochwald [1] under a mean power constraint on transmitted signal matrices. Unfortunately, the double maximization in (4) is generally very difficult since the inner integral is raised to a fractional exponent when $\mu \in (0, 1)$. The cut-off rate specified below is a lower bound on the error exponent $E_U(R)$ which is frequently simpler to analyze.

The random coding error exponent $E_U(R)$ can be lower bounded by the tangent line $Q(R)$ having slope -1: $Q(R) = R_o - R$, where R_o is this line's intercept on both the $E_U(R)$ and the R axes [23]. The point of tangency $R_c = \max_{R \in [0,C]}\{R : E_U(R) = Q(R)\}$ is the critical rate and the intercept R_o is called the computational cut-off rate. The cut-off rate is given by

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} \sqrt{p(X|S)} dP(S) \right]^2 dX, \quad \text{nats/symbol} \tag{5}$$

which is equivalent to the error exponent $E_U(R)$ in (4) with the maximization over $\mu \in [0, 1]$ replaced by the simpler evaluation at the point $\mu = 1$.

While use of cut-off rate for evaluating system performance has its limitations, see e.g. [3, 14], it satisfies useful properties [7] which in our opinion justify its application to space-time coding:

- $R_o \leq C$, i.e. it provides a lower bound on channel capacity.

- $E_U(R) \approx R_o - R$ when the symbol rate R is close to R_c. Thus the signal distribution $P = P^*$ which attains the maximum in the definition of R_o will attain optimal error rates for R near R_c.

- For sequential decoding strategies, R_o is the maximum practical symbol transmission rate. Specifically, even though Shannon's coding theorems ensure that there exists a low error decoding algorithm for all R such that $R < C$, if $R_o < R < C$ the probability of arbitrary long decision delays becomes significant [20].

- Combining (3), (4) and (5) an upper bound is obtained on the minimum probability of decoder error for transmission rates R less than R_o

$$P_e \leq \exp\{-L(R_o - R)\}, \quad R < R_o. \tag{6}$$

  Thus $R - R_o$ is an upper bound on the error rate $(\ln P_e)/L$ of the optimal decoder.

- When analysis of channel capacity C is difficult or intractable, R_o offers an alternative which is frequently easier to analyze.

3.1 R_o Representation for Rayleigh Flat Fading Channel

Here we give integral representations for the cut-off rate.

Lemma 1 *The cut-off rate for the spatio-temporal fading model (1) is given by*

$$R_o = \max_{P\in\mathcal{P}} -\ln \int_{S_1\in\mathcal{S}} dP(S_1) \int_{S_2\in\mathcal{S}} dP(S_2) \left[\frac{\sqrt{|I_T + \eta S_1 S_1^H|\,|I_T + \eta S_2 S_2^H|}}{|I_T + \frac{\eta}{2}(S_1 S_1^H + S_2 S_2^H)|}\right]^N, \quad \text{nats/symbol} \tag{7}$$

where the maximization is performed over the set $\mathcal{P}$ of distributions P of suitably constrained signal matrices.

Proof of Lemma 1

For notational convenience define the $T \times T$ matrix $A(S) = I_T + \eta S S^H$. Then the conditional p.d.f. (2) takes the form $p(X_i|S_i) = \exp\left(-\mathrm{tr}\{A(S_i)^{-1} X_i X_i^H\right) / (\pi^{TN}\,|A(S_i)|^N)$. The integral in (5) is the triple integral $\int dX \int dP(S_1) \int dP(S_2) \sqrt{P(X|S_1)P(X|S_2)}$. As $\eta S_i S_i^H$ is non-negative definite $|A(S_i)| \geq 1$ and therefore $p(X_i|S_i) \leq 1/\pi^{NT} < \infty$. Thus by Fubini we can interchange order of integration in the triple integral to obtain

$$R_o = \max_{P\in\mathcal{P}} -\ln \int_{S_1\in\mathcal{S}} dP(S_1) \int_{S_2\in\mathcal{S}} dP(S_2) \int_{X\in\mathcal{X}} dX \sqrt{p(X|S_1)p(X|S_2)} \tag{8}$$

The inner integral has the explicit form

$$
\begin{aligned}
&\int_{X\in\mathcal{X}} dX\,\sqrt{p(X|S_1)p(X|S_2)} \qquad (9)\\
&\qquad = \frac{1}{|A(S_1)|^{N/2}\,|A(S_2)|^{N/2}}\underbrace{\pi^{TN}\int \exp\left(-\tfrac{1}{2}\left[A^{-1}(S_1)+A^{-1}(S_2)\right]XX^H\right)}_{\left|\frac{1}{2}A^{-1}(S_1)+\frac{1}{2}A^{-1}(S_2)\right|^N}\\
&\qquad = \left(\frac{\sqrt{|A(S_1)|\,|A(S_2)|}}{|\frac{1}{2}A(S_1)+\frac{1}{2}A(S_2)|}\right)^N
\end{aligned}
$$

where in the first line the normalization constant of the multivariate complex Gaussian p.d.f. has been identified ($\int \exp\{-\mathrm{tr}(B^{-1}XX^H)\}dX = \pi^{TN}|B|^N$) and in the second line the determinant property $|BC||B^{-1}+C^{-1}| = |B+C|$ has been used. □

The form of the cut-off rate given in (7) suggests defining the pairwise dissimilarity measure for any two signal matrices $S_1, S_2 \in S$

$$D(S_1\|S_2) \quad \stackrel{\mathrm{def}}{=} \tfrac{1}{2}\ln\frac{\left|I_T+\frac{\eta}{2}(S_1S_1^H+S_2S_2^H)\right|^2}{\left|I_T+\eta S_1S_1^H\right|\left|I_T+\eta S_2S_2^H\right|}. \qquad (10)$$

Thus the cut-off rate can be equivalently expressed as

$$R_o = \max_{P\in\mathcal{P}} -\ln\int_{S_1\in\mathcal{S}} dP(S_1)\int_{S_2\in\mathcal{S}} dP(S_2)\, e^{-ND(S_1\|S_2)} \quad \text{nats/symbol.} \qquad (11)$$

As the integral (9) is upper bounded by one (Cauchy-Schwarz inequality) $D(S_1|S_2)$ is non-negative. By Lemma 8, given in the Appendix, $D(S_1\|S_2) = \eta^2/8\|S_1S_1^H - S_2S_2^H\|^2 + o(\eta^2)$, so that in the asymptotic regime of low SNR (η small), $D(S_1\|S_2)$ is to a good approximation proportional to the squared norm $\|S_1S_1^H - S_2S_2^H\|^2$ of the matrix outerproduct difference. We define the minimum distance (dissimilarity) of a set of signal matrices $\mathcal{S}^k = \{S_i\}_{i=1}^k$ as $D_{\min} = \min_{S_i,S_j\in\mathcal{S}^k: i\neq j} D(S_i\|S_j)$. In Proposition 2 we will give a lower bound on the largest possible value of $D_{\min}$ for a peak constrained constellation of K signal matrices. A set of signal matrices which is close to optimal, in terms of nearly attaining the cut-off rate, might be expected to have large value of $D_{\min}$. Indeed, Proposition 9 implies that the optimal peak constrained signal constellation maximizes $D_{\min}$ for low symbol-rates.

3.2 An Equalization Condition for Optimal Distribution

Define the following set of probability distributions over the set $\mathcal{S}$ of complex $T\times M$ matrices S:

$$\mathcal{P}_{eq} = \cup_{c\in[0,1]}\left\{P(S): P(S')\geq 0,\ \int_{S\in\mathcal{S}} dP(S) = 1,\ \int_{S\in\mathcal{S}} e^{-ND(S\|S')}dP(S) = c,\ \forall S'\in\mathcal{S}\right\}. \qquad (12)$$

Probability distributions lying in $\mathcal{P}_{eq}$ will be called "equalizer distributions".

The set $\mathcal{P}_{eq}$ is compact. Indeed, let $\{P^{(k)}\}_{k>0}$ be an arbitrary sequence of probability distributions in $\mathcal{P}_{eq}$ and let $\{c^{(k)}\}_{k>0}$, $c^{(k)} \stackrel{\mathrm{def}}{=} \int_{S\in\mathcal{S}} e^{-ND(S\|S')}dP^{(k)}(S)$, be the corresponding constants. If for some probability distribution $P^{(\infty)}$, $P^{(k)} \to P^{(\infty)}$ then, as $0 \leq e^{-ND(S\|S^{'})} \leq 1$, by Lebesgue dominated convergence $\int e^{-ND(S\|S^{'})} dP^{(\infty)}(S) = \lim_{k\to\infty} \int e^{-ND(S\|S^{'})} dP^{(k)}(S) = \lim_{k\to\infty} c^{(k)} = c^{(\infty)} \in [0,1]$. Thus $P^{(\infty)} \in \mathcal{P}_{eq}$. Next we give a sufficient condition for a distribution P to attain R_o.

Lemma 2 *Assume that $\mathcal{P}_{eq}$ is non-empty. Then*

$$\begin{aligned} R_o &= \max_{P\in\mathcal{P}_{eq}} \left\{ -\ln \int_{S\in\mathcal{S}} dP(S) \int_{S^{'}\in\mathcal{S}} dP(S^{'}) e^{-ND(S\|S^{'})} \right\} \\ &= \max_{P\in\mathcal{P}_{eq}} \left\{ -\ln \int_{S\in\mathcal{S}} e^{-ND(S\|S^{'})} dP(S) \right\}. \end{aligned} \tag{13}$$

As $\mathcal{P}_{eq}$ is compact, Lemma (2) asserts that if there exist signal distributions P which equalize the average dissimilarity function $\int \exp\{-ND(S\|S^{'})\} dP(S)$ over $S^{'} \in \mathcal{S}$, then one of these distributions achieves the cut-off rate R_o. The assumption that $\mathcal{P}_{eq}$ is non-empty is difficult to verify in general, except, as will be shown, when the optimal signal set is finite. Indeed continuous equalizer distributions P may not exist. Let f denote the probability density function of such a P. Existence of an equalizing distribution would require that the range of the integral operator $\int_{S\in\mathcal{S}} f(S) e^{-ND(S\|S^{'})} dS$ contain an everywhere constant function over $S^{'} \in \mathbb{C}^{T\times M}$, a condition which is quite difficult to verify even for $T = M = 1$.

Proof of Lemma 2

The proof of relation (13) begins with the following change of measure in the integral representation (11) for R_o

$$\begin{aligned} R_o &= \max_{P} -\ln \int_{S_1\in\mathcal{S}} \int_{S_2\in\mathcal{S}} dQ(S_1,S_2) e^{-ND(S_1\|S_2)} \frac{dP(S_1)dP(S_2)}{dQ(S_1,S_2)} \\ &= \max_{Q} -\ln E_Q \left[\exp\left\{ -ND(S_1\|S_2) - \ln \frac{dQ(S_1,S_2)}{dP(S_1)dP(S_2)} \right\} \right], \end{aligned}$$

where $E_Q[\cdot]$ denotes statistical expectation with respect to the joint probability distribution $Q(S_1,S_2)$ which is arbitrary except that its marginals are identically the probability distribution P. Note that the maximization over the marginal P has been replaced by a maximization over the joint Q. Now by Jensen's inequality

$$-\ln E_Q \left[\exp\left\{ -ND(S_1\|S_2) - \ln \frac{dQ(S_1,S_2)}{dP(S_1)dP(S_2)} \right\} \right] \leq N E_Q \left[D(S_1\|S_2) \right] + I(S_1,S_2) \tag{14}$$

where $I(S_1,S_2) = E_Q\left[\ln \frac{dQ(S_1,S_2)}{dP(S_1)dP(S_2)} \right]$ is the mutual information associated with Q, and the inequality in (14) is achieved if and only if

$$-ND(S_1\|S_2) - \ln \frac{dQ(S_1,S_2)}{dP(S_1)dP(S_2)} = a, \quad (a.s.\ Q),$$

for some real constant a. The above condition for equality implies that $e^a dQ(S_1, S_2) = e^{-ND(S_1\|S_2)} dP(S_1)dP(S_2)$ (a.s. Q) and therefore, integrating over S_2, we see that for all S_1 such that $dP(S_1) > 0$

$$\int_{S_2 \in \mathcal{S}} e^{-ND(S_1\|S_2)} dP(S_2) = c, \quad (a.s.\ P), \tag{15}$$

where $c = e^a$. This is the equalization condition defined in (12) satisfied by probability distributions in $\mathcal{P}_{eq}$. Hence, if $\mathcal{P}_{eq}$ is non-empty equality in (14) is achievable and, maximizing the right hand side of (14) over $\mathcal{P}_{eq}$ establishes (13). $\square$

We end this subsection with a result that parallels Theorems 1 and 2 of Marzetta and Hochwald [13], but covers the case of peak power constrained signal sets.

Proposition 1 *Assume that the transmitted signal S is constrained to satisfy the peak power constraint $\|S\|^2 \leq MT$. The peak power constrained cut-off rate attained with $M > T$ transmit antennas is the same as that attained with $M = T$ antennas. Therefore, there is no advantage to using more than T transmit antennas. Furthermore, for $M \leq T$ the signal matrix which achieves peak power constrained cut-off rate can be expressed as $S = V\Lambda$ where V is a $T \times T$ unitary matrix, $\Lambda = [\Lambda_M, O]^T$ is a $T \times M$ matrix, and Λ_M is a diagonal $M \times M$ matrix.*

Proof of Proposition 1

We first treat the case that $M > T$. Let S have singular value decomposition (SVD) $S = V\Lambda U^H$ where V is a $T \times T$ unitary, U is an $M \times M$ unitary and $\Lambda = [\Lambda_T, O]$ is a $T \times M$ matrix partitioned into a diagonal $T \times T$ matrix Λ_T and a $T \times (M - T)$ matrix of zeros. The fading model (1) for $L = 1$ specifies the received signal as the $T \times N$ matrix

$$X = \sqrt{\eta} SH + W,$$

where the $M \times N$ matrix H and the $T \times N$ matrix W are mutually independent matrices of of i.i.d. zero mean complex Gaussian random variables. Substituting the SVD of S into the model we obtain

$$\begin{aligned} X &= \sqrt{\eta} SH + W \\ &= \sqrt{\eta} V\Lambda U^H H + W \\ &= \sqrt{\eta} V\Lambda_T \, [I_T, O]U^H H + W \\ &= \sqrt{\eta} V\Lambda_T \tilde{H}_T + W. \end{aligned}$$

Note that, as U is unitary, $\tilde{H}_T = [I_T, O]U^H H$ is a $T \times N$ matrix of i.i.d. zero mean complex Gaussian r.v.s. Note also that, as $\|S\|^2 = \|V\Lambda_T\|^2$, the $T \times T$ transformed signal $V\Lambda_T$ satisfies the same peak power constraint as the original $T \times M$ signal S. Thus the signal X measured at the receiver after transmission of the signal S on M antennas is statistically equivalent to X received when the signal $V\Lambda_T$ is transmitted on only T antennas.

When $M \leq T$ the SVD of S is identical to $S = V\Lambda U^H$ above except that now $\Lambda = [\Lambda_M, O]^T$, where Λ_M is an $M \times M$ diagonal matrix. Therefore

$$X = \sqrt{\eta} V [\Lambda_M, O]^T \tilde{H} + W$$

where $\tilde{H} = U^H H$. □

Readers familiar with Theorems 1 and 2 of [13] might suspect that characterization of the statistical distribution P of the optimal cut-off achieving signal matrix S can be obtained. Indeed, paralleling the arguments of [13], it can be shown that, as $tr\{SS^H\} \leq TM$ is invariant to unitary premultiplication of S and as the maximization in the definition of R_o is over a concave function of P, the peak-power constrained cut-off rate is attained by random matrices of the form $S = V\Lambda$ where V is a $T \times T$ isotropically distributed matrix, $\Lambda = [\Lambda_M, 0]^T$ is a random $T \times M$ diagonal matrix, and V and Λ are statistically independent.

