Diversity and Degrees of Freedom in Wireless Communications

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Outline

1. MIMO systems and motivating examples
2. Link degrees-of-freedom (DOF) vs. diversity (DIV)
3. Error exponent analysis

Acknowledgments

This work was supported in part by a Graduate Fellowship to the first author from the Department of Electrical Engineering and Computer Science at the University of Michigan and by the Department of Defense Research & Engineering (DDR&E) Multidisciplinary University Research Initiative (MURI) on "Low Energy Electronics Design for Mobile Platforms" managed by the Army Research Office (ARO) under ARO grant DAAH04-96-1-0337.

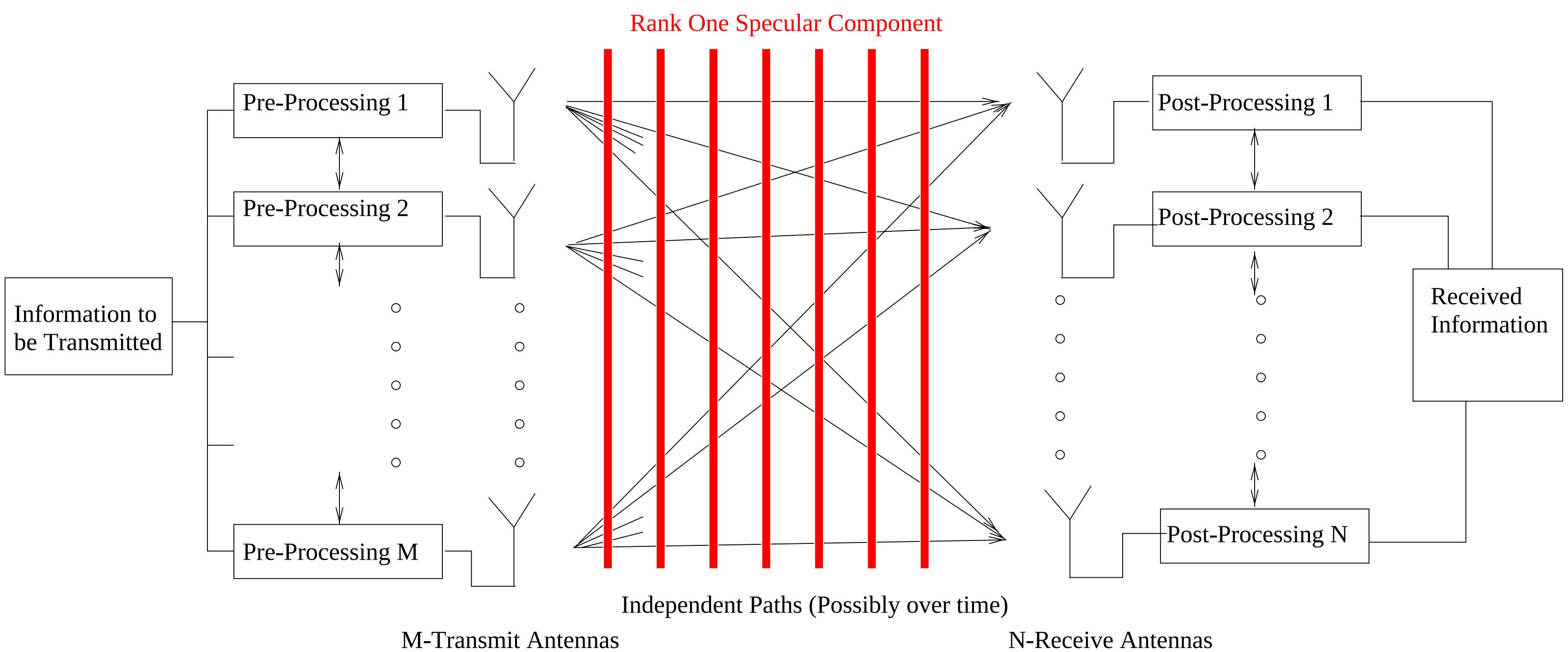

Figure 1: *Diagram of a multiple antenna communication system*

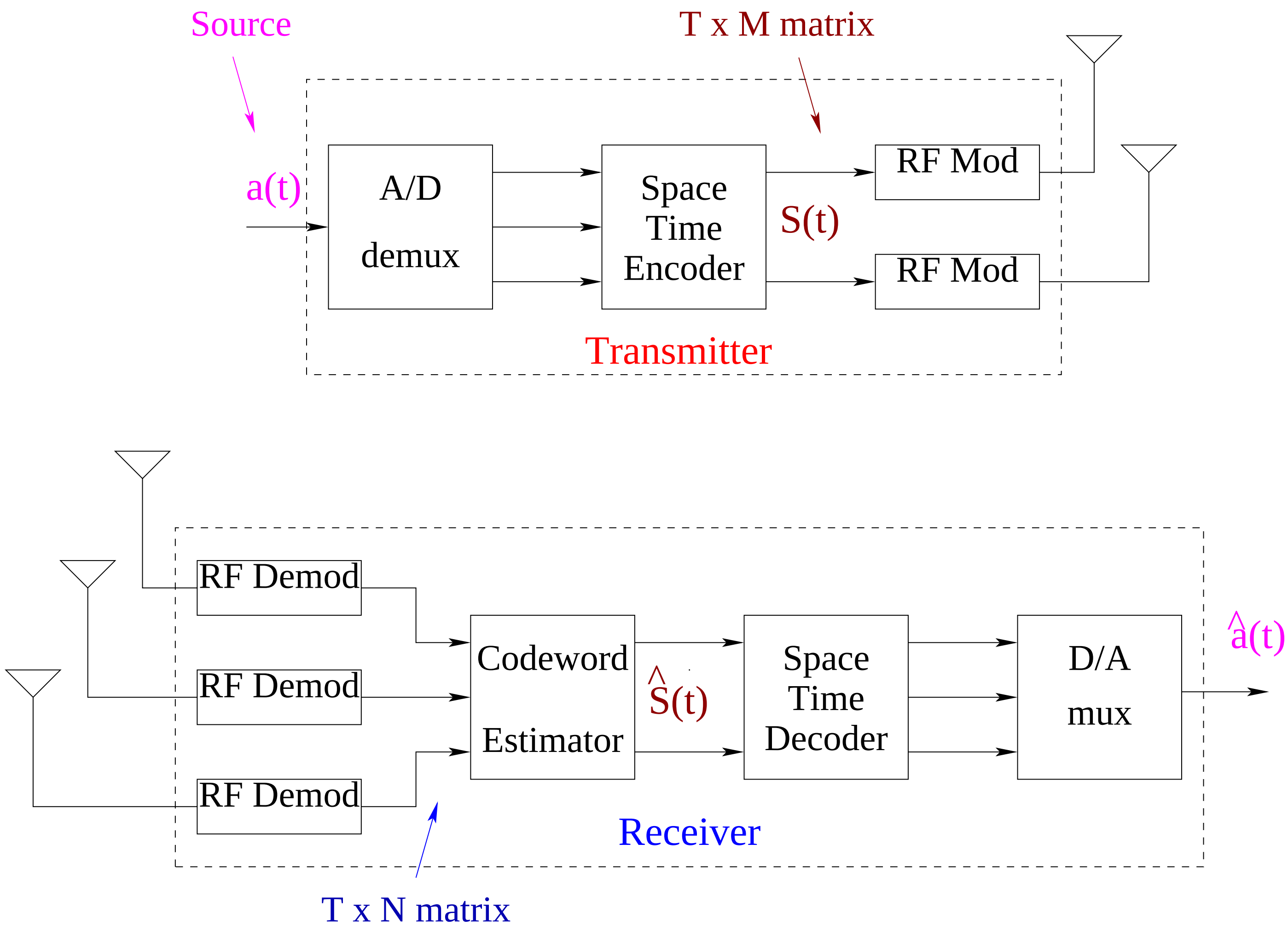

Figure 2: *Space-time transmitter/receiver.*

Diversity and Degrees-of-Freedom

Folk definitions:

- Degrees-of-Freedom (DOF):

  Aka *multiplexing gain*, *transmit diversity*

  DOF is number of independent channels exploited by the transmitter

- Diversity(DIV):

  Aka *processing gain*, *receive diversity*

  DIV is number of independent channels exploited by receiver

Example: SISO system

Received signal

$$x_l = \sqrt{\rho} h s_l + w_l, \;\; l = 1, \ldots, T$$

Channel capacity for energy-constrained signaling and informed receiver (IR) is:

$$C \approx T \log(\rho), \quad (\text{large } \rho)$$

⇒ ”DOF advantage is T”

Probability of decoding error for BPSK with ML decoding:

$$-\log P_e \approx T \log(\rho), \quad (\text{large } \rho)$$

⇒ ”DIV advantage is T”

Example: SIMO system

Received signal

$$x_l = \sqrt{\rho}\underline{h}s + w_l, \;\; l = 1, \ldots, T$$

Channel capacity for energy-constrained signaling and informed receiver (IR) is:

$$C \approx T \log(\rho), \quad (\text{large } \rho)$$

$\Rightarrow$ ”DOF advantage is NT”

