EECS 564
Estimation, Filtering, and Detection
(*Prof. Hero*)

Homework Assignment 4, Fall 2004
SOLUTIONS

1. [7.2] Let Z be a random variable with values in the interval $[-1,1]$ having density function

$$p_\theta(z) = \frac{1}{2}\frac{3}{3+\theta}\left(\theta z^2+1\right)$$

where $\theta > 0$. Note θ controls the deviation of p_θ from the uniform density p_0. You are to test Z against non-uniformity given a *single sample* Z.

(a) Assuming priors $p = P(H_1) = 1 - P(H_0)$ derive the minimum probability of error (MAP) test for the simple hypotheses $H_0 : \theta = 0$ vs. $H_1 : \theta = \theta_1$, where θ_1 is a fixed and known positive value.

`Soln:`

`The Bayes minimum` P_e `test is a LRT of the form`

$$\frac{p_{\theta_1}(z)}{p_0(z)} = \frac{3}{3+\theta}\left(\theta z^2+1\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

`where` $\eta = p/(1-p)$. `Equivalently, as` $\theta_1 > 0$ `the min Pe test is`

$$|z| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \overset{\text{def}}{=} \sqrt{\frac{3+\theta_1}{3\theta_1}\eta - 1/\theta_1}.$$

(b) Find an expression for the ROC curve and plot for $\theta_1 = 0, 1, 10$.

`Soln:`

`The false alarm probability of the LRT of part (a) is`

$$\alpha = P(|z| > \gamma | H_0) = 1 - P_0(-\gamma \le z \le \gamma | H_0) = 1 - \tfrac{1}{2}\int_{-\gamma}^{\gamma} dz = 1-\gamma, \quad (0 \le \gamma < 1)$$

`The detection probability is`

$$\begin{aligned}\beta &= P(|z| > \gamma | H_1) = 1 - \tfrac{1}{2}\frac{3}{3+\theta_1}\int_{-\gamma}^{\gamma}\left(\theta_1 z^2 + 1\right)\, dz \\ &= 1 - \frac{3}{3+\theta_1}[\gamma^3\theta_1/3 + \gamma].\end{aligned}$$

`The ROC curve` $\beta(\alpha)$ `is obtained by substitution of the expression` $\gamma = 1-\alpha$. `The plot is a straight line for` $\theta_1 = 0$ `and becomes concave with increasing |curvature| as` θ_1 `increases.`

(c) Now find the form of the min-max test. Show how you can use your answer to part b) to graphically determine the min-max threshold.

`Soln:`

The min-max test is obtained by solving the equation $\alpha = P_F(\gamma) = P_M(\gamma) = 1-\beta$ or, γ satisfies cubic equation

$$\frac{3}{3+\theta_1}[\gamma^3\theta_1/3+\gamma] - (1-\gamma) = 0$$

or

$$a_3\gamma^3 + a_1\gamma - 1 = 0$$

where $a_3 = \frac{\theta_1}{3+\theta_1}$ and $a_1 = \frac{6+\theta_1}{3+\theta_1}$. Use polynomial rooting (matlab roots.m) command to find solution $\gamma \in [0,1]$ for a specific value of θ_1. Alternatively, the slope of the ROC curve $\beta = \beta(\alpha)$ at the intersection of the diagonal line $\beta = 1-\alpha$ with the ROC curve $\beta(\alpha)$ gives the min-max value of the LRT threshold $\eta = \eta_{minmax}$. This value determines γ_{minmax} through the relation $\gamma_{minmax} = \sqrt{\frac{3+\theta_1}{3\theta_1}\eta_{minmax} - 1/\theta_1}$.

(d) Derive the MP test for the same simple hypotheses as in part (a). Is the MP test uniformly most powerful (UMP) for the composite hypotheses $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$?