In the following sections we specialize to the case of discrete signal constellations for which equalizer distributions are always optimal.

4 Finite (K) Dimensional Cut-off Rate

For practical coding schemes it is of interest to restrict attention to finite sets of signal matrices. Let K be a prespecified finite positive integer. The cut-off rate (11) restricted to discrete distributions concentrated on at most K signal matrices will be called the K dimensional cut-off rate and takes the form

$$\tilde{R}_o(K) = \max_{\{P_i, S_i\}_{i=1}^K \in \mathcal{G}^K} - \ln \tilde{Q}\left(\{P_i\}_{i=1}^K, \{S_i\}_{i=1}^K\right) = -\ln \min_{\{P_i, S_i\}_{i=1}^K \in \mathcal{G}^K} \tilde{Q}\left(\{P_i\}_{i=1}^K, \{S_i\}_{i=1}^K\right) \quad (16)$$

where $\tilde{Q}$ is the quadratic form

$$\tilde{Q}\left(\{P_i\}_{i=1}^K, \{S_i\}_{i=1}^K\right) = \sum_{i=1}^K P_i \sum_{j=1}^K P_j \, e^{-ND(S_i \| S_j)}, \quad (17)$$

and where $\mathcal{G}^K$ is a suitably constrained set of signal matrices $\{S_i\}_{i=1}^K$ and signal probabilities $\{P_i\}_{i=1}^K$ (see subsection below for examples).

When $P_i = 0$ the signal S_i is never transmitted which motivates a natural definition of a signal constellation.

Definition 1 *A set of matrices $\{S_i\}_{i=1}^K$ in $\mathbb{C}^{T\times M}$ is a signal constellation if all assigned signal probabilities P_i are strictly positive, $i = 1, \ldots, K$.*

As the optimization in (16) is performed over a restricted set $\tilde{R}_o(K)$ is a lower bound on R_o. Hence, in particular, $\tilde{R}_o(K)$ can be used in (6) to specify an upper bound on probability of decoding error or a lower bound on capacity. If the optimizing distribution P in (11) over $\mathcal{P}$ actually concentrates on a constellation consisting of a discrete finite number of signal matrices then the bound is tight, i.e. P is a discrete probability and $\tilde{R}_o(K) = R_o$ for some finite K. This hypothesis may not be unreasonable, e.g. when $N = M = 1$, the optimal P was shown to be discrete by Abou-Faycal and Hochwald [1] for codes that achieve the random coding error exponent. However, for cut-off rate and for more general values of N and M this property remains to be verified. On the other hand, when the signal matrices are generated under finite precision arithmetic the bound $\tilde{R}_o(K)$ is always tight for an appropriately chosen K, e.g., $K = 2T \times M2^B$ for B bit register resolution on each complex entry of the signal matrix.

4.1 Peak Power vs. Average Power Constraints

When an average transmitted power constraint is imposed, such as adopted in [13], the optimization (16) must be performed over the restricted set

$$\mathcal{G}^K = \mathcal{G}^K_{\text{avg}} = \left\{ \{P_i, S_i\}_{i=1}^K : P_i \geq 0,\ S_i \in \mathbb{C}^{TM},\ \sum_{i=1}^K P_i = 1,\ \sum_{i=1}^K P_i \|S_i\|^2 \leq TM \right\}$$

where TM is the average transmitter power budget. Thus the average power constraint introduces additional dependency between signal matrices and signal probabilities which complicates the optimization problem.

On the other hand, when a peak power constraint is imposed, the optimization (16) is performed over the simpler product set

$$\mathcal{G}^K = \mathcal{G}^K_{\text{peak}} = \left\{ \{P_i, S_i\}_{i=1}^K : P_i \geq 0, \sum_{i=1}^K P_i = 1, S_i \in \mathbb{C}^{TM}, \|S_i\|^2 \leq TM \right\} = \mathcal{P}^K \times \mathcal{S}^K_{peak} \tag{18}$$

where

$$\mathcal{P}^K = \left\{ \{P_i\}_{i=1}^K : P_i \geq 0, \sum_{i=1}^K P_i = 1 \right\} \tag{19}$$

and

$$\mathcal{S}^K_{\text{peak}} = \left\{ \{S_i\}_{i=1}^K : S_i \in \mathbb{C}^{TM}, \|S_i\|^2 \leq TM \right\}. \tag{20}$$

In particular, under the constraint (18) the optimization (16) decomposes into two nested minimizations

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K \in \mathcal{S}^K_{\text{peak}}} \tilde{Q}\left(\{P_i^*\}_{i=1}^K, \{S_i\}_{i=1}^K\right), \tag{21}$$

where, for fixed $\{S_1\}_{i=1}^K$, $P_i^* = P_i^*(S_1, \ldots, S_K)$, $i = 1, \ldots, K$, is the optimal probability assignment to the signal S_i

$$\{P_i^*\}_{i=1}^K = \text{argmin}_{\{P_i\}_{i=1}^K \in \mathcal{P}^K} \tilde{Q}\left(\{P_i\}_{i=1}^K, \{S_i\}_{i=1}^K\right). \tag{22}$$

4.2 Signal Dissimilarity Matrix

A more compact form for $\tilde{R}_o(K)$ (16) is obtained by putting the quadratic form into vector form with a dissimilarity matrix E_K. For K signal matrices $S_1, \ldots, S_K$, and for $e_K(i,j) = \exp(-ND(S_i \| S_j))$, $E_K = E_K(S_1, \ldots, S_K)$ is the $K \times K$ matrix

$$\begin{aligned} E_K &= ((e_K(i,j)))_{i,j=1}^K \\ &= \begin{bmatrix} 1 & e_{1,2} & \cdots & e_{1,K} \\ e_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_{K-1,K} \\ e_{K,1} & \cdots & e_{K,K-1} & 1 \end{bmatrix}. \end{aligned} \tag{23}$$

Note that E_K is symmetric with non-negative entries.

Lemma 3 *E_K is non-negative definite. If the K outerproduct matrices $\{S_i S_i^H\}_{i=1}^K$ are distinct then: (i) E_K is positive definite; and (ii) $e_K(i,j) < 1$, $i \neq j$.*

Proof of Lemma 3

From the defining relations (10) and representation (9) in the proof of Lemma 1 we have $e_K(i,j) = \int_{X \in \mathcal{X}} dX \sqrt{p(X|S_i)p(X|S_j)}$. For assertion (i) consider for any $\underline{a} \in \mathbb{R}^K$

$$\begin{aligned} \underline{a}^T E_K \underline{a} &= \sum_{i=1}^K \sum_{j=1}^K a_i e_K(i,j) a_j \\ &= \int_{X \in \mathcal{X}} dX \sum_{i=1}^K \sum_{j=1}^K a_i a_j \sqrt{p(X|S_i)p(X|S_j)} \\ &= \int_{X \in \mathcal{X}} dX \left(\sum_{i=1}^K a_i \sqrt{p(X|S_i)} \right)^2 \geq 0. \end{aligned}$$

Thus $\underline{a}^T E_K \underline{a}$ is equal to zero iff $\sum_{i=1}^K a_i \sqrt{p(X|S_i)} = 0$ almost everywhere (X). In view of (2): $\sum_{i=1}^K a_i \sqrt{p(X|S_i)} = \sum_{i=1}^K \tilde{a}_i \tilde{p}(X|S_i)$ where $\tilde{p}(X|S_i)$ is a multivariate circular complex Gaussian density with zero mean and covariance matrix $2[I_T + \eta S_i S_i^H]$ and $\tilde{a}_i = a_i 2^{TN} \pi^{TN/2} |I_T + S_i S_i^H|^{N/2}$. For a reference signal $S_1 \in \mathcal{S}$ an easy calculation establishes that for any $S_2 \in \mathcal{S}$ the Kullback-Leibler divergence [5] of $\tilde{p}(X|S_2)$ from $\tilde{p}(X|S_1)$ is

$$\begin{aligned} K(S_1 \| S_2) &= \int_{X \in \mathcal{X}} \tilde{p}(X|S_1) \ln\left(\tilde{p}(X|S_1)/\tilde{p}(X|S_2)\right) dX \\ &= N \sum_{i=1}^K [(\sigma_i - 1) - \ln \sigma_i] \end{aligned}$$

where $\{\sigma_i\}_{i=1}^K$ are the eigenvalues of the positive definite matrix $A = [I_T + \eta S_2 S_2^H]^{-\frac{1}{2}}[I_T + \eta S_1 S_1^H][I_T + \eta S_2 S_2^H]^{-\frac{1}{2}}$. As $(\sigma_i - 1) - \ln \sigma_i \geq 0$, with equality iff $\sigma_i = 1$, we see that $K(S_1 \| S_2) = 0$ iff $A = I_K$, as A is symmetric. Thus $K(S_1 \| S_2) = 0$ iff $S_1 S_1^H = S_2 S_2^H$ and therefore for distinct $\{S_i S_i^H\}_{i=1}^K$ the density functions $\{\tilde{p}(X|S_i)\}_{i=1}^K$ are almost everywhere linearly independent functions of X [11]. Gathering together the above results: if $\{S_i S_i^H\}_{i=1}^K$ are distinct then $\sum_{i=1}^K a_i \sqrt{p(X|S_i)} = 0$ (a.e.) implies $\{a_i\}_{i=1}^K$ are identically zero. Hence E_K is positive definite.

As for assertion (ii) of the Lemma we have by Cauchy-Schwarz

$$\begin{aligned} e_K^2(i,j) &= \left(\int_{X \in \mathcal{X}} dX \sqrt{p(X|S_i) p(X|S_j)}\right)^2 \\ &\leq \int_{X \in \mathcal{X}} p(X|S_i) dX \int_{X \in \mathcal{X}} p(X|S_j) dX \end{aligned}$$

with equality iff $p(X|S_i) = p(X|S_j)$ (a.e. X). As shown above, this equality condition can only occur if $S_i S_i^H = S_j S_j^H$. Hence $e_K(i,j) < 1$ under the hypotheses of the lemma. □

4.3 A Bound on Minimum Distance

Recall the definition $D_{\min} = \min_{i \neq j} D(S_i \| S_j)$. When $K \leq \lfloor T/M \rfloor$ we will see (Proposition 9) that a set of signal matrices $\{S_i\}_{i=1}^K$ can exist for which $D(S_i \| S_j) = D_{\min}$ for all $i \neq j$, and which simultaneously attains the cut-off rate $\tilde{R}_o(K)$ and attains the maximum possible value $D_{\min}$. Such constellations do not exist for $K > \lfloor T/M \rfloor$. The following result establishes a lower bound on the largest possible value $D_{\min}^{**}$ of $D_{\min}$. A constellation $\{S_i\}_{i=1}^K$ achieving $D_{\min} = D_{\min}^{**}$ is called a maximin constellation of dimension K.

Define

$$D_{\min}^{**} \stackrel{\text{def}}{=} \max_{\{S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K} \min_{i \neq j} D(S_i \| S_j).$$

For positive integer p let $\mathcal{L}_{p,T}$ be the uniform lattice of 2^{pT} points covering the T-dimensional unit cube: $\mathcal{L}_{p,T} = \{(i_1, \ldots, i_T) : \ i_l \in \{0, 1/(2^p - 1), 2/(2^p - 1), \ldots, 1\}\}$. Define the integer valued function

$$g_T(q) = \sum_{i_T=0}^{q} \sum_{i_{T-1}=0}^{q-i_T} \cdots \sum_{i_1=0}^{q-i_T-\cdots-i_2} .$$

For example, $g_2(q) = (q+1)(q+2)/2$ and $g_3(q) = (q+1)(q+2)(q+3)/6$. $g_T(2^p - 1)$ is the number of lattice points of $\mathcal{L}_{p,T}$ which are inside of the T-dimensional unit simplex $\left\{(u_1, \ldots, u_T) : \ \sum_{l=1}^T u_l \leq 1, \ u_l \in [0,1]\right\}$ (See Fig. 1). The following bounds are easily established

$$2^{(p-1)T} < g_T(2^p - 1) < 2^{pT}.$$

The upper bound follows from the fact that the number of points of $\mathcal{L}_{p,T}$ in the unit simplex must be less than the number (2^{pT}) of points of $\mathcal{L}_{p,T}$ in the unit cube containing the unit simplex. The lower bound follows from the fact that the number of points of $\mathcal{L}_{p,T}$ in the unit simplex must be greater than the number ($2^{(p-1)T}$) of points of $\mathcal{L}_{p,T}$ in the cube of side 1/2 and volume 2^{-T} which is contained inside the unit simplex.

Proposition 2 *For given $K > 1$ let p be the unique integer for which $g_T(2^{p-1}-1) < K \leq g_T(2^p-1)$. Then*

$$D^{**}_{\min} \geq \frac{\eta^2}{8}\frac{(TM)^2}{(2^p-1)^2} + o(\eta^2) > \frac{\eta^2(TM)^2}{128}K^{-2/T} + o(\eta^2),. \tag{24}$$

The first inequality is tight when $K = g_T(2^p-1)$ and the anti-diagonal matrices $Z_i = S_iS_i^H - \text{diag}(S_iS_i^H)$ satisfy $\|Z_i - Z_j\| = 0$, $i \neq j$.

The proposition says that, for sufficiently small SNR η^2, D^{**}_{min} cannot asymptotically decrease to zero faster than rate $K^{-2/T}$. This asymptotic rate can be achieved, for example, in the case that $M = T$, the $\{S_iS_i^H\}_{i=1}^K$'s are diagonal and form the constellation of a linearly constrained lattice code [4] in $\mathbb{R}^T$. Specifically, each S_i is a square matrix with orthogonal rows and the vectors of diagonals of the $S_iS_i^H$ matrices are locations of the tie points of the K point lattice shown in Fig. 1.

Proof of Proposition 2

Define the $T \times T$ non-negative definite Hermitian matrix $A_i = S_iS_i^H$. By Lemma 8 and Lemma 9 (see Appendix)

$$D(S_i\|S_j) = \eta^2/8\|A_i - A_j\|^2 + o(\eta^2) \geq \eta^2/8 \sum_{l=1}^T (a^i_{ll} - a^j_{ll})^2 + o(\eta^2),$$

where $\{a^i_{ll}\}_{l=1}^T$ are the diagonal entries of A_i. Therefore, recalling the definition of $\mathcal{S}^K_{\text{peak}}$ and using $\|S_i\|^2 = \sum_{l=1}^T a^i_{ll}$:

$$\max_{\{S_i\}_{i=1}^K \in \mathcal{S}^K_{\text{peak}}} \min_{i\neq j} D(S_i\|S_j) \geq \eta^2/8 \ D^2_* + o(\eta^2), \tag{25}$$

where

$$D^2_* = \max_{\underline{a}_i \in \mathcal{A}_{\text{peak}}} \min_{i\neq j} \sum_{l=1}^T (a^i_{ll} - a^j_{ll})^2,$$

and we have defined the T-element non-negative real vector $\underline{a}_i = [a^i_{11}, \ldots, a^i_{TT}]^T$ and the peak constrained feasible set

$$\mathcal{A}_{\text{peak}} = \left\{\underline{a}_i \in \mathbb{R}^T_+, \ \sum_{l=1}^T a^i_{ll} \leq TM, \ i = 1, \ldots, K\right\},$$

where $\mathbb{R}^T_+ = \{\underline{x} \in \mathbb{R}^T : x_i \geq 0, i = 1, \ldots, T\}$ is the non-negative orthant of $\mathbb{R}^T$.

