

FREQUENCY ESTIMATION DERIVED FROM A DYNAMICAL SYSTEM ANALYSIS

Cyril Hory, Michael Ting and Alfred Hero

Department of Electrical Engineering and Computer Science,
University of Michigan,
1301 Beal Avenue Ann Arbor, MI. 48109 USA.

ABSTRACT

The main concern of this paper is to build a method of characterization and identification of a dynamical system with several degrees of freedom when only one observation record is available. The analysis of the number of degrees of freedom provided by performing a Principal Component Analysis leads to an eigendecomposition-based method. The link between the proposed method and the MUSIC algorithm is then drawn. This approach gives a new insight on the use of this type of methods.

1. INTRODUCTION

Performing single spin detection with Magnetic Resonance Force Microscopy (MRFM) raises questions ranging from theoretical to experimental physics. Apart from the understanding of how quantum and classical mechanics systems interact [5], an important issue is the characterization of a non-linear dynamical system from a single observation. That is the problem this communication proposes to tackle. A MRFM device consists of coupling a single electron in resonance in a magnetic field with a ferromagnetic cantilever. Under hypothesis H_0 , no spin is present. The cantilever is a harmonic oscillator whose natural frequency depends on the magnetic field gradient. The motion of the cantilever is described by a dynamical system with two degrees of freedom. The introduction of the non-linear coupling term of the cantilever with a single spin under hypothesis H_1 induces an increasing of the number of degrees of freedom from two to five. A detection scheme is proposed in [10] that performs a parallel tracking of the output signal under both hypothesis. This method requires a simulation of the systems of dynamical equations, and thus the estimation of all the degrees of freedom. Here the spin detection task is reduced to determining how many degrees of freedom are necessary to

describe the system from the only observation of the output of the cantilever.

We first recall that the Takens Embedding method provides a phase space reconstructed trajectory of a dynamical system from a single observation [9]. Under suitable approximations, it can be shown [1], that the coupling of the single spin with the cantilever results in a shift of the natural frequency. Thus, the characterization task turns out to be the detection and estimation of this shift. Working with a 5-dimensional phase space, we propose to determine the actual number of degrees of freedom by performing a Principal Component Analysis. This approach gives a new insight on the eigendecomposition-based methods of frequency estimation [4], [6]. We finally present a comparison of the proposed algorithm with the well-known MUSIC algorithm [3], [7].

2. ANALYSIS OF A DYNAMICAL SYSTEM

Consider a system of p differential equations with variables $d_1(t), d_2(t), \dots, d_p(t)$

$$\begin{cases} \frac{d_1(t)}{dt} = \Phi_1(d_1(t), d_2(t), \dots, d_p(t)), \\ \frac{d_2(t)}{dt} = \Phi_2(d_1(t), d_2(t), \dots, d_p(t)), \\ \vdots \\ \frac{d_p(t)}{dt} = \Phi_p(d_1(t), d_2(t), \dots, d_p(t)). \end{cases} \quad (1)$$

Suppose the only observation available is the component $d_1(t)$. There exists a mapping from the actual phase space

$$(d_1(t), d_2(t), \dots, d_p(t)),$$

to the space spanning the set of p delayed versions of the observation [9]

$$(d_1(t), d_1(t - \tau), \dots, d_1(t - (p - 1)\tau)).$$

This space is called the reconstructed phase space. Parameter τ is called the embedding delay. Fraser shows in [2] that an optimal choice of τ in terms

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of information redundancy is the first zero of the auto-correlation function of $d_1(t)$. That is how we define τ here. The actual number p of degrees of freedom of the dynamical system is the dimension of the trajectory. If unknown, it can be estimated from the Principal Component Analysis (PCA) of the reconstructed phase space. The estimation of p is then given by the number of non-null eigenvalues of the covariance matrix of the reconstructed phase space.

Consider that the available observation is the discrete signal

$$x[n] = d_1(nT_s) + \nu[n], \quad n = 0 \dots N - 1 \quad (2)$$

where T_s is the sampling period and $\nu[n]$ is the measurement white Gaussian noise with variance σ^2 . Note

$$x_n^l = [x[n], x[n+l], \dots, x[n+(p-1)l]]^T, \quad (3)$$

the l -times under-sampled segment of the observation. Performing PCA of the phase space consists of deriving the eigendecomposition of the $p \times p$ covariance matrix

$$C^l = \frac{1}{P} \sum_{n=0}^{P-1} x_n^l x_n^{lT}, \quad (4)$$

where $P = N - l(p-1)$ is the number of under-sampled segments. The under-sampling rate l is related to the embedding delay τ by

$$lT_s = \tau + \varepsilon T_s, \quad (5)$$

where $\varepsilon \in (-1/2, 1/2)$ is the error to the estimation of the embedding delay, due to the discretization of the observation. As P tends to infinity, C^l tends to the matrix $\Gamma^l = E\{x_n^l x_n^{lT}\}$. Assuming wide sense stationary dynamics, this matrix is of Toeplitz form with the element in the k th diagonal $\Gamma_{i,k}^l = \Gamma_{1+(i-1)l, 1+(k-1)l}^1$.