Soln:

The MP test is identical to the Bayes test of part (a) except that the threshold γ is set to give $P_F = \alpha$, yielding

$$|z| \underset{H_0}{\overset{H_1}{\gtrless}} 1-\alpha$$

Note that no randomization is necessary here since we can attain arbitrary $\alpha \in [0,1]$ without randomizing. As the decision regions are independent of θ_1 for $\theta_1 > 0$, this is an UMP test.

2. [7.3] Let Z be a single observation having density function

$$p_\theta(z) = (2\theta z + 1 - \theta), \qquad 0 \le z \le 1$$

where $-1 \le \theta \le 1$.

(a) Find the most powerful test between the hypotheses

$$H_0 : \theta = 0$$
$$H_1 : \theta = 1$$

Be sure to express your test in terms of the false alarm level $\alpha \in [0,1]$. Plot the ROC curve for this test.

Soln:

MP test is LRT

$$\frac{p_1(z)}{p_0(z)} = 2z \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

where η is determined to give $P_F = \alpha$. This test is equivalent to

$$z \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

where γ satisfies

$$\alpha = P_O(z > \gamma) = \int_\gamma^1 dz = 1-\gamma$$

Note that no randomization is necessary here since we can attain arbitrary $\alpha \in [0,1]$ without randomizing.

The power of this test is

$$\beta = P_1(z > \gamma) = \int_\gamma^1 2z dz = 1 - \gamma^2 = 1 - (1-\alpha)^2$$

which specifies the ROC curve $\beta(\alpha)$.

(b) repeat part (a) for $H_1 : \theta = -1$.

Soln:

Now MP test is

$$\frac{p_{-1}(z)}{p_0(z)} = -2z + 2 \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad \eta$$

or

$$z \quad \overset{H_0}{\underset{H_1}{\gtrless}} \quad \gamma$$

with

$$\alpha = P_O(z \le \gamma) = \int_0^\gamma dz = \gamma$$

Power of test is

$$\beta = P_1(z \le \gamma) = \int_0^\gamma 2(1-z)dz = \int_\gamma^1 2zdz = 1 - \gamma^2 = 1 - \alpha^2$$

similarly to part (a).

3. [7.4] It is desired to test the following hypotheses based on a single sample x:

$$\begin{aligned} H_0 &: \quad x \sim f_0(x) = \frac{3}{2}\, x^2, -1 \le x \le 1 \\ H_1 &: \quad x \sim f_1(x) = \frac{3}{4}\, (1 - x^2), -1 \le x \le 1 \end{aligned}$$

(a) Under the assumption that the prior probabilities of H_0 and H_1 are identical, find the minimum probability of error (Bayes) test.

Soln:

Bayes test is

$$\frac{f_1(x)}{f_0(x)} = \frac{1-x^2}{2x^2} \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad \eta$$

where $\eta = 1$ for equiprobable hypotheses. Equivalently

$$1 - x^2 - 2x^2 = 1 - 3x^2 \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad 0$$

which gives the H_1 decision region as

$$x^2 \le 1/3 \quad \text{or} \quad -1/\sqrt{3} \le x \le 1/\sqrt{3}$$

(b) Find the Most Powerful test of level $\alpha \in [0,1]$.

`Soln:`

MP test is same as above except that η must be set to ensure $P_F = \alpha$. Therefore, mimicking the steps of part (a) we have the MP test

$$1 - x^2 - 2\eta x^2 = 1 - (1+2\eta)x^2 \quad \begin{smallmatrix} H_1 \\ > \\ < \\ H_0 \end{smallmatrix} \quad 0$$

or equivalently

$$|x| \quad \begin{smallmatrix} H_0 \\ > \\ < \\ H_1 \end{smallmatrix} \quad \sqrt{1/(1+2\eta)}$$

Defining $\gamma = \sqrt{1/(1+2\eta)}$, the false alarm probability is

$$P_F = P(-\gamma \le x \le \gamma | H_0) = (3/2)\int_{-\gamma}^{\gamma} x^2 dx = \gamma^3$$

so that, setting $P_F = \alpha$, $\gamma = \alpha^{1/3}$.