Since $2^{(p-1)T} < g_T(2^p-1) < 2^{pT}$, under the hypothesis $g_T(2^{p-1}-1) < K \leq g_T(2^p-1)$ of the Proposition we have $2^{(p-2)} < g_T^{1/T}(2^{p-1}-1) < K^{1/T}$. Hence, $2^p < 4K^{1/T}$ which establishes the second (strict) inequality on the RHS of (24) given the first inequality (alternatively see Fig. 2). Turning to the first inequality, define the uniform lattice $\mathcal{L}_{p,T}(TM)$ of 2^{pT} points in the T-dimensional cube of side TM which just contains $\mathcal{A}_{\text{peak}}$ (see Fig. 1). This lattice has knot locations $\{(n_1 TM/(2^p-1), \ldots, n_T TM/(2^p-1)) : n_1, \ldots, n_T \in \{0, 1, \ldots, 2^p-1\}\}$. Consider a test set of K codewords having associated vectors $\{\underline{a}_i\}_{i=1}^K$ which are placed on a subset of the $g_T(2^p-1)$ knot locations lying in the feasible set $\mathcal{A}_{\text{peak}}$ and indicated in Fig. 1. This set of $\underline{a}_i$'s has minimum Euclidean distance lower bounded by the interknot spacing $TM/(2^p-1)$ and this specifies a lower bound on D_*. When $K = g_T(2^p-1)$ this is also an upper bound since the set of $\underline{a}_i$'s fully populate a uniform lattice which covers the feasible set (any perturbation of the knot locations in a uniform lattice necessarily reduces the minimum distance). Substituting the lower bound on D_* into the RHS of the bound (25) establishes the inequality. □

5 General Properties of Solutions to R_o

In this and the following subsections we establish properties of solutions to (16) under the peak power constraint (18).

First we give a relation between the average distance of the optimal cut-off rate attaining constellation and the minimum distance of the maximin constellation.

Proposition 3 *Let $\{S_i^*\}_{i=1}^K$ be a constellation that attains the K dimensional cut-off rate $\tilde{R}_o(K)$. Then,*

$$\overline{D(S_i^* \| S_j^*)} \geq D_{\min}^{**},$$

where $\overline{D(S_i^* \| S_j^*)}$ is the mean value

$$\overline{D(S_i^* \| S_j^*)} = \frac{\sum_{i \neq j} P_i^* P_j^* D(S_i^* \| S_j^*)}{\sum_{i \neq j} P_i^* P_j^*}.$$

As $\overline{D(S_i^* \| S_j^*)} \leq \max_{i \neq j} D(S_i^* \| S_j^*)$ an immediate implication of Proposition 3 and Proposition 2 is

$$\max_{i \neq j} D(S_i^* \| S_j^*) \geq \frac{\eta^2 (TM)^2}{128} K^{-2/T} (1 + o(\eta^2)).$$

Proof of Proposition 3

Let $\{P_i^*\}_{i=1}^K$ achieve the minimum value $\tilde{Q}^*$ of the quadratic form $\tilde{Q}\left(\{P_i\}_{i=1}^K, \{S_i^*\}_{i=1}^K\right)$ in (22). First we show that

$$\frac{\sum_{i \neq j} P_i^* P_j^* e^{-ND(S_i^* \| S_j^*)}}{\sum_{i \neq j} P_i^* P_j^*} \leq e^{-ND_{\min}^{**}}, \tag{26}$$

Assume the contrary. Then

$$\begin{aligned}
\tilde{Q}^* &= \sum_{i\neq j}^{K} P_i^* P_j^* e^{-ND(S_i^*\|S_j^*)} + \sum_{i=1}^{K}(P_i^*)^2 \\
&> e^{-ND^{**}} \sum_{i\neq j}^{K} P_i^* P_j^* + \sum_{i=1}^{K}(P_i^*)^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{Q}^* &\leq \min_{\{S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K} \left\{ \sum_{i\neq j}^{K} P_i^* P_j^* e^{-ND(S_i\|S_j)} + \sum_{i=1}^{K}(P_i^*)^2 \right\} \\
&\leq \min_{\{S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K} \left\{ e^{-N \min_{i\neq j} D(S_i\|S_j)} \right\} \sum_{i\neq j}^{K} P_i^* P_j^* + \sum_{i=1}^{K}(P_i^*)^2 \\
&= e^{-ND_{\min}^{**}} \sum_{i\neq j}^{K} P_i^* P_j^* + \sum_{i=1}^{K}(P_i^*)^2,
\end{aligned}$$

which is a contradiction. The Proposition follows after application of Jensen's inequality to the left hand side of (26).

5.1 Optimality of Equalizer Distributions

The solution $\underline{P}_K^* = [P_1^*, \ldots, P_K^*]^T$ to the minimization (22) can be found by convex quadratic optimization subject to linear equality and inequality constraints. Define the Lagrangian

$$J(\underline{P}_K) = \underline{P}_K^T E_K \underline{P}_K - 2c(\underline{1}_K^T \underline{P}_K - 1), \tag{27}$$

where, $\underline{P}_K = [P_1, \ldots, P_K]^T$, $\underline{1}_K = [1, \ldots, 1]^T$ is a K-element vector of ones, and $c > 0$ is an undetermined multiplier that must be chosen to enforce the equality constraint $\underline{1}_K^T \underline{P} = 1$. As $J(\underline{P})$ is convex cup the Kuhn-Tucker conditions [19] assert that the minimum exists and must satisfy

$$\frac{\partial J(\underline{P}_K^*)}{\partial P_i^*} \begin{cases} = 0 & \text{if} \quad P_i^* > 0 \\ \geq 0 & \text{if} \quad P_i^* = 0 \end{cases}$$

or equivalently

$$\underline{\varepsilon}_K^T(i)\underline{P}_K^* = c \quad \text{if} \quad P_i^* > 0 \tag{28}$$

$$\underline{\varepsilon}_K^T(i)\underline{P}_K^* \geq c \quad \text{if} \quad P_i^* = 0 \tag{29}$$

where $\underline{\varepsilon}_K(i)$ is the i-th column of E_K. By reordering the entries of $\underline{P}_K^*$ and the rows and columns of E_K we can assume, without loss of generality, that the $\underline{\varepsilon}_K(i)$,'s satisfying (28) are the first k columns of E_K with the remainder satisfying (29), $k \leq K$. Making this assumption, (28) and (29) imply that $P_{k+1}^* = \ldots = P_K^* = 0$ and

$$E_k \underline{P}_k^* = c\underline{1}_k. \tag{30}$$

Furthermore, we can assume that $\{S_i S_i^H\}_{i=1}^K$ are distinct (see Lemma 10) and therefore, by Lemma 3, E_k is positive definite. Therefore, from (28)-(30) the strictly positive component of the minimizer $\underline{P}_K^*$ is given by

$$\underline{P}_k^* = cE_k^{-1}\underline{1}_k, \tag{31}$$

where c is determined via the constraint $\underline{1}_K^T \underline{P}_K^* = \underline{1}_k^T \underline{P}_k^* = 1$, or

$$c = 1/\underline{1}_k^T E_k^{-1} \underline{1}_k. \tag{32}$$

Note that $c > 0$ since E_k is positive definite and $c \le 1$ since $(\underline{1}_k^T E_k^{-1}\underline{1}_k)(\underline{1}_k^T E_k \underline{1}_k) \ge (\underline{1}^T \underline{1})^2 = k^2$ and $\underline{1}^T E_k \underline{1} \le k^2$ as the elements of E_k are ≤ 1. Thus, in view of (30), the strictly positive elements of $\underline{P}_K^*$ satisfy the finite dimensional analog to the equalization condition (12)

$$\sum_{j=1}^k P_j^* e^{-ND(S_j \| S_i)} = c, \qquad i = 1, \ldots, k,$$

where $\{S_i\}_{i=1}^k$ are the signal matrices in $\{S_i\}_{i=1}^K$ with strictly positive assigned probabilities $\{P_i^*\}_{i=1}^k$.

We have thus shown that optimal finite dimensional constellations, defined in Definition 1, must have equalizer distributions. These results are summarized in the following.

Proposition 4 *A constellation of dimension K achieves the K dimensional peak constrained cut-off rate (16) only if the optimal distribution $\{P_i^*\}_{i=1}^K$ over signal matrices in the constellation is an equalizer distribution of the form $\underline{P}_K^* = E_K^{-1}\underline{1}_K / \underline{1}_K^T E_K^{-1} \underline{1}_K$.*

An equivalent condition to (30) is that there exist a vector $\underline{x} = [x_1, \ldots, x_k]^T$, not identically zero, lying in the positive orthant $\mathbb{R}_+^k$ which satisfies

$$E_k \underline{x} = \underline{1}_k, \tag{33}$$

or, equivalently, $E_k^{-1}\underline{1}_k \in \mathbb{R}_+^k$. Define the feasibility set $\tilde{\mathcal{S}}_{\text{peak}}^K$ of K-dimensional constellations

$$\tilde{\mathcal{S}}_{\text{peak}}^K = \left\{ \{S_i\}_{i=1}^K : S_i \in \mathbb{C}^{TM},\ \|S_i\|^2 \le TM, E_K^{-1}\underline{1}_K \in \mathbb{R}_+^K,\ S_i S_i^H \ne S_j S_j^H, i \ne j \right\}. \tag{34}$$

By Proposition 4 $\tilde{R}_o(K)$ is attained by a constellation of dimension K if and only if $\tilde{\mathcal{S}}_{\text{peak}}^K$ is non-empty. Substitution of the form of the optimal probability vector $\underline{P}_K^*$ specified in Proposition (4) into (17) we obtain the following alternative characterization of $\tilde{R}_o(K)$.

Proposition 5 *Let K be a positive integer. The peak power constrained K dimensional cut-off rate is*

$$\tilde{R}_o(K) \quad = \quad \ln \left\{ \max_{0<k\le K} \max_{\{S_i\}_{i=1}^k \in \tilde{\mathcal{S}}_{\text{peak}}^k} \underline{1}_k^T E_k^{-1} \underline{1}_k \right\}. \tag{35}$$

Observe that by taking the limit $K \to \infty$ in Proposition 5, we obtain the cut-off rate of constellations of countable, but possibly infinite, dimension. The objective function $\underline{1}_k^T E_k^{-1} \underline{1}_k$ maximized in (35) is similar to the criterion used in Capon's method, also known as minimum variance distortionless response (MVDR), for adapting the weights of a beamforming array of antenna elements and for high resolution spectral estimation [12].

The cut-off rate $\tilde{R}_o(K)$ and the rate achieving signal set can be iteratively computed using (35) in the following steps: (1) select a candidate set of signal matrices $\{S_i\}_{i=1}^K$ in $\mathcal{S}_{\text{peak}}^K$, (2) use quadratic programming to find the zero entries of $\underline{P}_K^*$, specified by condition (29), e.g. using slack variable or active set methods [16]; (3) reorder the zero and non-zero entries of $\underline{P}_K^*$ and permute the rows and columns of E_K as described above; (4) solve $E_K \underline{x} = \underline{1}_K$ for $\underline{x}$; (5) compute inner product $\underline{1}^T \underline{x}$ and take its natural logarithm; (6) perturb the candidate set of signal matrices and repeat (1)-(5). This iterative procedure is repeated for each new set of candidate signal matrices until $1/\underline{1}_K^T E_K^{-1} \underline{1}_K$ attains a maximum. The procedure becomes computationally heavy when K becomes large. Proposition 5 suggests an alternative optimization method which bypasses steps (1) and (2) by preselecting the candidate set of signal matrices $\{S_i\}_{i=1}^K$ to lie in $\tilde{\mathcal{S}}_{\text{peak}}^K$.

It will thus be of interest to establish conditions on E_K which guarantee existence of a positive vector $\underline{x}$ satisfying (33) or, equivalently, which guarantee that the vector $E_K^{-1} \underline{1}_K$ lies in the positive orthant. The first result in this direction is that this always holds for a sufficiently large number N of receiver antennas.

5.2 Large Number of Receiver Antennas

Proposition 6 *Let $\{S_i\}_{i=1}^K$ be a set of peak constrained signal matrices with distinct outerproducts $\{S_i S_i^H\}_{i=1}^K$. Define the finite positive integer N_o*

$$N_o = \lfloor \frac{\ln(K-1)}{D_{\min}} \rfloor + 1, \tag{36}$$

where $D_{\min} = \min_{i \neq j} D(S_i \| S_j)$. Then, for $N \geq N_o$ the optimal distribution defined in (22) is

$$\underline{P}_K^* = E_K^{-1} \underline{1}_K / \underline{1}_K^T E_K^{-1} \underline{1}_K = (1 + O(\delta^{N-N_o+1}))\, \underline{1}_K / K, \tag{37}$$

where $\delta = \exp(-D_{\min})$.

The significance of the above proposition is that if K is no greater than $\exp(N D_{\min}) + 1$ any peak constrained set of signal matrices $\{S_i\}_{i=1}^K$ is in the feasible set $\tilde{\mathcal{S}}_{\text{peak}}^K$ and thus has an equalizer distribution $\underline{P}^*(S_1, \ldots, S_K)$. Furthermore, as $N \to \infty$, $\underline{P}_K^*$ converges to the equiprobable distribution $\underline{1}_K / K$. Finally, by Proposition 2 we have the useful bound on N_o for the constellation having best minimum distance properties: $N_o \leq \lfloor 128K^{2/T}\ln(K-1)/(\eta^2(TM)^2)\rfloor + 1$ (η^2 small).

As $\tilde{R}_o(K)$ is strictly increasing in K (see Lemma 11 in Appendix) Proposition 6 implies that the size of the optimal countable constellation increases without bound as N goes to infinity. Indeed, by Proposition 6, for any given K and $D_{\min} > 0$ there exists an N_o such that for $N \geq N_o$: $\{S_i\}_{i=1}^K \in \tilde{\mathcal{S}}_{\text{peak}}^K$ for all constellations $\{S_i\}_{i=1}^K$ having $D(S_i\|S_j) \geq D_{\min}$, $i \neq j$. Optimizing over all such constellations gives $\tilde{R}_o(K)$ by Proposition 5.

Proof of Proposition 6

First observe that under the hypothesis of the proposition $N_o < \infty$ since, by Lemma 3, $D(S_i\|S_j) > 0$, $i \neq j$. Define the symmetric matrix $\Delta = E_K - I_K$. The elements $\{d_{i,j}\}$ of Δ are $d_{i,j} = e^{-ND(S_i\|S_j)} \leq \delta^N$, $i \neq j$, and $d_{i,i} = 0$. Therefore $0 \leq d_{i,j} < 1$. When $N \geq N_o$ the eigenvalues σ_i^Δ of Δ are bounded

$$|\sigma_i^\Delta| \leq \delta^N(K-1) < 1, \quad i = 1, \ldots, K. \tag{38}$$

The left hand inequality of (38) follows from the sequence of inequalities

$$\begin{aligned}
\max_i |\sigma_i^\Delta| &\leq \max_{\underline{y}\in\mathbb{R}^K} \frac{|\underline{y}^T \Delta \underline{y}|}{\underline{y}^T \underline{y}} = \max_{\underline{y}\in\mathbb{R}^K} \frac{\left|\sum_{i,j=1}^K y_i y_j d_{i,j}\right|}{\sum_{i=1}^K y_i^2} \\
&\leq \max_{\underline{y}\in\mathbb{R}^K} \frac{\left|\sum_{i=1}^K y_i^2 \sum_{j=1}^K d_{i,j}\right|}{\sum_{i=1}^K y_i^2} \\
&\leq \max_i \sum_{j=1}^K d_{i,j} \\
&\leq \delta^N(K-1).
\end{aligned}$$

The inequality on the second line follows from Cauchy-Schwarz

$$\left(\sum_{i,j=1}^K (y_i\sqrt{d_{i,j}})(y_j\sqrt{d_{i,j}})\right)^2 \leq \left(\sum_{i,j=1}^K (y_i\sqrt{d_{i,j}})^2\right)^2,$$

and the inequality on the third line follows from non-negativity of $d_{i,j}$. The right hand inequality of (38) follows from

$$\delta^N(K-1) = \delta^{N_o}(K-1)\delta^{N-N_o} \leq \delta^{N-N_o+1} < 1, \quad N \geq N_o. \tag{39}$$

Now (33) can be expressed as the perturbed linear system by expressing $E_K = I_K + \Delta$

$$[I_K + \Delta]\underline{x} = \underline{1}_K.$$

As E_K is positive definite $[I_K + \Delta]$ is invertible. Furthermore, from (38) the eigenvalues of Δ^2 are strictly less than one and by elementary matrix manipulations

$$\underline{x} = [I_K + \Delta]^{-1}\underline{1}_K = [I_K - \Delta^2]^{-1}[I_K - \Delta]\underline{1}_K.$$

The i-th element of the vector $[I_K - \Delta]\underline{1}_K$ is $2 - \sum_{j=1}^K d_{i,j} \geq 1 - (K-1)\delta^N$ which by (39) is positive for $N \geq N_o$. Also, $[I_K - \Delta^2]^{-1} = I_K + \sum_{m=1}^\infty \Delta^{2m}$ has positive entries. This implies that for $N \geq N_o$, $\underline{x} \in \mathbb{R}_+^K$ and the i-th element of $\Delta\underline{1}_K$ is upper bounded by $(K-1)\delta^N$ and consequently, as it is non-negative, of order $O(\delta^{N-N_o+1})$ by (39). Hence, $\underline{x} = [I_K + \Delta]^{-1}\underline{1}_K = \underline{1}_K + O(\Delta\underline{1}_K) = \underline{1}_K(1 + O(\delta^{N-N_o+1}))$. $\square$

From the proof of Proposition 6 we can identify a weaker condition on the number of receiver antennas necessary to ensure $\underline{P}_K^*$ be an equalizer distribution: N be sufficiently large to make E_K a diagonally dominant matrix, i.e. to make $\sum_{j=1}^K e^{-ND(S_i\|S_j)} < 1$, $i = 1, \ldots, K$.