Probability of decoding error for BPSK with ML decoding:

$$-\log P_e \approx NT \log(\rho), \quad (\text{large } \rho)$$

$\Rightarrow$ ”DIV advantage is NT”

Towards Formal Definition of DOF/DIV

1. Fix a rate-R coder and min-P_e decoder: denoted C

2. Define

Definition 1

$$\mathrm{DIV}_{M,N,T}(C) = \lim_{\rho\to\infty} \frac{\log P_e(\rho; M, N, T, C)}{\log \inf_C P_e(\rho; 1, 1, 1, C)} \tag{1}$$

and

$$\mathrm{DOF}_{M,N,T}(C) = \lim_{\rho\to\infty} \frac{R(\rho; M, N, T, C)}{\sup_C R(\rho; 1, 1, 1, C)}. \tag{2}$$

Note: These definitions depend on C

C-independent definitions of DOF/DIV of a channel:

Definition 2

$$\mathrm{DIV}_{M,N,T}(C) = \lim_{\rho\to\infty} \frac{\inf_C \log P_e(\rho; M,N,T,C)}{\inf_C \log P_e(\rho;1,1,1,C)} \tag{3}$$

and

$$\mathrm{DOF}_{M,N,T}(C) = \lim_{\rho\to\infty} \frac{\sup_C R(\rho; M,N,T,C)}{\sup_C R(\rho;1,1,1,C)} = \lim_{\rho\to\infty} \frac{C(\rho;M,N,T)}{C(\rho;1,1,1)}. \tag{4}$$

Note: for IR Rayleigh fading channels, consistent with

- Heath and Paulraj's DOF (2001): asymptotic rate of minimum Euclidean distance in ρ
- Zheng and Tse's DOF (2002): asymptotic rate of C_{outage} in $\log\rho$
- Tarokh et. al.'s DIV (1998): asymptotic exponent of min-P_e in $\frac{1}{\rho}$

Application: Rayleigh fading MIMO system

$$X = \sqrt{\frac{\rho}{M}} SH + W$$

- X: $T \times N$; S: $T \times M$; and H: $M \times N$
- T: time interval over which H remains constant
- H i.i.d. Gaussian known to receiver (IR)
- W i.i.d. AWGN
- Constraint on C: $\mathrm{tr}\{E[SS^\dagger]\} \leq TM$.

1. DOF: Channel capacity is

$$C = TE[\log\det(I_N + \rho H^\dagger H)] = TE[\log\det(I_M + \rho H H^\dagger)]$$

which is attained by $S \sim \mathcal{N}(0, cI_T \bigotimes I_M)$

Hence:

$$\text{DOF}(M,N,T) = T\min(M,N)$$

2: DIV: Let transmitter adopt the space-time QPSK signal adopted by Tarokh etal (1999) and consider MAP receiver

Find:

$$\text{DIV}(M,N,T,C) = N\min(M,T)$$

DOF-DIV via error exponent approach

Gallager's Random Coding Error Exponents

- Establishes bound on probability of error P_e for any communication system having information rate R and transmitted distribution $P(S)$

$$P_e(\rho, M, N, T) \leq \exp\left(-LE_U(P(S), \rho)\right)$$

- Error exponent: $E_U(P(S), \rho) =$

$$\max_{\gamma \in [0,1]} \max_{P \in P} \left\{ -\gamma R - \underbrace{\ln \int_{X \in X} \left[\int_{S \in S} \left(p(X|S)\right)^{1/(1+\gamma)} dP(S) \right]^{1+\gamma} dX}_{E_0(\gamma, P_G(S), \rho)} \right\}$$

- Special case $\gamma = 1$ involves cutoff rate R_o: $P_e \leq \exp\{-(R_o - R)\}$

$$R_o(\rho, M, N, T) = \max_{P \in P} \ln \int_{X \in X} \left[\int_{S \in S} \left(p(X|S)\right)^{1/2} dP(S) \right]^2 dX$$

DIV and Cutoff Rate

$$R_o(H) = \max_{P_{S|H}} -\ln \int\int_{S_1,S_2\in\mathbb{C}^{T\times M}} dP_{S|H}(S_1)dP_{S|H}(S_2) \; e^{-ND(S_1\|S_2)}$$

1. T/R Informed cutoff rate: H known to both T/R

$$D(S_1\|S_2) = \frac{\rho}{4}\,\mathrm{tr}\left(H^\dagger(S_1-S_2)^\dagger(S_1-S_2)H\right) \Rightarrow \mathrm{DIV} = N min(T,M)$$