(c) Derive and plot the ROC curve for these tests.

`Soln:`

The ROC's for the Bayes and MP tests are identical since they differ only in their threshold specifications. With $\beta = P_D$ it is seen easily that

$$\begin{aligned} \beta \quad &= P(-\gamma \le x \le \gamma | H_1) \\ &= (3/4)\int_{-\gamma}^{\gamma} (1-x^2)dx \\ &= \frac{\gamma}{2}(3-\gamma^2) \\ &= \frac{\alpha^{1/3}}{2}(3-\alpha^{2/3}) \end{aligned}$$

which specifies the ROC.

4. [8.1] The observations $\{x_i\}_{i=1}^n$ are i.i.d. exponential $x_i \sim f_\theta(x) = \beta e^{-\beta x}$, where $x, \beta \ge 0$. Consider testing the following single sided hypotheses

$$\begin{aligned} H_0 &: \beta = \beta_0 \\ H_1 &: \beta > \beta_0 \end{aligned}$$

(a) First find the MP test of level α for the simple alternative $H_1 : \beta = \beta_1$ where $\beta_1 > \beta_0$. Express the threshold in terms of the Gamma distribution (distribution of n i.i.d. exponential r.v.s). Next establish that your test is UMP for the single sided composite H_1 above.

`Soln:`

MP-LRT:

$$\Lambda = \frac{f(\underline{x};\beta_1)}{f(\underline{x};\beta_0)} = \prod_{i=1}^{n} \frac{\beta_1 e^{-\beta_1 x_i}}{\beta_0 e^{-\beta_0 x_i}} = \left(\frac{\beta_1}{\beta_0}\right)^n \ e^{(\beta_0-\beta_1)t} \quad \begin{smallmatrix} H_1 \\ > \\ < \\ H_0 \end{smallmatrix} \quad \eta$$

where $t = \sum_{i=1}^n x_i$ (a sufficient statistic). As $\beta_1 > \beta_0$ an equivalent test is

$$t = \sum_{i=1}^n x_i \overset{H_0}{\underset{H_1}{\gtrless}} \gamma$$

For any value of β the statistic t is the sum of n i.i.d. exponential r.v.s so it has a Gamma density (actually a special case of Gamma called the Erlang density) which you can call $f(t;\beta)$ (although you don't need to know this to solve the problem, it happens to be of the form $f(t;\beta) = \beta^n t^{n-1} e^{-\beta t}/\Gamma(n)$ where $\Gamma(n) = (n-1)!$). Define the gamma cdf $\Gamma_{n,\beta_0}(u) \stackrel{\text{def}}{=} \int_0^u f(t;\beta_0)dt$. This cdf is continuous since a sum of exponential r.v.s is a cts r.v. With this definition we have

$$P_F = P(t \leq \gamma | \beta = \beta_0) = \Gamma_{n,\beta_o}(\gamma)$$

so that for a specified level $\alpha \in [0,1]$ the threshold of the MP-LRT is $\gamma = \Gamma^{-1}_{n,\beta_0}(\alpha)$ yielding the MP-LRT

$$t = \sum_{i=1}^n x_i \overset{H_0}{\underset{H_1}{\gtrless}} \Gamma^{-1}_{n,\beta_0}(\alpha)$$

which is UMP as the form of the test does not depend on β_1 as long as $\beta_1 > \beta_0$.

(b) Specialize the results of (a) to the case of a single observation $n = 1$ and derive the ROC curve. Plot your curve for $\beta_1/\beta_0 = 1, 5, 10$.