6 Characterization of Feasible Set of Constellations

Here we derive sufficient conditions and necessary conditions for $E_K^{-1}\underline{1}_K$ to lie in the positive orthant. Several of these conditions will be defined recursively in K. So as to not confuse the reader, and to improve clarity of the equations, we will use lower case k throughout this section to distinguish it from the fixed index K.

For a square $k \times k$ matrix A consisting of columns $\underline{a}_1, \ldots, \underline{a}_k$ the positive polyhedral cone generated by A is [2, 19]

$$\text{cone}(A) = \{A\underline{x} : \underline{x} \in \mathbb{R}_+^k\} = \left\{\sum_{i=1}^k \underline{a}_i x_i : x_i > 0,\ i = 1, \ldots, k\right\}.$$

This cone is convex and each of its k faces are contained in one of the hyper-planes $\text{span}\{\underline{a}_1, \ldots, \underline{a}_{i-1}, \underline{a}_{i+1}, \ldots, \underline{a}_k\}$. The extremals of cone$(A)$ are the k positive rays $\{c\underline{a}_i : c > 0\}$, $i = 1, \ldots, k$.

It is evident that if cone(E_k) contains the vector $\underline{1}_k$ then, as E_k is positive definite, $\underline{x} = E_k^{-1}\underline{1}$ must have positive elements, which would guarantee that $\underline{P}_k^* = \underline{x}/\underline{1}_k^T\underline{x}$ is the optimal equalizer distribution.

6.1 A Necessary Condition and a Sufficient Condition

Here we give two conditions, one sufficient and one necessary, to ensure $\underline{1}_k \in \text{cone}(E_k)$. The sufficient condition reduces to specifying the largest inscribed right circular cone which fits inside cone(E_k) while the necessary condition is equivalent to specifying the smallest right circular cone which contains cone(E_k). This sufficient condition is slightly weaker than the sufficient condition presented in Lemma 5 of the next section and obtained by recursive inscription of a polyhedral cone. On the other hand, as contrasted with the condition of Lemma 5, the sufficient condition in this section is not easy to express recursively in k

As in (28)-(29), $\underline{\varepsilon}_k(i)$ will denote the i-th column of $E_k = ((e_k(i,j)))_{i,j=1}^k$. For $i = 1, \ldots, k$, define

$E_k(-i)$ the $k \times (k-1)$ matrix obtained by deleting its i-th column. Define

$$\Pi_{E_k(-i)} = E_k(-i)[E_k^T(-i)E_k(-i)]^{-1}E_k^T(-i) \tag{40}$$

the idempotent $k \times k$ matrix which orthogonally projects vectors in $\mathbb{R}^k$ onto the column span of $E_k(-i)$.

Lemma 4 *Let $\underline{u}$ be any vector in* $\text{cone}(E_k)$. *A sufficient condition for $\underline{1}_k$ to be contained in* $\text{cone}(E_k)$ *is*

$$\left\| \underline{u} - \frac{\underline{u}^T \underline{1}_k}{\|\underline{1}_k\|^2} \underline{1}_k \right\|^2 < \min_i \underline{u}^T [I - \Pi_{E_k(-i)}] \underline{u}. \tag{41}$$

A necessary condition for $\underline{1}_k$ to be contained in $\text{cone}(E_k)$ *is*

$$\left\| \underline{u} - \frac{\underline{u}^T \underline{1}_k}{\|\underline{1}_k\|^2} \underline{1}_k \right\|^2 < \max_i \left\| \underline{u} - \frac{\underline{u}^T \underline{\varepsilon}_k(i)}{\|\underline{\varepsilon}_k(i)\|^2} \, \underline{\varepsilon}_k(i) \right\|^2. \tag{42}$$

Both of the conditions specified in Lemma 4 require that the test vector $\underline{u}$ be close to the ray $\{c\underline{1}_k : c > 0\}$. As a particularly simple application of the Lemma, consider setting $\underline{u} = \underline{\bar{\varepsilon}}_k = E_k \underline{1}_k / k$; the arithmetic mean of the columns of E_k. In this case, as all but the i-th column of E_k are orthogonal to $I_k - \Pi_{E_k(-i)}$, the sufficient condition becomes

$$\left\| \underline{\bar{\varepsilon}}_k - \frac{\underline{\bar{\varepsilon}}_k \underline{1}_k}{\|\underline{1}_k\|^2} \underline{1}_k \right\|^2 < \frac{1}{k^2} \; \min_i \underline{\varepsilon}_k^T(i) [I - \Pi_{E_k(-i)}] \underline{\varepsilon}_k(i).$$

This condition requires that the distance between $\underline{1}_k$ and $\underline{\bar{\varepsilon}}_k$ be less than k^{-2} times the prediction error squared of the most predictable column of E_k using the remaining columns of E_k as predictor vectors.

Proof of Lemma 4

Fix $c > 0$ and let $\underline{z} = c\underline{u}$. As $\underline{z} \in \text{cone}(E_k)$ there exists a sphere centered at $\underline{z}$ that fits inside $\text{cone}(E_k)$ (See Figure 3). Such a sphere can have radius b no larger than the distance between $\underline{z}$ and the closest face of $\text{cone}(E_k)$. As the k faces of $\text{cone}(E_k)$ are contained in the column span of $E_k(-i)$, $i = 1, \ldots, k$ and $\text{cone}(E_k)$ is convex, the radius b of the largest sphere is given by the distance between $\underline{z}$ and its orthogonal projection onto this span

$$\begin{aligned} b^2 &= \min_i \left\{ \min_{\underline{\hat{y}} \in \text{span}\{E_k(-i)\}} \{ \|\underline{z} - \underline{\hat{y}}\|^2 \} \right\} \\ &= \min_i \underline{z}^T [I - \Pi_{E_k(-i)}] \underline{z} \\ &= c^2 \; \min_i \underline{u}^T [I - \Pi_{E_k(-i)}] \underline{u}, \end{aligned}$$

which is the right hand side of (41). If the positive ray $\{a\underline{1}_k : a > 0\}$ passes through this sphere then, by convexity of $\text{cone}(E_k)$, $\underline{1}_k$ must lie in $\text{cone}(E_k)$. This occurs iff the distance from $\underline{z}$ to this ray is less than b^2, i.e.

$$\left\| \underline{z} - \frac{\underline{z}^T \underline{1}_k}{\|\underline{1}_k\|^2} \; \underline{1}_k \right\|^2 = c^2 \left\| \underline{u} - \frac{\underline{u}^T \underline{1}_k}{\|\underline{1}_k\|^2} \; \underline{1}_k \right\|^2 < b^2$$

Since $c > 0$ is arbitrary, we obtain the inequality (41).

As for the necessary condition, for any $c > 0$ there exists a sphere centered at $\underline{z} = c\underline{u}$ through which all positive rays in cone(E_k) must pass (See Figure 3). The smallest such sphere has radius equal to the maximum distance between $\underline{z}$ and the faces of cone(E_k). Points at maximum distance must occur along one of the rays $\{a\underline{\varepsilon}_k(i) : a > 0\}$, $i = 1, \ldots, k$, which are the extremals of cone(E_k). Therefore the radius of this smallest sphere is

$$d^2 = \max_i \left\| \underline{z} - \frac{\underline{z}^T \underline{\varepsilon}_k(i)}{\|\underline{\varepsilon}_k(i)\|^2} \, \underline{\varepsilon}_k(i) \right\|^2 = c^2 \, \max_i \left\| \underline{u} - \frac{\underline{u}^T \underline{\varepsilon}_k(i)}{\|\underline{\varepsilon}_k(i)\|^2} \, \underline{\varepsilon}_k(i) \right\|^2 .$$

Finally, if $\underline{1}_k \in$ cone(E_k) this sphere must intersect the ray $\{a\underline{1}_k : a > 0\}$ in which case $c^2 \left\| \underline{u} - (\underline{u}^T \underline{1}_k)/\|\underline{1}_k\|^2 \, \underline{1}_k \right\|^2 < d^2$. □

6.2 A Recursive Necessary and Sufficient Condition

Here the objective will be to specify conditions for which $\underline{1}_{k+1}$ lies in cone(E_{k+1}) when it is known that $\underline{1}_k$ lies in cone(E_k). To proceed we will need a recursive update for $E_{k+1}^{-1}\underline{1}_{k+1}$ in terms of $E_k^{-1}\underline{1}_k$. For a set of signal matrices $S_1, \ldots, S_{k+1}$ let the $(k+1) \times (k+1)$ matrix E_{k+1}be partitioned as

$$E_{k+1} = \begin{bmatrix} E_k & \underline{e}_k \\ \underline{e}_k^T & 1 \end{bmatrix}, \tag{43}$$

where $\underline{e}_k$ is the vector of pairwise dissimilarity functions

$$\underline{e}_k = \left[e^{-ND(S_1 \| S_{k+1})}, \ldots, e^{-ND(S_k \| S_{k+1})} \right]^T . \tag{44}$$

We also recall the partitioned matrix inverse identity for the case that E_{k+1} is positive definite

$$E_{k+1}^{-1} = \begin{bmatrix} E_k^{-1} + \frac{1}{|E_{k+1}|} E_k^{-1} \underline{e}_k \underline{e}_k^T E_k^{-1} & -\frac{1}{|E_{k+1}|} E_k^{-1} \underline{e}_k \\ -\frac{1}{|E_{k+1}|} \underline{e}_k^T E_k^{-1} & \frac{1}{|E_{k+1}|} \end{bmatrix} \tag{45}$$

where $|E_{k+1}| = 1 - \underline{e}^T E_k^{-1} \underline{e} > 0$. Let $\underline{x}^k = E_k^{-1}\underline{1}_k$ have partitioned form $\underline{x}^k = [\underline{x}_{k-1}^k, x_k^k]^T$ where x_k^k is a scalar.

Proposition 7 *For a given set of signal matrices $\{S_i\}_{i=1}^{k+1}$ with distinct outerproducts $\{S_i S_i^H\}_{i=1}^{k+1}$*

$$\begin{bmatrix} \underline{x}_k^{k+1} \\ x_{k+1}^{k+1} \end{bmatrix} = \begin{bmatrix} E_k^{-1}(\underline{1}_k - \alpha_k \underline{e}_k) \\ \alpha_k \end{bmatrix} = \begin{bmatrix} \underline{x}^k - \alpha_k E_k^{-1} \underline{e}_k \\ \alpha_k \end{bmatrix}, \tag{46}$$

where

$$\alpha_k = \frac{1 - \underline{e}_k^T E_k^{-1} \underline{1}_k}{1 - \underline{e}_k^T E_k^{-1} \underline{e}_k}. \tag{47}$$

Proof of Proposition 7

By Lemma 3 E_{k+1} is positive definite. Applying the partitioned matrix inverse identity (45) to the right hand side of $\underline{x}^{k+1} = E_{k+1}^{-1}\underline{1}_{k+1}$,

$$\begin{aligned} E_{k+1}^{-1}\underline{1}_{k+1} &= \begin{bmatrix} E_k^{-1} + \frac{1}{|E_{k+1}|}E_k^{-1}\underline{e}_k\underline{e}_k^T E_k^{-1} & -\frac{1}{|E_{k+1}|}E_k^{-1}\underline{e}_k \\ -\frac{1}{|E_{k+1}|}\underline{e}_k^T E_k^{-1} & \frac{1}{|E_{k+1}|} \end{bmatrix} \begin{bmatrix} \underline{1}_k \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} E_k^{-1}\underline{1}_k + E_k^{-1}\underline{e}_k \frac{\underline{e}_k^T E_k^{-1}\underline{1}_k - 1}{|E_{k+1}|} \\ \frac{1 - \underline{e}_k^T E_k^{-1}\underline{1}_k}{|E_{k+1}|} \end{bmatrix}. \end{aligned}$$

Using the definitions of $|E_{k+1}|$ and α_k yields (46). □

Corollary 1 *For a given set of signal matrices $\{S_i\}_{i=1}^k$ with distinct outerproducts $\{S_iS_i^H\}_{i=1}^k$ assume that the k-element vector $\underline{x}^k = E_k^{-1}\underline{1}_k$ lies in $\mathbb{R}_+^k$. Let S_{k+1} be a signal such that $S_{k+1}S_{k+1}^H \notin \{S_iS_i^H\}_{i=1}^k$. Then the $(k+1)$-element vector $\underline{x}^{k+1} = E_{k+1}^{-1}\underline{1}_{k+1}$ is in $\mathbb{R}_+^{k+1}$ iff (a) $\alpha_k > 0$ and (b) $E_k^{-1}(\underline{1}_k - \alpha_k\underline{e}_k)$ is in $\mathbb{R}_+^k$. Furthermore (b) implies that $\alpha_k \leq 1$.*

Proof of Corollary 1

It only need be shown that condition (b) of the Corollary implies that $\alpha_k \leq 1$. If this condition holds then $\underline{e}_k^T E_k^{-1}(\underline{1}_k - \alpha_k\underline{e}_k) \geq 0$ as $\underline{e}_k$ has non-negative elements. Therefore

$$\alpha_k\underline{e}_k^T E_k^{-1}\underline{e}_k \leq \underline{e}_k^T E_k^{-1}\underline{1}_k. \tag{48}$$

By definition (47) $\alpha_k(1 - \underline{e}_k^T E_k^{-1}\underline{e}_k) = 1 - \underline{e}_k^T E_k^{-1}\underline{1}_k$. Adding this latter equation to the inequality (48) gives $\alpha_k \leq 1$. □

Corollary 1 motivates an iterative procedure for generating an monotonically improving sequence of constellations lying in the feasibility set $\tilde{\mathcal{S}}_{\text{peak}}^k$ defined in Proposition 5 which is specified in Hero and Marzetta [9]. At iteration k assume that a k-dimensional peak constrained constellation $S_1, \ldots, S_k$ has been constructed whose optimal distribution $\underline{P}_k^*$ attaining the minimum in (22) is an equalizer distribution: $\underline{P}_k^* = cE_k^{-1}\underline{1}_k$. Consider adding a candidate signal S_{k+1}, with $S_{k+1}S_{k+1}^H$ distinct from $\{S_iS_i^H\}_{i=1}^k$, having associated dissimilarity vector $\underline{e}_k$ given by (44). The two conditions (a) and (b) of the corollary may be used to select S_{k+1} to ensure that the optimal distribution $\underline{P}_{k+1}^*$ for the updated constellation $S_1, \ldots, S_{k+1}$ is also an equalizer distribution.