2. R informed cutoff rate: H known to R only

$$D(S_1\|S_2) = \ln\left|I_T + \frac{\rho}{4}\,(S_1-S_2)(S_1-S_2)^\dagger\right| \Rightarrow \mathrm{DIV} = N min(T,M)$$

3. Uninformed cutoff rate: H unknown to either T/R

$$D(S_1\|S_2) = \ln\frac{\left|I_T + \frac{\rho}{2}(S_1S_1^\dagger + S_2S_2^\dagger)\right|}{\sqrt{\left|I_T+\rho S_1S_1^\dagger\right|\;\left|I_T+\rho S_2S_2^\dagger\right|}} \Rightarrow \mathrm{DIV} = N(min(T,2M) - min(T,M))$$

Error Exponent: DOF-DIV Tradeoff Analysis

For IR and Gaussian $P(S)$ satisfying $E[\mathrm{tr}\{S^\dagger S\}] \leq MT$

$$E_0(\gamma, P_G(S), \rho) = -\log E\left[\det\left(I_M + \frac{1}{1+\gamma}\frac{\rho}{M}HH^\dagger\right)^{-\gamma T}\right]$$

Hence:

$$\underbrace{-\log P_e(\rho, M, N, T)}_{DIV(M,N,T)} \geq E_0(\gamma, P(S), \rho) - \gamma \underbrace{R(\rho; M, N, T)}_{DOF(M,N,T)}$$

where

$$R(\rho, M, N, T) = \frac{\partial E_0(\gamma, P(S), \rho)}{\partial \gamma}$$

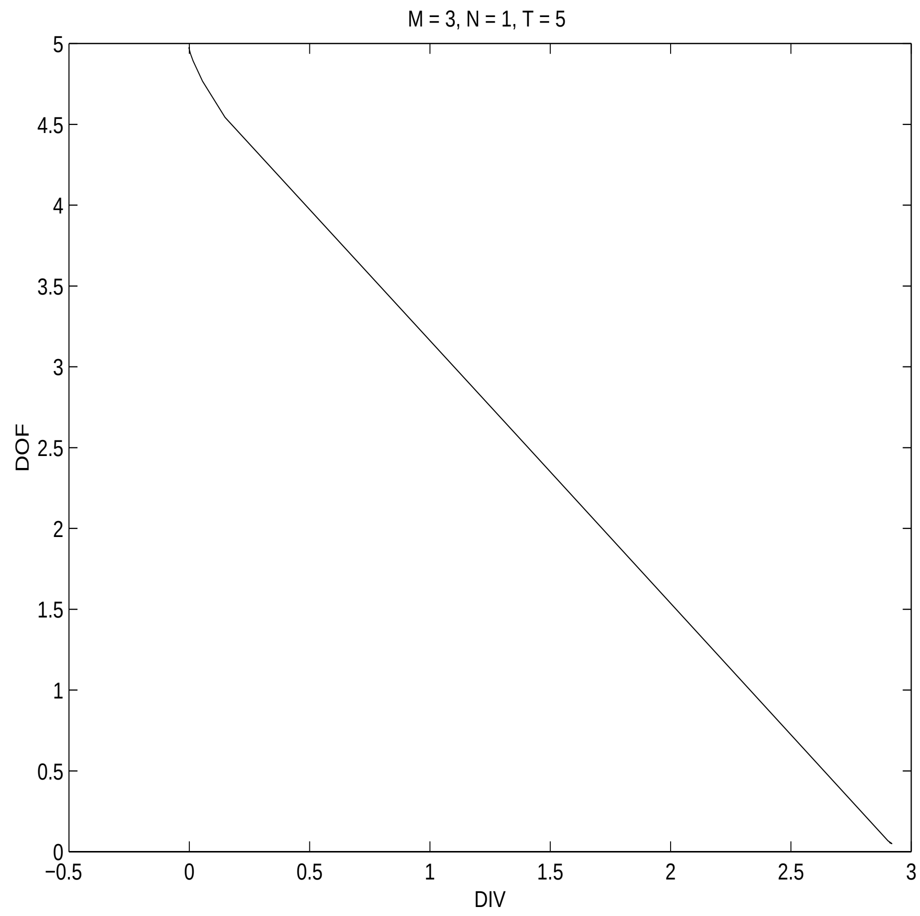

Figure 3: DOF(C_γ) as a function of DIV(C_γ). The y-intercept is $T\min(M,N)$ and the x-intercept is $N\min(M,T)$

Conclusions

- DOF and DIV are convenient encapsulations of link performance
- General system-dependent and -independent definitions for DOF and DIV given
- Our definitions specialize to previous definitions for Rayleigh channels
- Random coding exponents can be used to study design tradeoffs between DOF and DIV