Soln:

In this case $t = x_1$ is exponential with cdf $\Gamma_{1,\beta}(u) == \int_0^u \beta e^{-\beta_0 x} dx = 1 - e^{-\beta u}$. Therefore the threshold γ is given by setting $\Gamma_{1,\beta_0}(\gamma) = \alpha$ or

$$\gamma = \Gamma^{-1}_{1,\beta_0}(\alpha) = -\frac{1}{\beta_o}\ln(1-\alpha)$$

The ROC is determined from

$$P_D = P(x_1 \leq \gamma | \beta = \beta_1) = 1 - e^{-\beta_1 \gamma} = 1 - (1-\alpha)^{\beta_1/\beta_0}$$

(c) Derive the locally most powerful test (LMPT) for the single sided hypotheses (maximize slope of power subject to FA constraint) and verify that it is identical to the UMP test.

Soln:

LMPT is

$$\frac{d}{d\beta_0} \ln f(\underline{x}; \beta_0) \overset{H_1}{\underset{H_0}{\gtrless}} \gamma$$

The derivative is simply computed as $n/\beta_0 - t$, with t defined in part (a). Thus, absorbing the constant n/β_0 into γ, the LMPT is equivalent to the test obtained part (a)

(d) Now consider testing the double sided hypotheses

$$H_0 : \beta = \beta_0$$
$$H_1 : \beta \neq \beta_0$$

Derive the LMPT (maximize curvature of power subject to FA constraint and zero slope condition). Derive the ROC for $n = 1$ and compare to the ROC of part (b) over the region $\beta > \beta_0$.

Soln:
LMPT is

$$\frac{\frac{d^2}{d\beta_0^2} f(\underline{x};\beta_0)}{f(\underline{x};\beta_0) + \rho \frac{d}{d\beta_0} f(\underline{x};\beta_0)} \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad \gamma$$

where we have to determine ρ and γ to meet the FA constraint and the zero derivative condition $\frac{d}{d\beta} P_D|_{\beta=\beta_0} = 0$ on the power function of the test.
After computing the required derivatives and cancelling some terms, it is seen that the general form of the LMP test statistic is

$$\frac{\frac{d^2}{d\beta_0^2} f(\underline{x};\beta_0)}{f(\underline{x};\beta_0) + \rho \frac{d}{d\beta_0} f(\underline{x};\beta_0)} = \frac{t^2 - 2(n/\beta_0)t + \frac{n(n-1)}{\beta_0^2}}{1 + \rho(n/\beta_0 - t)}$$

which implies that the LMPT is equivalent to the quadratic test

$$t^2 - [2n/\beta_0 - \gamma\rho]t + \frac{n(n-1)}{\beta_0^2} - \gamma(1 + \rho n/\beta_0) \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad 0$$

We now specialize to the case of $n = 1$. The LMPT test above reduces to

$$x_1^2 - [2/\beta_0 - \gamma\rho]t - \gamma(1 + \rho/\beta_0) \quad \overset{H_1}{\underset{H_0}{\gtrless}} \quad \eta$$

The H_0 decision region is thus an interval on x_1 with endpoints given by the two roots of the quadratic

$$r_1, r_2 = \frac{2/\beta_o - \gamma\rho}{2} \pm \sqrt{\left(\frac{2/\beta_o - \gamma\rho}{2}\right)^2 + \gamma(1 + \rho/\beta_0)}$$

Note that if $\rho \geq 0$ there is only one positive root, t_1 say, and therefore, as $x_1 \geq 0$, the LMPT reduces to a linear test $x_1 \overset{H_1}{\underset{H_0}{\gtrless}} t_1$. Since it is not possible to satisfy the zero derivative constraint for such a test it must be that $\rho < 0$. Thus, unlike in Sec. 8.7, we cannot meet the constraints by assuming the simpler case $\rho = 0$.
We can solve for the threshold γ by replacing ρ with a new independent variable $u = \gamma\rho/2 < 0$. In this case, after some algebra, the two roots simplify to

$$r_1, r_2 = 1/\beta_o - u \pm \sqrt{1/\beta_o^2 + u^2 + \gamma}$$

and the false alarm constraint is

$$\alpha = P_F = 1 - P(r_1 \leq x_1 \leq r_2 | \beta = \beta_0) = 1 - (1 - e^{-\beta_0 r_1}) + (1 - e^{-\beta_0 r_2}) = 1 - 2e^{1 - u\beta_0} \sinh(\beta_0 \sqrt{1/\beta_o^2 + u^2 + \gamma}).$$