6.3 Feasibility via Polyhedral Inscribed Cones

Here we specify a simpler positive cone which is inscribed inside cone(E_k) and is generated by the positive definite $k \times k$ matrix

$$F_k(\delta) \stackrel{\text{def}}{=} I_k(1-\delta) + \delta\underline{1}_k\underline{1}_k^T, \tag{49}$$

where, $\delta \in [0,1)$. We recall the Sherman-Morrissey-Woodbury identity for the inverse

$$F_k^{-1}(\delta) = \frac{1}{1-\delta}\left[I_k - \frac{\delta}{1+(k-1)\delta}\underline{1}_k\underline{1}_k^T\right]. \tag{50}$$

The set $\mathrm{cone}(F_k(\beta))$ is centered along the positive ray $\{c\underline{1}_k : c > 0\}$. The set $\mathrm{cone}(F_k(\beta))$ is equal to the positive orthant $\mathbb{R}_+^k$ for $\beta = 0$ while it approaches the aforementioned positive ray as β approaches 1. Furthermore $\mathrm{cone}(F_k(\beta))$ is decreasing in β in the sense that $\mathrm{cone}(F_k(\beta)) \subset \mathrm{cone}(F_k(\beta'))$ for $\beta \geq \beta'$.

Define the maximum, minimum, mean, and rms values of a real vector $\underline{a} = [a_1, \ldots, a_k]^T$

$$\min(\underline{a}) = \min_{1\leq i\leq k} a_i, \qquad \max(\underline{a}) = \max_{1\leq i\leq k} a_i, \qquad \mathrm{avg}(\underline{a}) = \underline{1}_k^T\underline{a}/k, \qquad \mathrm{rms}(\underline{a}) = \sqrt{\underline{a}^T\underline{a}/k}.$$

Lemma 5 *Let $\{S_i\}_{i=1}^k$ be a set of signal matrices with distinct outerproducts $\{S_iS_i^H\}_{i=1}^k$ such that $E_k^{-1}\underline{1}_k \in \mathbb{R}_+^k$. Assume that $\mathrm{cone}(F_k(\beta)) \subset \mathrm{cone}(E_k)$ for $0 \leq \beta < 1$. Let S_{k+1} be a signal such that $S_{k+1}S_{k+1}^H \notin \{S_iS_i^H\}_{i=1}^k$. Then the $(k+1)$-element vector $\underline{x}^{k+1} = E_{k+1}^{-1}\underline{1}_{k+1}$ lies in $\mathbb{R}_+^{k+1}$ if*

$$(a) \quad 0 < \alpha_k \leq 1, \quad \text{and} \quad (b) \quad \max(\underline{e}_k) + \tfrac{k\beta}{1-\beta}\left[\max(\underline{e}_k) - \mathrm{avg}(\underline{e}_k)\right] < 1 \tag{51}$$

where $\underline{e}_k$ and α_k are defined in (44) and (47).

An alternative form for (b) in Lemma 5 is:

$$(1-\max(\underline{e}_k))/(1-\mathrm{avg}(\underline{e}_k)) > \theta(\beta), \tag{52}$$

where $\theta(z) \stackrel{\text{def}}{=} kz/(1+(k-1)z) \ \in [0,1)$ is monotonic increasing over $z \in [0,1)$ and, as by Lemma 3 $\mathrm{avg}(\underline{e}_k) < 1$, under condition (a): $1 - \alpha_k\mathrm{avg}(\underline{e}_k) \geq 1 - \mathrm{avg}(\underline{e}_k) > 0$.

Interpretation of the sufficient conditions in Lemma 5 is similar to the interpretation of Lemma 4. Condition (a) is equivalent to

$$\underline{e}_k^T E_k^{-1}\underline{e}_k \leq \underline{1}_k^T E_k^{-1}\underline{e}_k < 1. \tag{53}$$

If $\underline{e}_k$ lies close to the column span of E_k then $1 - \underline{e}_k^T E_k^{-1}\underline{e}_k$ is small and condition (a) restricts the inner product of $\underline{e}_k$ and the previous un-normalized probability vector $\underline{x}^k = E_k^{-1}\underline{1}_k$ to a narrow range near 1. Thus we can expect that constraint (a) will become active only for densely packed signal constellations (large $K/(TM)$). When α_k is close to one, which occurs for the case of small values of $N\max_i D(S_i|S_{k+1})$ (low SNR), condition (b) places restrictions on the elements of feasible vectors $\underline{e}_k$ to ensure low variation about the mean value $\mathrm{avg}(\underline{e}_k)$. This can be ensured if S_{k+1} can be selected to minimize the variation of its pairwise dissimilarities $\{D(S_i\|S_{k+1})\}_{i=1}^k$.

Proof of Lemma 5

Condition (a) of Lemma 5 obviously implies condition (a) of Corollary 1. Under the assumption $\text{cone}(F_k(\beta)) \subset \text{cone}(E_k)$ we will show conditions that (a) and (b) of Lemma 5 jointly imply condition (b) of Corollary 1. First, observe that $E_k^{-1}(\underline{1}_k - \alpha_k \underline{e}_k)$ is in $\mathbb{R}_+^k$ iff $\underline{1}_k - \alpha_k \underline{e}_k \in \text{cone}(E_k)$. Second, since $\text{cone}(E_k)$ contains $\text{cone}(F_k(\beta))$ it will suffice to show that $\underline{1}_k - \alpha_k \underline{e}_k \in \text{cone}(F_k(\beta))$ or, equivalently, $[F_k(\beta)]^{-1}(\underline{1}_k - \alpha_k \underline{e}_k)$ is in $\mathbb{R}_+^k$, under the conditions of Lemma 5. Applying the Sherman-Morrison-Woodbury identity (50) to $[F_k(\beta)]^{-1}$

$$[F_k(\beta)]^{-1}(\underline{1}_k - \alpha_k \underline{e}_k) \quad = \quad \frac{1}{1-\beta}\left[\underline{1}_k - \alpha_k \underline{e}_k - \frac{k\beta}{1+(k-1)\beta}(1-\alpha_k \text{avg}(\underline{e}_k))\underline{1}_k\right]. \tag{54}$$

The minimum element of the vector (54) is

$$\begin{aligned} \min\left([F_k(\beta)]^{-1}(\underline{1}_k - \alpha_k \underline{e}_k)\right) &= \frac{1}{1-\beta}\left[1 - \alpha_k \max(\underline{e}_k) - \frac{k\beta}{1+(k-1)\beta}(1-\alpha_k \text{avg}(\underline{e}_k))\right] \\ &= \frac{1}{1+(k-1)\beta}\left[1 - \alpha_k\left(\max(\underline{e}_k) + \frac{k\beta}{1-\beta}[\max(\underline{e}_k) - \text{avg}(\underline{e}_k)]\right)\right] \end{aligned}$$

which is positive under conditions (a) and (b) of the Lemma. Thus $[F_k(\beta)]^{-1}(\underline{1}_k - \alpha_k \underline{e}_k)$ is in the positive orthant which establishes condition (b) of Corollary 1. □

Since E_k is positive definite and $\text{cone}(F_k(\beta))$ and $\text{cone}(E_k)$ are convex, $\underline{1}_k$ does not lie on the boundary of $\text{cone}(E_k)$ and there always exists a value $\beta_o > 0$ such that $\text{cone}(F_k(\beta)) \subset \text{cone}(E_k)$ for all $\beta \in [\beta_o, 1)$. As the set $\text{cone}(F_k(\beta))$ is monotone decreasing in β the largest possible inscribed cone is obtained by using the minimum possible value of β. This gives the least restrictive sufficient condition (b) in (51). The form of this minimum β is specified for $k \geq 2$ in the Lemma below.

Lemma 6 *For a given set of signal matrices $\{S_i\}_{i=1}^k$ with distinct outerproducts $\{S_i S_i^H\}_{i=1}^k$ assume that the k-element vector $\underline{x}^k = E_k^{-1}\underline{1}_k$ lies in $\mathbb{R}_+^k$. For $k \geq 2$ the minimum value of $\beta \in [0,1)$ ensuring* $\text{cone}(F_k(\beta)) \subset \text{cone}(E_k)$ *is the value β_k given by*

$$\beta_k = \begin{cases} \frac{(k-1) - k\sqrt{(k-1)\gamma_{\min}(1-\gamma_{\min})}}{(k-1)(1-k\gamma_{\min})}, & \gamma_{\min} \neq \frac{1}{k} \\ \frac{1}{2} - \frac{\gamma_{\min}}{2}\frac{k}{k-1} & \gamma_{\min} = \frac{1}{k} \end{cases} \tag{55}$$

where $\gamma_{\min} = \min_i \gamma_k(i)$, $0 \leq \gamma_{\min} \leq 1/k$, and

$$\gamma_k(i) \quad = \quad \frac{1}{k}\underline{1}_k^T[I_k - \Pi_{E_k(-i)}]\underline{1}_k \tag{56}$$

and $\Pi_{E_k(-i)}$ is the orthogonal projector defined in (40).

Proof of Lemma 6

Fix a value $c > 0$. Let b be the distance between the vector $c\underline{1}_k$ and the closest face of $\text{cone}(E_k)$ (See Fig. 4). As each face is contained in the column span of $E_k(-i))$ and as $\text{cone}(E_k)$ is convex and contains $c\underline{1}_k$, b is specified by the projection theorem

$$b^2 \;=\; c^2 \min_i \{\underline{1}_k^T[I_k - \Pi_{E_k(-i)}]\underline{1}_k\} = c^2 k\, \gamma_{\min}, \tag{57}$$

Since $I_k - \Pi_{E_k(-i)}$ is an idempotent matrix $\gamma_k(i) = \|[I_k - \Pi_{E_k(-i)}]\underline{1}_k\|^2/k \geq 0$. As the elements of E_k are non-negative the projection error norm $\underline{1}_k^T[I_k - \Pi_{E_k(-i)}]\underline{1}_k$ is maximum when cone(E_k) is largest, which occurs for $E_k = I_k$. In this case $I_k - \Pi_{E_k(-i)} = \mathrm{diag}(0,\ldots,0,1,0,\ldots,0)$ and $\gamma_k(i) = 1/k$.

Next let $d(\beta)$ be the maximum distance from $c\underline{1}_k$ to cone$(F_k(\beta))$ (See Figure 4). As the extremals of cone$(F_k(\beta))$ are symmetric about the ray $\{a\underline{1}_k : a > 0\}$ each extremal has a point at identical maximum distance from $c\underline{1}_k$. Consider the extremal $\{a\underline{f}_1 : a > 0\}$ where $\underline{f}_1$ is the first column of $F_k(\beta)$. We thus have

$$\begin{aligned} d^2(\beta) \;&=\; c^2 \left\| \underline{1}_k - \frac{\underline{1}_k^T \underline{f}_1}{\|\underline{f}_1\|^2}\, \underline{f}_1 \right\|^2 \\ &=\; c^2 k \left[1 - \frac{(1+(k-1)\beta)^2}{k(1+(k-1)\beta^2)}\right]. \end{aligned} \tag{58}$$

Evidently cone$(F_k(\beta)) \subset$ cone(E_k) as long as $d^2(\beta) \leq b^2$. Let β_k be the minimum corresponding value of $\beta \in [0,1)$ for which $d^2(\beta_k) = b^2$. Equating (57) and (58) yields the following quadratic equation for β_k

$$(k-1)(\zeta(k-1)-1)\beta_k^2 + 2(k-1)\zeta\beta_k + \zeta - 1 = 0$$

where $\zeta = (1 + \gamma_{\min}/(1-\gamma_{\min}))/k$. For $\gamma_{\min} \in [0, 1/k)$ the quadratic equation has two non-negative solutions only one of which is contained in $[0,1)$. For $\gamma_{\min} = 1/k$, $\zeta = 1/(k-1)$ and there is one solution in $[0,1)$. In both cases this solution is given by (55). $\square$

7 Low Rate Optimality of Unitary Orthogonal Constellations

When the number of signal matrices K to be considered is sufficiently small significant simplification of the optimization (21) is possible. In particular, one obtains optimality of a set of scaled mutually orthogonal unitary signal matrices and a simple form for $\tilde{R}_o(K)$.

The first result specifies the solution to optimization of the dissimilarity measure $D(S_i\|S_j)$ defined in (10).

For given η, T and M define the integer M_o

$$M_o = \mathrm{argmax}_{m\in\{1,\ldots,M\}} \left\{ m \ln \frac{(1+\eta TM/(2m))^2}{1+\eta TM/m} \right\}. \tag{59}$$

We will see below that under some conditions M_o is the rank of the signal matrices S_i in the optimal K-dimensional constellation.

Proposition 8 *Let $2M \leq T$. Then*

$$D_{\max} \stackrel{\text{def}}{=} \max_{S_1,S_2 \in \mathcal{S}^K_{\text{peak}}} D(S_1\|S_2) = M_o \ln \frac{(1+\eta TM/(2M_o))^2}{1+\eta TM/M_o}. \tag{60}$$

Furthermore, the optimal signal matrices which attain $D_{\max}$ can be taken as scaled rank M_o mutually orthogonal unitary $T \times M$ matrices of the form

$$S_1 = \sqrt{TM/M_o}\,\Phi_1, \qquad S_2 = \sqrt{TM/M_o}\,\Phi_2$$

where, for $j = 1, 2$,

$$\Phi_j^H \Phi_j = I_{M_o}, \quad \text{and} \quad \Phi_i^H \Phi_j = 0, \;\; i \neq j.$$

The assumption $2M \leq T$ is critical and ensures that the singular vectors of S_1 and S_2 can be chosen as mutually orthogonal for any set of singular values.

The rank M_o of the optimal matrices S_1 and S_2 increases from 1 to M as the SNR parameter ηTM increases from 0 to ∞ (see Fig. 5). Numerical evaluation has shown that the functional relationship between M_o and SNR is well approximated by the relation

$$M_o \approx \max\left(1, \lfloor a\eta TM + b + 0.48 \rfloor\right)$$

where a, b are the slope and intercept of the least squares linear fit to the function $y(x) = \text{argmax}_{m=1,2,\ldots} m \ln[(1+x/(2m))^2/(1+x/m)]$. The approximation is a lower bound and underestimates the exact value of M_o, given by (59), by at most 1 over less than 0.5% of the SNR range shown in Fig. 5 ($0 < \eta TM \leq 120$). If the SNR is sufficiently large, e.g. (from Fig. 5) $\eta TM \geq 17$ for $M = 6$ and $T \geq 12$, $M_o = M$ and the optimal signal matrices utilize all M transmit antennas. On the other hand for small SNR, i.e. (from Fig. 5) $\eta TM < 4$, $M_o = 1$ and the optimal signal matrices apply all available transmit power to a single antenna element over the coherent fade interval T.

Proof of Proposition 8

Let S_1 and S_2 have the singular value decompositions

$$S_1 = V_1 \Lambda_1 U_1^H, \qquad S_2 = V_2 \Lambda_2 U_2^H, \tag{61}$$

where V_1, V_2 are $T \times M$ unitaries, i.e. $V_1^H V_1 = V_2^H V_2 = I_M$, U_1, U_2 are $M \times M$ unitaries, and Λ_1, Λ_2 are $M \times M$ real diagonal matrices of singular values $\{\lambda_{1i}\}_{i=1}^M$ and $\{\lambda_{2i}\}_{i=1}^M$, respectively.

