Robust Detection of SAR/IR Targets via Invariance

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Abstract

One of the most challenging problems in automatic target recognition is reliable detection of targets in high clutter backgrounds. When the clutter statistics are unknown or highly variable, the false alarm rate of classical detection algorithms, e.g. the matched filter, cannot be controlled and target detections become unreliable. The reason for this is lack of robustness of the test statistics to clutter variations. Here we apply *maximal invariants* to design target detection algorithms which have constant false alarm rate yet maintain high target detection rate. Numerical comparisons will be presented for multispectral and multisnapshot radar images which illustrate that significant gains are achievable using invariance approaches.

I. Introduction

We apply the method of invariance and exact robustness for performing optimal and robust target detection in images with Gaussian clutter. The basis for this method is a Gaussian GMANOVA (Generalized Multivariate ANalysis Of VAriance) model for the image. The method of invariance involves expressing uncertainty in the image parameters, e.g. unknown clutter covariance or target amplitude, as resulting from set of algebraic actions on the image by an appropriate operator algebra, i.e. a group of transformations. Once the uncertainty in the image has been mapped to such an algebra, one can often identify statistics whose statistical distributions are functionally invariant to unknown noise parameters yet entail minimum loss of target discrimination capability. On the basis of these statistics optimal invariant tests which have constant rate can often be specified. These statistics are called "maximal invariants," and they index the constant contours, called orbits or level sets, of the likelihood function associated with the invariant target detection problem.

The application of invariance methods to image and signal processing has been hampered by the mathematically abstract nature of operator algebras. This often makes it difficult to find maximal invariants and their statistical distributions for many target detection problems. Nonetheless, as will be shown in this paper, the payoff for the extra effort can be quite significant: invariant detection procedures can outperform mainstay methods such as the generalized likelihood ratio test (GLRT).

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II. Statistical image model

Let $\{\mathbf{X}_i\}_{i=1}^n$ be n statistically independent $m \times 1$ Gaussian vectors constructed by raster scanning a set of n 2D images. We call each of these vectors sub-images and we will assume that they each have identical $m \times m$ (clutter) covariance matrices $\mathbf{R}$ but with possibly different mean vectors (targets). Concrete examples are: multi-spectral images where subimages correspond to a scene at m different optical or radar wavelengths, multiple pulse SAR images where repeated probing of a scene produces a sequence of n subimages, polarized L or C band SAR where subimages correspond to $m = 3$ polarization components (HH, HV, VV) at n different spatial locations, or n disjoint spatial cells of a single IR/SAR image.

By concatenating these n vectors we obtain the matrix $\mathbf{X} = [\underline{X}_1, \ldots, \underline{X}_n]$:

$$\mathbf{X} = a\ \underline{s}\ \underline{b}^T + \mathbf{N}$$

where $\underline{s} = [s_1, \ldots, s_m]^T$ is the (scanned) target signature vector $\underline{b} = [b_1, \ldots, b_n]^T$ is a target location vector ($b_k = 1$ indicating that the k-th subimage possibly contains a target), and $\mathbf{N}$ is a multivariate Gaussian matrix: $\mathrm{vec}\{\mathbf{N}\} \sim \mathcal{N}\left(\underline{0}, \mathbf{R} \bigotimes \mathbf{I}_n\right)$ with i.i.d columns.

We denote the unknowns in the model by the unknown parameter vector $\theta = \{a, \underline{s}, \underline{b}, \mathbf{R}\} \in \Theta$, where Θ is the prior parameter range of uncertainty. We are interested in target detection in an image with unknown clutter covariance $\mathbf{R}$. The general form for this detection problem can be expressed via the two mutually exclusive hypotheses of target absent H_0 and target present H_1

$$\begin{aligned} H_0 &: \mathbf{X} \sim f(X; \theta_0), \quad \theta_0 \in \Theta_0 \\ H_1 &: \mathbf{X} \sim f(X; \theta_1), \quad \theta_1 \in \Theta_1 \end{aligned}$$

where Θ_0 and Θ_1 partition the parameter space into target absent and target present scenarios.

Now the detection objective can be stated as finding a test function to maximize the family of *ROC* curves (P_{FA}, P_D) where: $P_{FA} = P_{FA}(\theta_0)$ is the probability of false alarm and $P_D = P_D(\theta_1)$ is the probability of target detection. These two probabilities generally depend on unknown parameters $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$ and therefore no optimal *uniformly most powerful* (UMP) detector generally exists. Here we use the *invariance* approach [1] to designing suboptimal tests. In particular, the objective will be to find detectors which are most powerful among the restricted class of CFAR (constant false alarm rate) detectors (P_{FA} constant over Θ_0). The invariance approach is essentially a method of data reduction which is explained in the next section.