Therefore, solving for γ we obtain

$$\gamma = (\sinh^{-1}((1-\alpha)e^{u\beta_0 - 1}/2)/\beta_0)^2 - 1/\beta_0^2 - u^2.$$

And the ROC curve is now a function of α and u:

$$P_D = 1 - P(r_1 \leq x_1 \leq r_2 | \beta = \beta_1) = 1 - 2e^{(1/\beta_0 - u)\beta_1} \sinh\left(\frac{\beta_1}{\beta_0} \sinh^{-1}((1-\alpha)e^{u\beta_0 - 1}/2)\right)$$

The derivative constraint can now be satisfied by differentiating P_D wrt β_1, evaluating at $\beta_1 = \beta_0$, and using numerical root finding over u to solve for $u = \rho\gamma/2$ from which we can determine ρ.

(e) Derive the GLRT for the double sided hypotheses of part (d). Compare to your answer obtained in part (d).

`Soln:`

`GLRT is`

$$\Lambda_{GLR} = \frac{\max_{\beta \neq \beta_0} f(\underline{x}; \beta)}{f(\underline{x}; \beta_0)} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \eta$$

Using the expression for the derivative of the log likelihood function $\ln f(\underline{x}; \beta)$ derived in part (c) it is easily seen that the MLE of β under H_1 is $\hat{\beta} = 1/\overline{x_i}$ and the GLR test statistic is

$$\Lambda_{GLR} = \left(\frac{1}{\beta_0/\hat{\beta}} e^{\beta_0/\hat{\beta}} \right)^n e^{-n}$$

As the function $1/u\, e^u$ is convex over $u > 0$ (recall Sec. 9.3, Fig. 134) the GLRT is equivalent to a test with H_0 decision region

$$\gamma_- \leq \beta_0/\hat{\beta} \leq \gamma_+$$

or equivalently

$$t_1 \leq \overline{x_i} \leq t_2$$

where $t_1 = \gamma_-/\beta_0$, $t_2 = \gamma_+/\beta_0$, which for $n = 1$ is an ''interval test'' similar to the LMPT derived in part (d).

5. [8.3] A random variable X has density

$$f(x; \theta) = \frac{1 + \theta x}{2}, \quad -1 \leq x \leq 1$$

where $\theta \in [-1, 1]$.

(a) Find the MP test of level α for testing the simple hypotheses

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta = \theta_1 \end{aligned}$$

based on a single sample x, where $\theta_0 \in [-1, 0]$ and $\theta_1 \in (0, 1]$ are known. Derive and plot the ROC when $\theta_0 = 0$.

`Soln:`

The MP test of level α is the LRT:

$$\frac{f(x; \theta_1)}{f(x; \theta_0)} = \frac{1 + \theta_1 x}{1 + \theta_0 x} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \eta$$

where $\eta > 0$ is such that $P_{FA} = \alpha$. Equivalently,

$$(\theta_1 - \eta\theta_0)x \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} (\eta - 1)$$

As $\theta_0 \leq 0$, $\theta_1 - \eta\theta_0 > 0$ and the MP LRT reduces to the linear test

$$x \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \gamma$$

Now as x is a continuous r.v. we can set the threshold γ to achieve level α without randomization

$$P_{FA} = \int_{\gamma}^{1} (1+\theta_0)/2dx = \frac{1-\gamma+\theta_0(1-\gamma^2)/2}{2} = \alpha$$