The function $D(S_1\|S_2)$ of $S_1 S_1^H$ and $S_2 S_2^H$ only depends on S_1 and S_2 through $V_1, \Lambda_1, V_2, \Lambda_2$. Thus unitaries U_1 and U_2 can be arbitrarily chosen without affecting $D(S_1\|S_2)$ and hence we can choose them as I_M. We denote this functional dependence by writing $D(S_1\|S_2) = D(V_1,\Lambda_1\|V_2,\Lambda_2)$. Therefore

$$\max_{S_1,S_2} D(S_1\|S_2) = \max_{\Lambda_1,\Lambda_2}\max_{V_1,V_2} D(V_1,\Lambda_1\|V_2,\Lambda_2). \tag{62}$$

Using the form for D given in Lemma 7 (see the Appendix)

$$\begin{aligned} D(V_1,\Lambda_1\|V_2,\Lambda_2) &= \ln\frac{\left|I_M+\frac{\eta}{2}S_1^HS_1\right|\left|I_M+\frac{\eta}{2}S_2^HS_2\right|}{\sqrt{\left|I_M+\eta S_1^HS_1\right|\left|I_M+\eta S_2^HS_2\right|}} + \ln\left|I_M-\kappa^H\kappa\right| \\ &= \sum_{i=1}^M \ln\frac{(1+\frac{\eta}{2}\lambda_{1i}^2)(1+\frac{\eta}{2}\lambda_{2i}^2)}{\sqrt{(1+\eta\lambda_{1i}^2)(1+\eta\lambda_{2i}^2)}} + \ln\left|I_M-\kappa^H\kappa\right| \end{aligned} \tag{63}$$

For fixed arbitrary Λ_1,Λ_2 first consider the inner maximization in (62), i.e. maximization over V_1,V_2 of the right hand side of (63). Observe that the first term in (63) depends only on the singular values Λ_1,Λ_2 and not on V_1,V_2. Observe that as $S_1^HS_2 = \Lambda_1V_1^HV_2\Lambda_2$, for any Λ_1,Λ_2 the product $S_1^HS_2$ can be forced to zero by selecting matrices V_1 and V_2 to be mutually orthogonal – possible under the assumption $2M \leq T$. Recall that $\kappa = \tilde{S}_2^H\tilde{S}_1$ where $\tilde{S}_1$ and $\tilde{S}_2$, defined in Lemma 7, have the same column spaces as S_1 and S_2. Therefore, when V_1 and V_2 are mutually orthogonal $\kappa = 0$ and the second term in (63) is equal to its maximum value (0). This establishes that the optimum signal matrices S_1 and S_2 have orthogonal singular vectors V_1 and V_2 and that

$$\max_{\Lambda_1,\Lambda_2}\max_{V_1,V_2} D(V_1,\Lambda_1\|V_2,\Lambda_2) = \max_{\{\lambda_{1i}\}}\sum_{i=1}^M \ln\frac{1+\frac{\eta}{2}\lambda_{1i}^2}{\sqrt{1+\eta\lambda_{1i}^2}} + \max_{\{\lambda_{2i}\}}\sum_{i=1}^M \ln\frac{1+\frac{\eta}{2}\lambda_{2i}^2}{\sqrt{1+\eta\lambda_{2i}^2}}.$$

Due to the assumed peak power constraint each of the above maximizations on the right hand side are performed over the inequality constraint sets $\left\{\lambda_{j1},\ldots,\lambda_{jM} : \sum_{i=1}^M \lambda_{ji}^2 \leq TM\right\}$, $j = 1,2$. As these maximizations are of identical form,

$$\max_{\Lambda_1,\Lambda_2}\max_{V_1,V_2} D(V_1,\Lambda_1\|V_2,\Lambda_2) = 2\max_{\{\lambda_i\}}\sum_{i=1}^M \ln\frac{1+\frac{\eta}{2}\lambda_i^2}{\sqrt{1+\eta\lambda_i^2}} = \max_{\{\lambda_i\}}\sum_{i=1}^M \ln\frac{(1+\frac{\eta}{2}\lambda_i^2)^2}{1+\eta\lambda_i^2} \tag{64}$$

where the maximization on the right hand side is subject to $\sum_{i=1}^M \lambda_i^2 \leq TM$. Observe that as each summand is monotone increasing in λ_i^2 the inequality constraint in the maximization is always met with equality. Consider the Lagrangian

$$J(\lambda_1,\ldots,\lambda_M) = \sum_{i=1}^M \left(2\ln(1+\tfrac{\eta}{2}\lambda_i^2) - \ln(1+\eta\lambda_i^2)\right) - \alpha\sum_{i=1}^M \lambda_i^2$$

where $\alpha \geq 0$ is an undetermined multiplier. For a suitable value of α, the maximum of J over unconstrained $\{\lambda_i\}$ is identical to the maximum of the right hand side of (64) over power constrained $\{\lambda_i\}$. The derivative of J with respect to λ_i is

$$\frac{\partial J}{\partial \lambda_i} = -\alpha\eta^2\,\lambda_i\frac{(\lambda_i^2)^2 + 2\lambda_i^2\,v/\eta + 2/\eta^2}{(1+\eta\lambda_i^2)(1+\frac{\eta}{2}\lambda_i^2)}$$

where $v = \frac{3\alpha-\eta}{2\alpha}$. Therefore, for each λ_i, there are five possible stationary points of J given by either $\lambda_i = 0$ or by $\pm\sqrt{\lambda_i^2}$ where λ_i^2 is one of the roots

$$\lambda_i^2 = (-v \pm \sqrt{v^2-2})/\eta, \quad i = 1,\ldots,M. \tag{65}$$

With respect to (65) there are three cases which must be considered. For $v \geq \sqrt{2}$ both roots are real negative and for $-\sqrt{2} < v < \sqrt{2}$ both roots are complex. For $v \leq -\sqrt{2}$ both roots are real positive. Thus only the latter case is relevant. It can be verified that, when λ_i^2 is equal to one of these two positive roots, the second derivative of J is $\partial^2 J/\partial\lambda_i^2 = -4\alpha\eta^2\lambda_i^2 \frac{\lambda_i^2+v/\eta}{(1+\eta\lambda_i)(1+\frac{\eta}{2}\lambda_i)}$ which is negative only for the root $\lambda_i^2 = (-v+\sqrt{v^2-2})/\eta$. Therefore, we can restrict attention to a candidate maximizer $\{\lambda_1, \ldots, \lambda_M\}$ of J for which each λ_i^2 can take on either of two values: zero or $c = (-v+\sqrt{v^2-2})/\eta$. Let $m \in \{1, \ldots, M\}$ denote the number of nonzero valued λ_i for one of these candidate maximizers. Then, invoking the constraint $\sum_{i=1}^M \lambda_i^2 = TM$ and noting that the ordering of the indices of λ_i^2 is irrelevant, any candidate maximizer of J can be put in the form

$$\lambda_i^2 = \begin{cases} TM/m, & i = 1, \ldots, m \\ 0, & i = m+1, \ldots, M \end{cases}, \tag{66}$$

and now m is the only remaining free parameter. Substituting this expression for $\nu_i^2 = \lambda_i^2$ into (64) we obtain the expression (60) with M_o equal to the optimal value of m given by (59). Using the SVD representation (61) with $U_1 = U_2 = I_M$ it is easily seen that the optimal signal matrices, corresponding to (66) with $m = M_o$, have the form $S_j = V_j\Lambda_j = \sqrt{TM/M_o}\ \Phi_j, \quad j = 1, 2$, where the Φ_j's are rank M_o mutually orthogonal rectangular unitaries as defined in the statement of the Proposition. □

Proposition 8 implies that for low symbol rate, the signal constellations having best minimum distance properties are constellations of scaled mutually orthogonal unitary matrices of rank $M_o \leq M$.

Corollary 2 *Let $2M \leq T$ and let M_o be as defined in (59). Assume $M_o \leq \min\{M, T/K\}$ and define the set of signal matrices $\{S_i^*\}_{i=1}^K$ by $S_i^* = \sqrt{TM/M_o}\ \Phi_i$ where $\{\Phi_i\}_{i=1}^K$ are a set of $T \times M$ mutually orthogonal rectangular unitary matrices of rank M_o ($\Phi_i^H\Phi_i = I_{M_o}$ and $\Phi_i^H\Phi_j = 0,\ i \neq j$). This set of signal matrices are equidistant in the sense $D(S_i^*\|S_j^*) = D_{\max} = D_{\min},\ i \neq j$, and they attain the maximum possible value of $D_{\min} = \min_{l\neq m} D(S_l\|S_m)$ over all signal sets $\{S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K$ of dimension K.*

Proof of Corollary 2

First observe that if $M_o \leq \min\{M, T/K\}$ then such an orthogonal unitary set $\{\Phi_i\}_{i=1}^K$ exists, e.g. the set of K mutually orthogonal permutation matrices of dimension $T \times M$ whose columns are formed from K disjoint M_o-dimensional subsets of the columns of I_T. The set $\{S_i^*\}_{i=1}^K$ satisfies the peak power constraint $\|S_i^*\|^2 \leq TM,\ i = 1, \ldots, K$, with equality and, by Proposition 8, it has the property that $D(S_i^*\|S_j^*) = D_{\max} \stackrel{\text{def}}{=} \max_{S_1,S_2} D(S_1\|S_2),\ i \neq j$. Let $\{S_i\}_{i=1}^K$ be any set of K signal matrices in $\mathbb{C}^{T\times M}$ satisfying the same power constraint. Then $\min_{i\neq j} D(S_i\|S_j) \leq D_{\max} = \min_{i\neq j} D(S_i^*\|S_j^*)$ and thus $\{S_i^*\}_{i=1}^K$ maximizes $D_{\min}$. □

It can be easily verified that the mutually orthogonal unitary constellation of Corollary 2 satisfies the sufficient conditions (a) and (b) of Lemma 5 when $\tilde{\mathcal{S}}^K_{\text{peak}}$ is not empty. For this constellation $E_K = F_K(\delta)$, $\delta = e^{-ND_{\max}} < 1$, and

$$\begin{aligned} \alpha_K &= (1 - \underline{1}_K^T [F_K(\delta)]^{-1} \underline{1}_K \delta)/(1 - \underline{1}_K [F_K(\delta)]^{-1} \underline{1}_K \delta^2) \\ &= (1-\theta)/(1-\delta\theta). \end{aligned}$$

where $\theta = \theta(\delta) \in [0, 1)$ is defined below (52) with $k = K$. Therefore $0 < \alpha_K \leq 1$. Furthermore, as $\max(\underline{e}_K) = \text{avg}(\underline{e}_K) = \delta$, condition (b) reads: $\alpha_K \delta < 1$, which is satisfied regardless of the value of β.

The final result of this section is an expression for the cut-off rate.

Proposition 9 *Let $2M \leq T$ and let M_o be as defined in (59). Suppose that $M_o \leq \min\{M, T/K\}$. Then the peak constrained K dimensional cut-off rate (16) is*

$$\tilde{R}_o(K) = \ln\left(\frac{K}{1 + (K-1)e^{-ND_{\max}}}\right)$$

and $D_{\max}$ is given by (60). Furthermore, the optimal constellation attaining $\tilde{R}_o(K)$ is the set of K rank M_o mutually orthogonal matrices specified in Corollary 2 and the optimal probability assignment is uniform: $P_i^* = 1/K$, $i = 1, \ldots, K$.

Any unitary transformation on the columns (spatial coordinates) of a set of signal matrices produces a set of signal matrices with identical $D_{\min}$. In particular, any set of K mutually orthogonal $T \times M_o$ permutation matrices, specified in the course of proof of Corollary 2, has optimal distance properties. This simple set of signal matrices corresponds to transmitting energy on a single antenna element at a time, among a total of $M_o \leq M$ elements, in each of the available T time slots. Since $\tilde{R}_o(K)$ is increasing in K the maximum cut-off rate achievable using these mutually orthogonal unitary matrices is obtained by using the maximum possible number of them: $K = \lfloor T/M_o \rfloor$. Observe that the resulting optimal constellation may correspond to a code of quite low symbol rate, e.g. for $M_o = M = T/2$ the symbol rate is only 1 bit-per-symbol.

It is noteworthy that the optimal peak constrained signal constellation specified by Proposition 9 does not include the zero valued signal matrix $S_i = O$. Including zero in the signal constellation would allow signalling using on-off keying. On-off keying is often proposed for average power constrained signalling over low SNR channels since it permits energy discrimination at the receiver. As contrasted with on-off keying all signals in the optimal peak constrained signal set have equal power. We conjecture that the zero signal would result from replacing the peak power constraint with an average power constraint in Proposition 9.

Proof of Proposition 9

Define $\delta = e^{-ND_{\max}}$ which is strictly less than one under the hypotheses of the Proposition. For any set of signal matrices $\{S_i\}_{i=1}^K$ satisfying the assumptions of Proposition 9, Proposition 8 asserts that $D(S_i\|S_j) \leq D_{\max}$, $i \neq j$, with equality when $\{S_i\}_{i=1}^K$ consists of the specified mutually orthogonal matrices. Using this inequality and the fact $D(S_i\|S_j) = 0$, $i = j$,

$$\begin{aligned} \min_{\{S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K} \sum_{i=1}^K P_i \sum_{j=1}^K P_j\, e^{-ND(S_i\|S_j)} &= e^{-ND_{\max}} \sum_{i\neq j} P_i P_j + \sum_{i=1}^K P_i^2 \\ &= \underline{P}_K^T F_K(\delta) \underline{P}_K. \end{aligned} \tag{67}$$

Now, using the identity (50), $F_K(\delta)\underline{x} = \underline{1}_K$ has a unique solution in the positive orthant

$$\begin{aligned} \underline{x} &= [F_K^{-1}(\delta)]^{-1}\underline{1}_K \\ &= \frac{1}{1-\delta}\left(1 - \frac{K\delta}{1+(K-1)\delta}\right)\underline{1}_K \\ &= \frac{1}{(1+(K-1)\delta)}\underline{1}_K. \end{aligned}$$

Thus the minimizer $\underline{P}_K^*$ of $\underline{P}_K^T F_K(\delta)\underline{P}_K$ is the uniform distribution $\underline{P}_K = \underline{1}_K/K$. Substitution of this solution back into (67) establishes that

$$\min_{\{P_i\}_{i=1}^K \in \mathcal{P}^K} \min_{\{,S_i\}_{i=1}^K \in \mathcal{S}_{\text{peak}}^K} \sum_{i=1}^K P_i \sum_{j=1}^K P_j\, e^{-ND(S_i\|S_j)} = \frac{1}{K^2}\, \underline{1}_K^T F_K(\delta)\underline{1}_K = \frac{(1-\delta)+\delta K}{K}$$

which when substituted into (16) yields the expression given for $\tilde{R}_o(K)$ specified in Proposition 9. □

8 Conclusions and Future Work

We have derived representations for the single user computational cut-off rate for space time coding under the Rayleigh flat fading channel model under a peak transmitted power constraint. For finite dimensional constellations the cut-off rate and the optimal signal distribution were specified as a solution to a quadratic optimization problem and it was shown that optimal constellations have codeword distributions which satisfy an equalization condition. This characterization of optimality motivated us to study properties of the set of feasible constellations which satsify the equalization property. Easily verifiable necessary and sufficient conditions were given for validating that a given signal constellation lies in the feasible set.

9 Appendix: Properties of $D(S_1\|S_2)$

Here we collect various properties of the signal dissimilarity measure (10).

9.1 Alternative form for $D(S_1\|S_2)$

$D(S_1\|S_2)$ can be equivalently expressed in terms of a signal multiple correlation matrix κ.