3. THE PHASE-SPACE TRAJECTORY OF THE MRFM DEVICE

The dynamics of a cantilever in a MRFM device is described by a system of $p = 5$ equations which variables are the position and speed of the cantilever tip and the three components of the spin [1], [10]. The available observation d_1 is the position of the cantilever measured by a laser interferometer. Under hypothesis H_0 (no spin), the cantilever is an harmonic oscillator and the number of degrees of freedom reduces to $p = 2$. The variable d_1 is, in this case, a sinusoid with frequency f_0 equal to the natural frequency of the oscillator:

$$H_0 : d_1[n] = A \sin(2\pi f_0 n T_s + \psi). \quad (6)$$

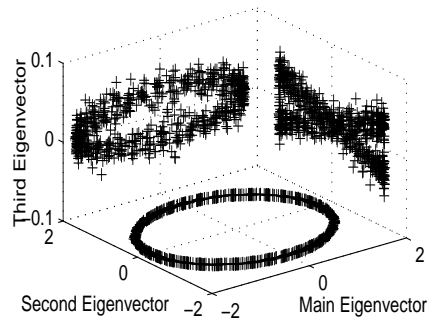


Fig. 1. Projection of the phase space trajectory on the three first eigenvector space. The central frequency is $f_0 = 7000$ Hz and the shift is $\delta f_0 = 30$ Hz.

In MRFM, the amplitude A of the signal and phase ψ are parameters controlled by the user. We assume they are known with values $A = 1$ and $\psi = 0$. Thus the frequency is the only parameter to be estimated. The trajectory in the 5-dimensional reconstructed phase space is an ellipsoid with radii and orientation depending on f_0 . This trajectory lies on a two-dimensional subspace.

The cantilever is driven by an external force which produces a cyclic adiabatic inversion (CAI) of the spin moment. It is shown in [1] that the effect of the CAI on the spin-cantilever coupling can be approximated as cyclically alternating positive and negative shift δf_0 of the natural frequency of the cantilever:

$$f = f_0 \pm \delta f_0. \quad (7)$$

Thus, under H_1 , the trajectory in the phase space is composed of two ellipsoids with different radii and different orientations (see figure 1). Under H_1 , the trajectory spans more than two dimensions.

4. THE EIGENDECOMPOSITION-BASED METHOD

The detection scheme we develop consists of tracking the orientation of the trajectory over a moving time window. This orientation is provided by the eigenvectors of the covariance matrix C^l rather than by the eigenvalues. As the output of the cantilever is a sinusoid, its period is approximately $4l$ where l is the discrete embedding delay defined by expression (5). We propose to perform a PCA of the reconstructed trajectory on successive sequences $[x_n^l, x_{n+1}^l, \dots, x_{n+M-1}^l]$. The eigenvectors of C^l are asymptotically unbiased estimators of the eigenvectors $\gamma_1^l, \gamma_2^l, \dots, \gamma_p^l$ of Γ^l [3]. The eigenvector γ_1^l corresponding to the highest eigen-

value is expressible as:

$$\gamma_1^l = [e(f_0)^4, e(f_0)^3, e(f_0)^2, e(f_0), 1]^T, \quad (8)$$

where $e(f_0) = \exp\{-i2\pi f_0 l T_s\}$. The PCA of the phase space provides an unbiased estimation of $\hat{e}(f_0)$. The estimated frequency \hat{f}_0 is the angular position of $\hat{e}(f_0)$:

$$\hat{f}_0 = \frac{1}{2\pi l T_s} \Im\{\log\{\hat{e}(f_0)\}\}, \quad (9)$$

where $\Im\{x\}$ is the imaginary part of the complex value x .

5. COMPARISON WITH MUSIC

The frequency estimation method described in the previous section is based on the eigendecomposition of the covariance matrix Γ^l . It can be seen as a member of the class of subspace methods [6]. One of the most popular method belonging to this class is the well-known MUSIC algorithm introduced first by Pisarenko in [4] for frequency estimation and extended to the problem of direction of arrival estimation (see for example [3] or [7] for analysis of performances of MUSIC and related algorithms).

Unlike our proposed method, MUSIC is based on the eigendecomposition of the entire covariance matrix Γ^1 of the data following model (2) with

$$d_1[n] = \sum_{j=1}^k A_j \exp\{i2\pi f_j n T_s + \psi_j\}, \quad (10)$$

sum of k cisoids. The estimates \hat{f}_j of the frequencies are solution of

$$a^*(f)G^1 G^{1*} a(f) = 0, \quad (11)$$

where

$$a(f) = [1, \dots, \exp\{i2\pi f(m-1)T_s\}]^T \quad (12)$$

and G^1 is the matrix of the noise subspace eigenvectors:

$$G^1 = [\gamma_{k+1}^1, \dots, \gamma_m^1], \quad (13)$$

namely, the eigenvectors with eigenvalues $\lambda_{k+1} = \dots = \lambda_m = \sigma^2$.