This is a quadratic equation $\theta_0\gamma^2 + 2\gamma + (4\alpha - 2 - \theta_0) = 0$ in γ. If $\theta_0 = 0$ then $\gamma = 1 - 2\alpha$. If $\theta_0 < 0$ then the solution is

$$\gamma = \gamma_\pm = \frac{-1 \pm \sqrt{1 - 4\theta_0(4\alpha - 2 - \theta_0)}}{2\theta_0}$$

Only the ''+'' solution is in the required region $[-1, 1]$ so the MP LRT is simply

$$x \overset{H_1}{\underset{H_0}{\gtrless}} \begin{cases} \frac{-1+\sqrt{1-4\theta_0(4\alpha-2-\theta_0)}}{2\theta_0}, & \theta_0 < 0 \\ 1 - 2\alpha, & \theta_0 = 0 \end{cases}$$

The ROC for $\theta_0 = 0$ is simply

$$P_D = \beta = \int_{\gamma}^{1} (1+\theta_1)/2dx = \alpha + \theta_1(1 - (1-2\alpha)^2)/4$$

which is a concave quadratic as a function of α.

(b) Is there a UMP test of level α, and if so what is it, for the following hypotheses?

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta > 0 \end{aligned}$$

Soln:

From part (a), when $\theta_0 = 0$ the MP LRT is $x \overset{H_1}{\underset{H_0}{\gtrless}} 1 - 2\alpha$ which does not depend on θ_1. Hence the MP LRT is in fact UMP for the composite alternative and simple null hypotheses above.

(c) Now consider testing the doubly composite hypotheses

$$\begin{aligned} H_0 &: \theta \leq 0 \\ H_1 &: \theta > 0 \end{aligned}$$

Find the GLRT for the above hypotheses. Derive the threshold of the GLRT that ensures the level α condition $\max_{\theta\in[-1,0]} P_{FA}(\theta) \leq \alpha$.

Soln:

The GLRT is

$$\frac{\max_{\theta_1\in(0,1]} f(x;\theta_1)}{\max_{\theta_0\in[-1,0]} f(x;\theta_0)} = \frac{\max_{\theta_1\in(0,1]}(1+\theta_1 x)}{\max_{\theta_0\in[-1,0]}(1+\theta_0 x)} \overset{H_1}{\underset{H_0}{\gtrless}} \eta$$

where $\eta > 0$ is such that $P_{FA}(\theta_0) = \alpha$ for all $\theta_0 \in [-1, 0]$. If $x > 0$ then the maxima of the numerator and denominator occur for $\theta_1 = 1$ and $\theta_0 = 0$, respectively, and vice versa if $x < 0$ (ignore the case $x = 0$ since this has probability zero). Thus, we have

$$\frac{\max_{\theta_1\in(0,1]}(1+\theta_1 x)}{\max_{\theta_0\in[-1,0]}(1+\theta_0 x)} = \begin{cases} 1+x, & x > 0 \\ \frac{1}{1-x}, & x < 0 \end{cases}$$

Or equivalently, the MP LRT is

$$(1+|x|)^{\mathrm{sgn}(x)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Now for determining the threshold we need to compute

$$\begin{aligned}
P_{FA}(\theta_0) &= P_0(1+x>\eta|x>0)P_0(x>0)+P_0(1/(1-x)>\eta|x<0)P_0(x<0)\\
&= \int_{\max(0,\eta-1)}^{1} \frac{1+\theta_0 x}{2}dx + \int_{\min(0,1-1/\eta)}^{1} \frac{1+\theta_0 x}{2}dx\\
&= 1-\frac{\max(0,\eta-1)+\min(0,1-1/\eta)}{2}+\theta_0\frac{(1-\max(0,\eta-1))^2+(1-\min(0,1-1/\eta))^2}{4}
\end{aligned}$$

The maximum of P_{FA} over θ must be set to α. As the factor multiplying θ_0 is always non-negative, the maximum of the last line occurs for $\theta_0=0$ and

$$\max_{\theta_0\in[-1,0]} P_{FA}(\theta_0) = 1-\frac{\max(0,\eta-1)+\min(0,1-1/\eta)}{2}=\alpha$$

Therefore, if we select $\eta>1$ we have $\eta=2(1-\alpha)$ and GLRT is equivalent to the linear test

$$x \underset{H_0}{\overset{H_1}{\gtrless}} 1-2\alpha$$

as in part (b).