Lemma 7 *The signal dissimilarity measure (10) for the spatio-temporal fading model (1) has the equivalent form*

$$D(S_1\|S_2) = \frac{1}{2}\ln\frac{\left|I_M+\frac{\eta}{2}S_1^HS_1\right|^2\left|I_M+\frac{\eta}{2}S_2^HS_2\right|^2}{\left|I_M+\eta S_1^HS_1\right|\left|I_M+\eta S_2^HS_2\right|}\left|I_M-\kappa^H\kappa\right|^2,$$

where κ is the $M\times M$ multiple correlation matrix

$$\kappa = \tilde{S}_2^H\tilde{S}_1,$$

$\tilde{S}_1$ and $\tilde{S}_2$ are the "prewhitened" signal matrices

$$\tilde{S}_1 = \sqrt{\tfrac{\eta}{2}}S_1[I_M+\tfrac{\eta}{2}S_1^HS_1]^{-\frac{1}{2}}, \qquad \tilde{S}_2 = \sqrt{\tfrac{\eta}{2}}S_2[I_M+\tfrac{\eta}{2}S_2^HS_2]^{-\frac{1}{2}},$$

and $A^{\frac{1}{2}}$ denotes the positive definite square root factor of positive definite square matrix A.

Proof of Lemma 7

Consider the quantity in the numerator of expression (10)

$$\begin{aligned}\left|I_T+\tfrac{\eta}{2}(S_1S_1^H+S_2S_2^H)\right| &= \left|I_T+\tfrac{\eta}{2}[S_1\ \ S_2]\begin{bmatrix}S_1^H\\ S_2^H\end{bmatrix}\right| \\ &= \left|I_T+\tfrac{\eta}{2}\begin{bmatrix}S_1^H\\ S_2^H\end{bmatrix}[S_1\ \ S_2]\right| \\ &= \left|\begin{bmatrix}I_M+\frac{\eta}{2}S_1^HS_1 & \eta S_1^HS_2\\ \eta S_2^HS_1 & I_M+\frac{\eta}{2}S_2^HS_2\end{bmatrix}\right|,\end{aligned}$$

where in the second line we have used the property $|I+AB| = |I+BA|$.

For A and C Hermitian positive definite matrices we recall the following determinant identity for block partitioned matrices

$$\left|\begin{bmatrix}A & B\\ B^H & C\end{bmatrix}\right| = |A|\ |C|\ \left|I-B^HA^{-1}BC^{-1}\right|.$$

Furthermore, denoting by $C^{\frac{1}{2}}$ the invertible square root factor of C, where $C = C^{\frac{H}{2}}C^{\frac{1}{2}}$,

$$\begin{aligned}\left|I-B^HA^{-1}BC^{-1}\right| &= \left|C^{\frac{H}{2}}(I-C^{-\frac{H}{2}}B^HA^{-\frac{1}{2}}A^{-\frac{H}{2}}BC^{-\frac{1}{2}})C^{-\frac{H}{2}}\right| \\ &= \left|I-\kappa^H\kappa\right|\end{aligned}$$

where $\kappa = C^{-\frac{H}{2}}B^HA^{-\frac{1}{2}}$. Identifying $A = I_M+\frac{\eta}{2}S_1^HS_1$, $C = I_M+\frac{\eta}{2}S_2^HS_2$, and $B = S_1^HS_2$ finishes the proof of Lemma 7. □

9.2 Bounds on $D(S_1\|S_2)$

Lemma 8 *The dissimilarity measure $D(S_1\|S_2)$ satisfies*

$$D(S_1\|S_2) = \frac{\eta^2}{8}\|S_1S_1^H - S_2S_2^H\|^2 + o(\eta^2)$$

where $o(\eta^2)$ is a non-negative function of η such that $\lim_{\eta\to 0} o(\eta^2)/\eta^2 = 0$.

Proof of Lemma 8

Let $\{\nu_{1i}\}_{i=1}^T$ and $\{\nu_{2i}\}_{i=1}^T$ denote the eigenvalues of $A = S_1S_1^H$ and $B = S_2S_2^H$ respectively and let $\{\sigma_i\}_{i=1}^T$ denote the eigenvalues of $\frac{1}{2}(A+B)$. Since these are Hermitian non-negative definite matrices these eigenvalues are real and positive. Thus the dissimilarity measure $D(S_1\|S_2)$, given in (10), can be written as

$$D(S_1\|S_2) \;=\; \tfrac{1}{2}\sum_{i=1}^T \left(2\ln(1+\eta\sigma_i) - \ln(1+\eta\nu_{1i}) - \ln(1+\eta\nu_{2i})\right).$$

Using the relation $\ln(1+a) = a - a^2/2 + o(a^2)$ and the fact that $\sum_{i=1}^T(2\sigma_i - \nu_{1i} - \nu_{2i}) = 0$

$$\begin{aligned} D(S_1\|S_2) &= \frac{\eta^2}{4}\sum_{i=1}^T\left(\nu_{1i}^2 + \nu_{2i}^2 - 2\sigma_i^2\right) + o(\eta^2) \\ &= \frac{\eta^2}{4}\left(\mathrm{tr}(AA^H) + \mathrm{tr}(BB^H) - 2\mathrm{tr}\left(\frac{(A+B)}{2}\frac{(A+B)^H}{2}\right)\right) + o(\eta^2) \\ &= \frac{\eta^2}{8}\mathrm{tr}\left((A-B)(A-B)^H\right) + o(\eta^2). \end{aligned}$$

Since $\mathrm{tr}((A-B)(A-B)^H) = \|A-B\|^2$ Lemma 8 is established. □

Lemma 9 *Let A and B be $T\times T$ complex matrices. Then the trace norm of $A-B$ satisfies*

$$\|A-B\|^2 = \mathrm{tr}((A-B)(A-B)^H) \geq \sum_{i=1}^T |a_{ii} - b_{ii}|^2, \tag{68}$$

where $\{a_{ii}\}_{i=1}^T$ and $\{b_{ii}\}_{i=1}^T$ are the diagonal elements of A and B, respectively. Equality occurs in (68) iff the trace norm of the anti-diagonal matrix $A - B - \mathrm{diag}(A-B)$ is equal to zero.

Proof of Lemma 9

Let $\Delta = A - B$ have diagonal elements $\{d_{ii}\}_{i=1}^T$ and consider the decomposition $\Delta = D + Z$ into diagonal matrix $D = \mathrm{diag}(d_{ii})$ and anti-diagonal matrix Z. Then

$$\|\Delta\|^2 \;=\; \|D\|^2 + \|Z\|^2 + \mathrm{tr}(DZ^H) + \mathrm{tr}(ZD^H).$$

As $\mathrm{tr}(DZ^H) = \mathrm{tr}(ZD^H) = 0$,

$$\|\Delta\|^2 \geq \|D\|^2 = \sum_{i=1}^T d_{ii}^2,$$

with equality iff $\|Z\| = 0$. □

10 Appendix: Miscellaneous results

The following establishes that it is safe to assume that the signal matrices in the optimal constellation have distinct outerproducts. While this result can also be obtained from the statistical invariance of the Rayleigh fading model, the lemma below is proven using a more elementary non-statistical argument.

Lemma 10 *Let the set of signal matrices $\{S_i\}_{i=1}^K$ have dissimilarity matrix E_K and assume that for some $i \neq j$: $S_i S_i^H = S_j S_j^H$. Then there exists a $K-1$ dimensional subset of $\{S_i\}_{i=1}^K$ having dissimilarity matrix E_{K-1} such that*

$$\min_{\underline{P}_K \in \mathcal{P}^K} \underline{P}_K^T E_K \underline{P}_K = \min_{\underline{P}_{K-1} \in \mathcal{P}^{K-1}} \underline{P}_{K-1}^T E_{K-1} \underline{P}_{K-1}$$

Proof of Lemma 10

Without loss of generality we can assume that $S_{K-1} S_{K-1}^H = S_K S_K^H$. As $D(S_i \| S_j)$ is a function of S_i and S_j only through $S_i S_i^H$ and $S_j S_j^H$, E_K takes the form

$$E_K = \begin{bmatrix} E_{K-2} & \underline{e} & \underline{e} \\ \underline{e}^T & 1 & 1 \\ \underline{e}^T & 1 & 1 \end{bmatrix}, \qquad ,$$

where $\underline{e}$ is a vector with entries $\exp(-ND(S_i \| S_K)) = \exp(-ND(S_i \| S_{K-1}))$, $i = 1, \ldots, K-2$. Let $\underline{P}_{K-2} = [P_1, \ldots, P_{K-2}]^T$ and $\tilde{Q}(\{P_i\}_{i=1}^K, \{S_i\}_{i=1}^K) = \underline{P}_K^T E_K \underline{P}_K$. Then using (10)

$$\begin{aligned} \tilde{Q}\left(P_1, \ldots, P_{K-1}, P_K, S_1, \ldots, S_{K-1}, S_K\right) &= (P_K + P_{K-1})^2 + 2[\underline{P}_{K-2}]^T \underline{e}(P_K + P_{K-1}) + [\underline{P}_{K-2}]^T E_{K-2} \underline{P}_{K-2} \\ &= \tilde{Q}\left(P_1, \ldots, P_{K-2}, (P_K + P_{K-1}), S_1, \ldots, S_{K-2}, S_{K-1}\right) \end{aligned}$$

Since

$$\min_{\underline{P}_K \in \mathcal{P}^K} \tilde{Q}\left(P_1, \ldots, P_{K-2}, (P_K + P_{K-1}), S_1, \ldots, S_{K-2}, S_{K-1}\right) = \min_{\underline{P}_{K-1} \in \mathcal{P}^{K-1}} \tilde{Q}\left(P_1, \ldots, P_{K-1}, S_1, \ldots, S_{K-1}\right)$$

the Lemma follows. □

The next result establishes that if $\tilde{\mathcal{S}}_{\text{peak}}^{k+1}$ is non-empty, i.e. a candidate S_{k+1} can be found such that $\underline{x}^{k+1}$ is in the positive orthant, then $S_1, \ldots, S_{k+1}$ is an improvement over $S_1, \ldots, S_k$.

Lemma 11 *For a given set of signal matrices $\{S_i\}_{i=1}^k$ with distinct outerproducts $\{S_i S_i^H\}_{i=1}^k$ assume that its optimal distribution $\underline{P}_k^*$ is an equalizer distribution. Let S_{k+1} be a signal such that $S_{k+1} S_{k+1}^H \notin \{S_i S_i^H\}_{i=1}^k$. If the conditions (a) and (b) of Corollary 1 are satisfied then the optimal distribution $\underline{P}_{k+1}^*$ for $S_1, \ldots, S_{k+1}$ is an equalizer distribution and $\underline{1}_{k+1}^T E_{k+1}^{-1} \underline{1}_{k+1} > \underline{1}_k^T E_k^{-1} \underline{1}_k$.*

Lemma implies that $\tilde{R}_o(k)$ is monotonically increasing as a function of k if the feasibility set $\tilde{\mathcal{S}}^{k+1}_{\text{peak}}$ is non-empty. Indeed, assume that a k dimensional constellation $\{S_i\}_{i=1}^k \in \tilde{\mathcal{S}}^k_{\text{peak}}$ achieves $\tilde{R}_o(k)$. Then, if there exists a signal S_{k+1} such that (i) $S_{k+1}S_{k+1}^H \notin \{S_iS_i^H\}_{i=1}^k$, (ii) $\{S_i\}_{i=1}^{k+1} \in \tilde{\mathcal{S}}^{k+1}_{\text{peak}}$, and (iii) $\alpha_k > 0$, then, by Proposition 5, and the above Lemma

$$\tilde{R}_o(k+1) \quad \geq \quad \ln \underline{1}_{k+1}^T E_{k+1}^{-1} \underline{1}_{k+1} > \ln \underline{1}_k^T E_k^{-1} \underline{1}_k = \tilde{R}_o(k), . \tag{69}$$

Proof of Lemma 11

First consider $(\underline{x}^{k+1})^T E_{k+1} \underline{x}^{k+1}$ where $\underline{x}^{k+1}$ is as given in (46) of Proposition 7. Using the partitioned matrix representation (43) of E_{k+1}

$$\begin{aligned} (\underline{x}^{k+1})^T E_{k+1} \underline{x}^{k+1} &= \alpha_k^2 + 2\alpha_k \underline{e}_k^T E_k^{-1}(\underline{1}_k - \alpha_k \underline{e}_k) + (\underline{1}_k - \alpha_k \underline{e}_k)^T E_k^{-1}(\underline{1}_k - \alpha_k \underline{e}_k) \\ &= \alpha_k^2(1 - \underline{e}_k^T E_k^{-1} \underline{e}_k) + \underline{1}_k^T E_k^{-1} \underline{e}_k. \end{aligned} \tag{70}$$

Now, by Corollary 1, under the cited conditions (a) and (b) $\underline{x}^{k+1} = E_{k+1}^{-1}\underline{1}_{k+1}$ is in $\mathbf{R}_+^{k+1}$. Thus, the optimal signal probabilities are $\underline{P}_{k+1}^* = \underline{x}^{k+1}/\underline{1}^T \underline{x}^{k+1}$. Using (70) and (47) this yields

$$\begin{aligned} (\underline{P}_{k+1}^*)^T E_{k+1} \underline{P}_{k+1}^* &= \frac{\alpha_k^2(1 - \underline{e}_k^T E_k^{-1} \underline{e}_k) + \underline{1}_k^T E_k^{-1} \underline{e}_k}{\left(\alpha_k + \underline{1}_k^T E_k^{-1}(\underline{1}_k - \alpha_k \underline{e}_k)\right)^2} \\ &= \frac{1}{\underline{1}_k^T E_k^{-1} \underline{1}_k + \alpha_k^2(1 - \underline{e}_k^T E_k^{-1} \underline{e}_k)} \\ &< \frac{1}{\underline{1}_k^T E_k^{-1} \underline{1}_k} = (\underline{P}_k^*)^T E_k \underline{P}_k^*, \end{aligned} \tag{71}$$

where in the second line we have used the identity $\alpha_k^2(1 - \underline{e}_k^T E_k^{-1} \underline{e}_k) = \alpha_k(1 - \underline{1}_k^T E_k^{-1} \underline{e}_k)$, and in the third line we have used the positivity of $\alpha_k^2(1 - \underline{e}_k^T E_k^{-1} \underline{e}_k)$.