For a comparison with the MUSIC algorithm, a sinusoid of $f_0 = 7000$ Hz embedded in an additive white Gaussian noise of variance $\sigma^2 = 1$ has been simulated. Figure 2 presents the projection of the trajectory of the signal on the span of the three first eigenvectors $(\gamma_1^l, \gamma_2^l, \gamma_3^l)$ of the under-sampled

covariance matrix. Subsequently, the signal is projected on the signal subspace estimated by the MUSIC algorithm. The phase space trajectory of this filtered signal is projected on the space $(\gamma_1^l, \gamma_2^l, \gamma_3^l)$. One can observe that the projections of the two signals on the span of the two first eigenvectors space are similar. However, the trajectory of the filtered signal is nearly orthogonal to e_3 . The same observation can be made for the two last eigenvectors. This shows that the signal-space derived from the eigendecomposition of the under-sampled covariance matrix is bi-dimensional, as predicted by the analysis of the dynamical system.

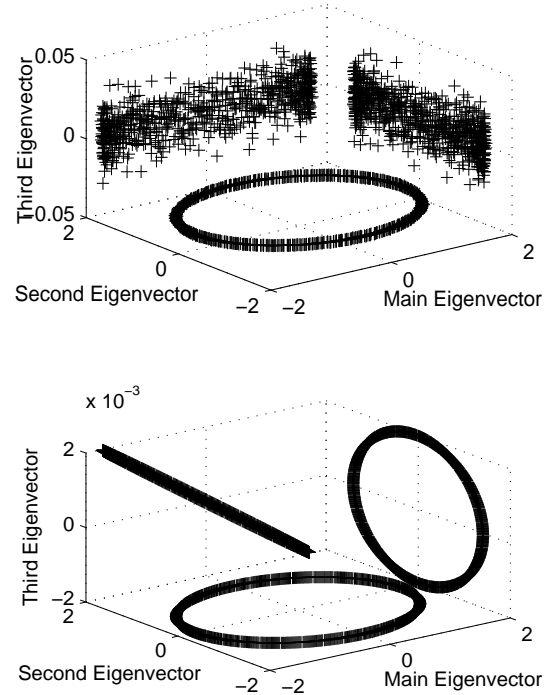


Fig. 2. Projections of the phase space trajectories on the three first eigenvector space of a sinusoid (top) and of its projection on the MUSIC signal space (bottom). The frequency is $f_0 = 7000$ Hz.

A comparative study of the performances of MUSIC and our proposed method is presented on figure 3. The frequency of a cisoid embedded in a white Gaussian noise of variance $\sigma^2 = 1$ is estimated by both methods using C^l for several under sampling rates l . The proposed method is applied with a constant length of snapshot m and decreasing dimension p as l increases such that $m = (p-1)l + 1$.

The bias of the MUSIC frequency estimator is almost constant over the delay l . However, the variance of MUSIC increases as l increases. This is due to the decreasing of the dimension p , given that

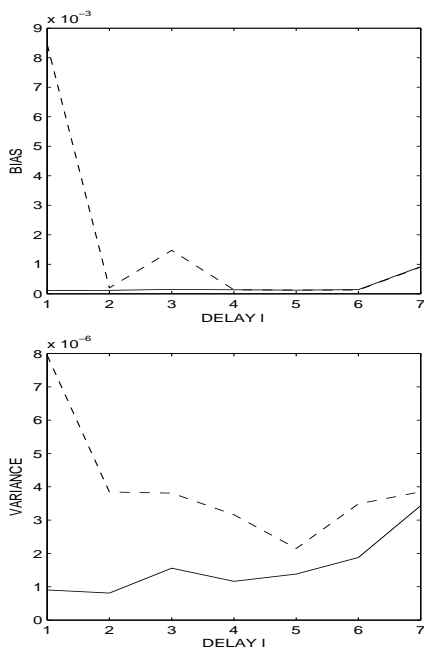


Fig. 3. Comparison between frequency estimates from MUSIC (plain lines) and from the proposed estimator (dashed lines) in terms of the delay l . The signal is a cisoid of normalized frequency $f_0 = 0.05$ with length $N = 200$ points. Each snapshot is of length $m = 21$ points before sub-sampling. The signal to noise ratio is $SNR = 0$ dB.

the variance of MUSIC does not depend on the normalized frequency [8]. Note that the performance of the method derived from the Takens embedding improves when l increases. The bias of Takens embedding is similar to the bias of MUSIC when $l > 4$, while the variance is still slightly higher. However one can see that the variance of the proposed method reaches a minimum value when $l = 5$. This delay is the value suggested by Fraser in [2].

6. CONCLUSION

The analysis of the phase space of a dynamical system has led us to the formulation of an eigendecomposition-based method of frequency estimation. This approach gives a new insight into the use of this type of method. The proposed one is directly derived from dynamical system analysis. The output signal model is non-stationary. Therefore, a straight application of MUSIC or related estimators is not valid. The use of the under-sampled covariance matrix suggested by the Takens embedding approach is better suited to non-stationary signal analysis and more specifically, estimation of shift in frequency.

7. REFERENCES

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