6. [8.5] Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the marginal density $X_i \sim f(x) = \epsilon g(x) + (1-\epsilon)h(x)$, where $\epsilon \in [0,1]$ is a non-random constant and $g(x)$ and $h(x)$ are known density functions. It is desired to test the composite hypotheses

$$\begin{aligned}
H_0 &: \epsilon = 1/2 \quad (1)\\
H_1 &: \epsilon > 1/2 \quad (2)
\end{aligned}$$

(a) Find the most powerful (MP) test between H_0 and the simple hypothesis $H_1: \epsilon=\epsilon_1$, where $\epsilon_1 > 1/2$ (you needn't solve for the threshold). Is your MP test a UMP test of the composite hypotheses (2)?

Soln:

MP test is the LRT

$$\frac{f(\underline{x};\epsilon)}{f(\underline{x};\frac{1}{2})} = \prod_{i=1}^{n} \frac{\epsilon g(x_i)+(1-\epsilon)h(x_i)}{\frac{1}{2}(g(x_i)+h(x_i))} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Can't take this any further since ϵ cannot be factored out of numerator for $n>1$. As the decison regions for this test depend on ϵ it cannot be UMP.

(b) Find the locally most powerful (LMP) test for (2). Show how you can use the CLT to set the threshold for large n.

Soln:

LMP is the LRT

$$\frac{d\ln f(\underline{x};\epsilon)}{d\epsilon}\Big|_{\epsilon=\frac{1}{2}} = \frac{d}{d\epsilon}\sum_{i=1}^{n}\ln(\epsilon g(x_i)+(1-\epsilon)h(x_i))\Big|_{\epsilon=\frac{1}{2}}$$

$$= \sum_{i=1}^{n} \frac{g(x_i) - h(x_i)}{\epsilon g(x_i) + (1-\epsilon)h(x_i)}\Big|_{\epsilon=\frac{1}{2}}$$

$$= 2\sum_{i=1}^{n} \frac{g(x_i) - h(x_i)}{g(x_i) + h(x_i)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The summands of this test, call them $Z_i = 2\frac{g(x_i)-h(x_i)}{g(x_i)+h(x_i)}$, are all i.i.d. random variables taking values in the interval $[-2, 2]$. Thus the mean μ_z and variance σ_z^2 of the Z_i's exist and by the CLT: $n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i \to \mathcal{N}(\mu_z, \sigma_z^2)$. Thus we can set the threshold η for $P_F = \alpha$ as

$$\eta = \sqrt{n}\sigma_z \mathcal{N}^{-1}(1-\alpha) + n\mu_z$$

(c) Find the generalized LRT (GLRT) test for (2) in the case of $n = 1$. Compare to your answer in part (b).

Soln:

GLRT is

$$\Lambda_{GLR} = \frac{\max_{\epsilon > \frac{1}{2}} f(x;\epsilon)}{f(x;\frac{1}{2})} = \frac{\max_{\epsilon>\frac{1}{2}}(\epsilon g(x) + (1-\epsilon)h(x))}{\frac{1}{2}(g(x)+h(x))} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Now as $\epsilon g(x) + (1-\epsilon)h(x)$ is linear in ϵ it takes its maximum value at $\epsilon = \frac{1}{2}$ if $h(x) > g(x)$ and at $\epsilon = 1$ otherwise. Therefore, the GLRT test statistic becomes

$$\Lambda_{GLR} = \begin{cases} \frac{g(x)}{\frac{1}{2}(g(x)+h(x))}, & h(x) < g(x) \\ 1, & h(x) > g(x) \end{cases}$$

The GLR is **not** the same as the LMPT.