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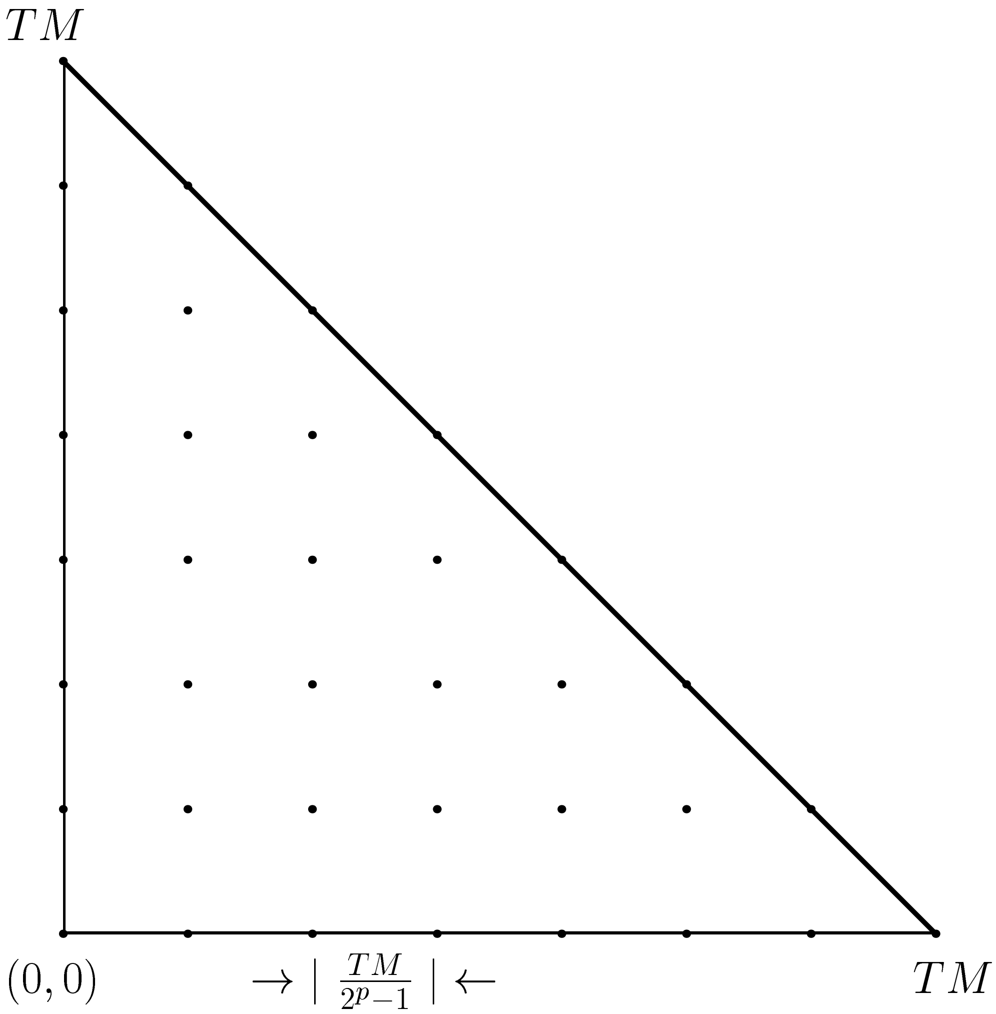

Figure 1: For $g_T(2^{p-1}-1) < K \leq g_T(2^p-1)$ and p a positive integer, we define a K-point test signal constellation by placing the vectors of diagonals $\underline{a}_i$ of each signal matrix $S_i S_i^H$ on K of the equispaced grid points of the largest triangular equispaced grid of $g_T(2^p-1)$ points contained in the peak constrained triangular region $\mathcal{A}_{\text{peak}}$. The Euclidean minimum distance of any such placement of the $\underline{a}_i$'s is $D_{\min} = TM/(2^p-1)$. The constellation is optimal (maximum $D_{\min}$) when there are exactly $K = g_T(2^p-1)$ signals in the constellation. In the figure $T=2$ and $p=3$. This signal constellation gives the minimum distance specified in the first inequality of Proposition 2.

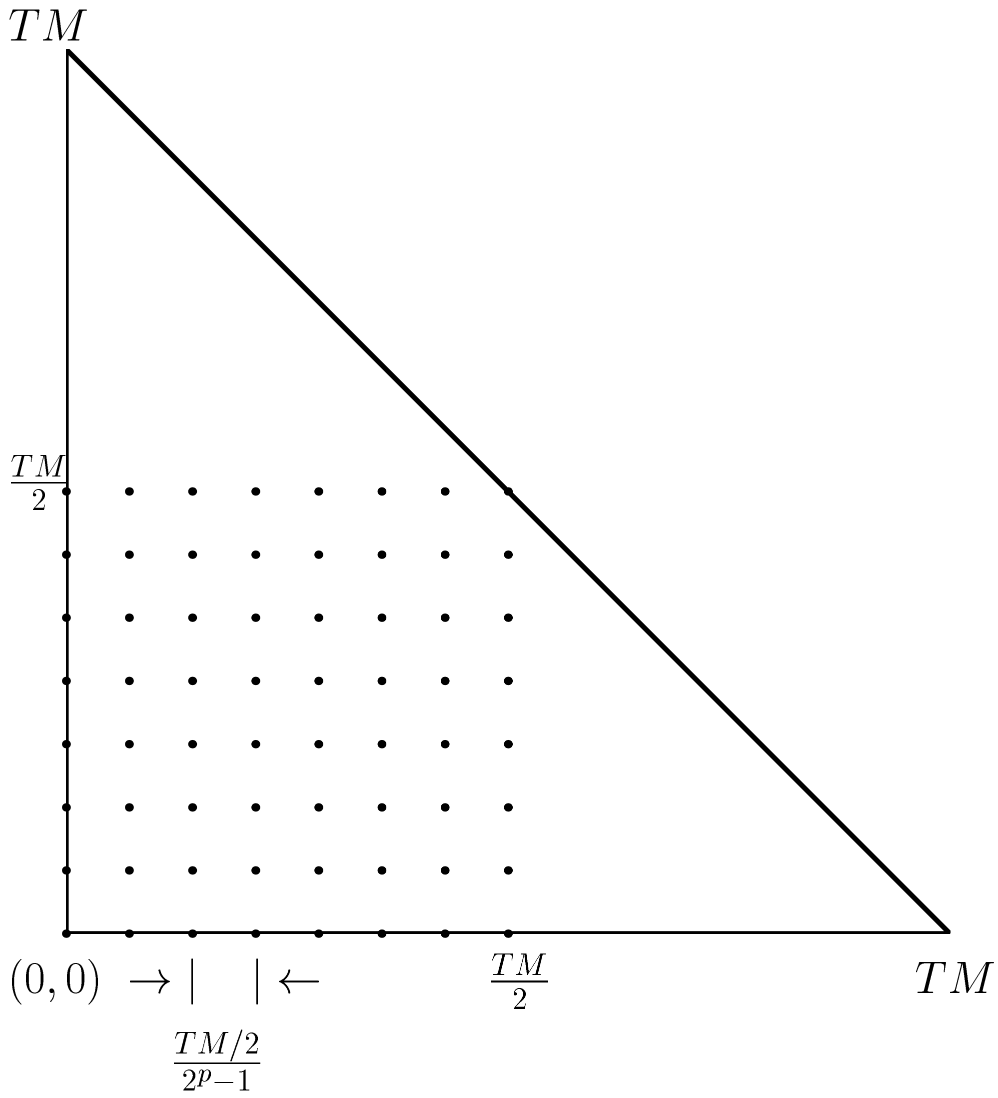

Figure 2: For $2^{(p-1)T} < K \leq 2^{pT}$, p a positive integer, an alternative suboptimal K-point test signal constellation is defined by placing the $\underline{a}_i$ vectors on K of the grid points of the largest square equispaced grid of 2^p points contained in the peak constrained region $\mathcal{A}_{\text{peak}}$. The minimum Euclidean distance is is now only one half of that shown in Fig 1: $D_{\min} = (TM/2)/(2^p - 1)$. This gives the minimum distance specified in the second (strict) inequality in Proposition 2.

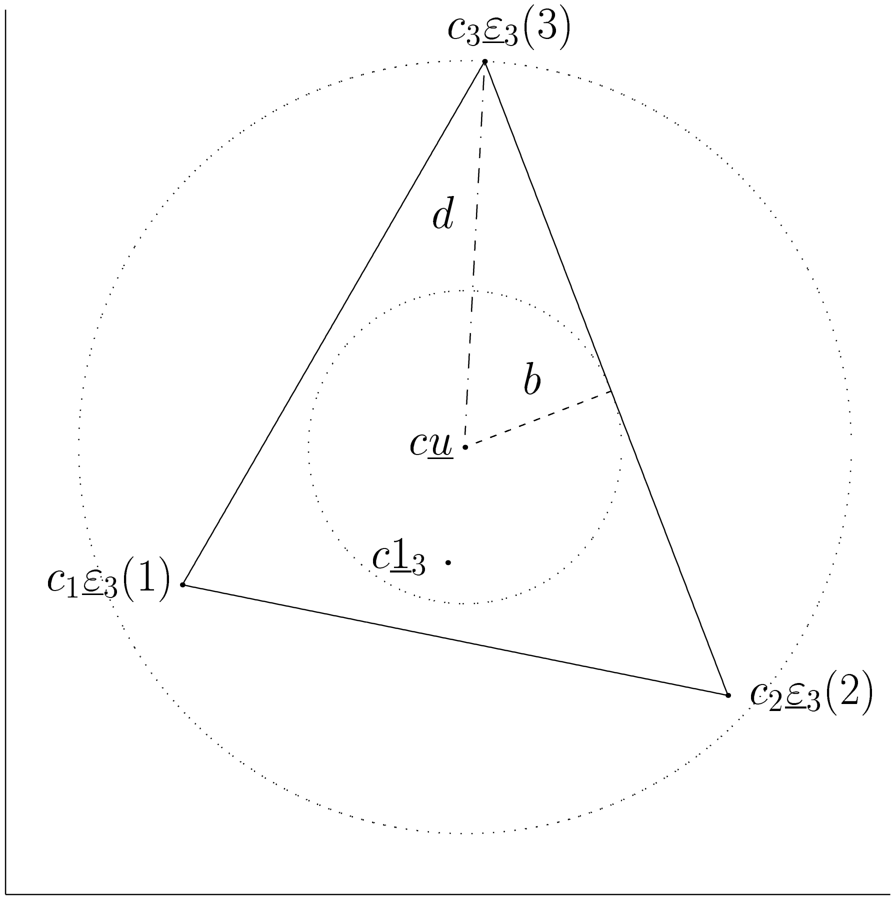

Figure 3: Two dimensional slice of three dimensional cone cone(E_k) $(k = 3)$ with inscribed and circumscribed three dimensional spheres used to establish sufficient and necessary conditions, respectively, for establishing that $\underline{1}_k \in$ cone(E_k). Interior of triangle is slice of cone(E_k) in a plane orthogonal to the vector $c\underline{u}$. Vertices of triangle are located at points $\{c_i\underline{\varepsilon}_k(i)\}_{i=1}^3$ along extremal rays of cone(E_k). The inner dotted circle is the corresponding slice of the largest possible sphere centered at $c\underline{u}$ that can be inscribed in cone(E_k). The outer dotted circle is the slice of the smallest possible sphere centered at $c\underline{u}$ through which all rays in cone(E_k) must pass.

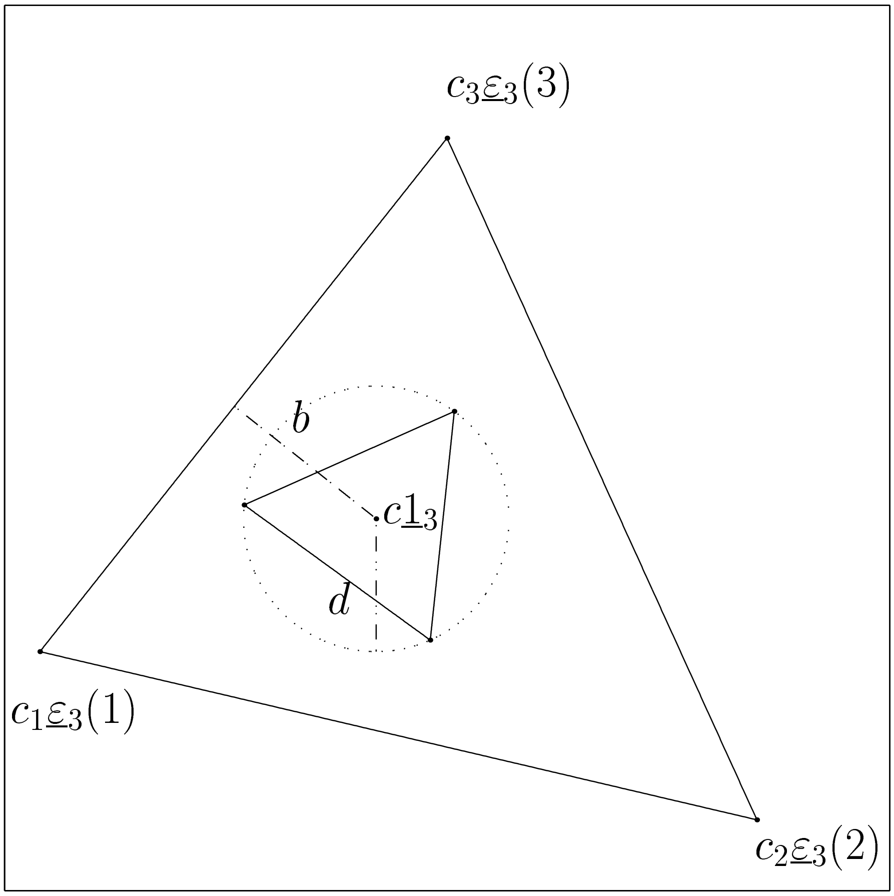

Figure 4: Interior of outer triangle is slice of cone(E_k) for $k = 3$ in a plane orthogonal to the vector $c\underline{1}_k$. Vertices of outer triangle are located at points $\{c_i\underline{\varepsilon}_3(i)\}_{i=1}^{3}$ along the extremal rays of cone(E_k). The interior of the smaller equilateral triangle in the center is the slice of cone$(F_k(\beta_k))$ within the same plane. The equilateral triangle has center point $c\underline{1}_k$. The dotted circle is the slice of a sphere with radius d equal to the distance between $c\underline{1}_k$ and the vertices of the smaller triangle. b is the distance of $c\underline{1}_k$ to the closest face of cone(E_k). cone$(F_k(\beta_k)) \subset$ cone(E_k) as long as the smaller triangle is inscribed in the larger triangle; i.e. $d \leq b$.

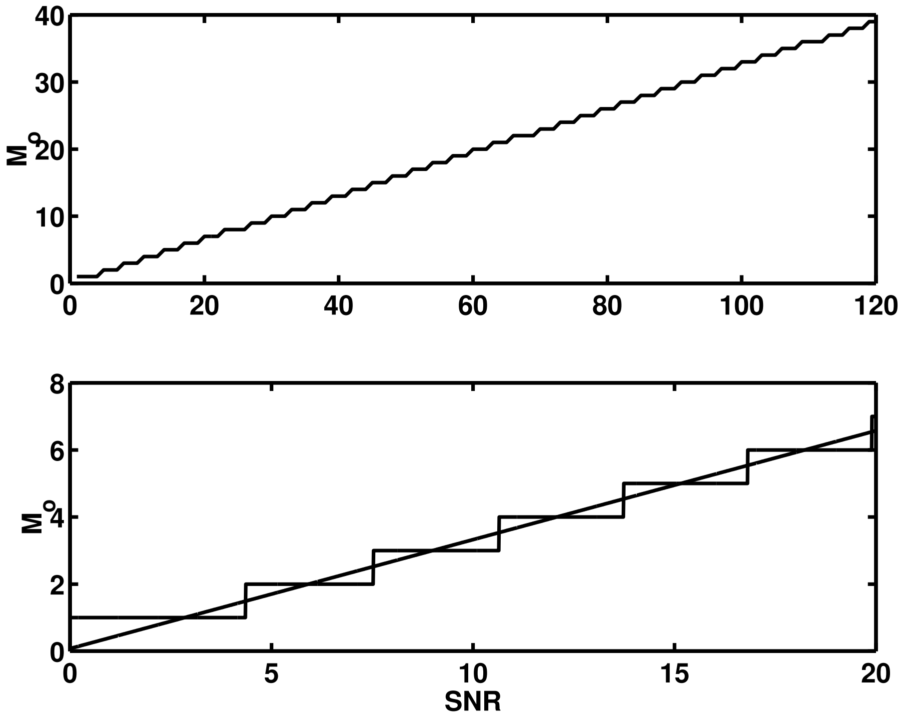

Figure 5: Top panel shows M_o given by (59) as a function of the SNR parameter ηTM. Bottom panel is blow up of first panel over a reduced range of SNR. The straight line is a least squares linear fit to the upper panel. The linear approximation has slope 0.32 and zero intercept 0.08. Average residual error between linear fit and exact M_o vs SNR step function is less than 0.09 and maximum error is less than 0.52. By Corollary 2, for T, K, M, M_o satisfying $T \geq KM_o$ and $M_o \leq M$, the curve gives the number of antennas utilised by the optimal constellation for various SNR's