III. Data Reduction via Invariance

It is instructive to first consider the mechanism of data reduction associated with the *Sufficient Statistic*. Recall that $\mathbf{T} = T(\mathbf{X})$ is a sufficient statistic for testing between H_0 and H_1 if for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$ the likelihood ratio Λ depends on $\mathbf{X}$ only through $T(\mathbf{X})$:

$$\Lambda(\mathbf{X}, \theta_0, \theta_1) \stackrel{\text{def}}{=} \frac{f(\mathbf{X}; \theta_1)}{f(\mathbf{X}; \theta_0)} = g(T(\mathbf{X}), \theta_1, \theta_0)$$

The sets $\{\mathbf{X} : T(\mathbf{X}) = t\}_t$ can be thought of as **orbits** of $\mathbf{X}$ which specify constant contours of $\Lambda(\mathbf{X})$. Thus the sufficient statistic $T(\mathbf{X})$ indexes the orbits and preserves all information needed to discriminate between H_0 vs H_1.

The main idea behind data reduction via invariance is to find a statistic $\mathbf{Z} = Z(\mathbf{X})$, called a maximal invariant, which indexes the values (which we can think of as constant contours) of the set function

$$\tilde{\Lambda}(X) \stackrel{\text{def}}{=} \{\Lambda(X, \theta_0, \theta_1) : \theta_0 \in \Theta_0, \theta_1 \in \Theta_1\}$$

To make this practical a tractable mathematical characterization of this set function must be adopted. This can be accomplished when the probability model has functional invariances which can be characterized by group actions on the measurement space $\mathbf{X}$ and the parameter space θ.

Let $\mathcal{G}$ be a group of transformations $g : \mathcal{X} \to \mathcal{X}$ acting on $\mathbf{X}$. Assume that for each $\theta \in \Theta$ there exists a unique $\overline{\theta} = \overline{g}(\theta)$ such that $f_\theta(g(\mathbf{X})) = f_{\overline{\theta}}(\mathbf{X})$. $\overline{g}$ is called the induced group action on Θ. The above relation implies that natural invariances which exist in the parameter space of θ imply a natural invariance in the space of measurements $\mathbf{X}$. If we further assume that $\overline{g}(\Theta_0) = \Theta_0 \quad \overline{g}(\Theta_1) = \Theta_1$, then the model and the decision problem are said to be *invariant to G*. The *orbits* of $\mathbf{X}$ under actions of $\mathcal{G}$ are defined by: $\mathbf{X} \equiv \mathbf{Y}$ if $\exists\, g \in \mathcal{G} : \mathbf{Y} = g(\mathbf{X})$. The orbits of θ under actions of $\overline{\mathcal{G}}$ are similarly defined.

The principle of invariance stipulates that any optimal decision rule should only depend on $\mathbf{X}$ through the *maximal invariant* $\mathbf{Z} = Z(\mathbf{X})$ which indexes these orbits in the sense that:

1. $\mathbf{Z}$ constant along orbits of $\mathbf{X}$ (invariant property): $\mathbf{Z}(g(\mathbf{X})) = \mathbf{Z}(\mathbf{X})$, and
2. $\mathbf{Z}$ discriminates between orbits (maximal property): $Z(\mathbf{X}) = Z(\mathbf{Y}) \Rightarrow \mathbf{Y} = g(\mathbf{X}), \quad g \in \mathcal{G}$

It can be shown that the probability density $f(Z; \delta)$ of $\mathbf{Z}$ only depends on θ through a reduced set of parameters $\delta = \delta(\theta)$, which is the induced maximal invariant under $\overline{\mathcal{G}}$. Use of $\mathbf{Z}$ for detection gives the equivalent set of hypotheses:

$$\begin{aligned} H_0 &: \mathbf{Z} \sim f(Z; \delta(\theta_0)), \quad \theta_0 \in \Theta_0 \\ H_1 &: \mathbf{Z} \sim f(Z; \delta(\theta_1)), \quad \theta_1 \in \Theta_1. \end{aligned}$$

Since δ is generally a dimension reducing function of θ, use of the reduced data $\mathbf{Z}$ gives us better chances of finding a most powerful CFAR test. In particular, when $\delta(\theta_0)$ is constant over Θ_0 the distribution of $\mathbf{Z}$ under H_0 is fixed and therefore any test based on $\mathbf{Z}$ will automatically be CFAR.

IV. IR/SAR Target detection applications

Case I: Unknown target and clutter:
Here the signal signature $\underline{s}$ is a completely unknown $m \times 1$ vector and the clutter covariance $\mathbf{R}$ is an unknown $m \times m$ positive definite matrix. We first consider the case of known target location, i.e. the vector $\underline{b}$ is known. In this case we can left multiply $\mathbf{X}$ by an $n \times n$ unitary matrix V to put $\mathbf{X}$ in canonical form

$$\mathbf{X} \;=\; \underline{s} \cdot \underline{e}_1^T + \mathbf{N} = [\underline{x}_1, \;\; \mathbf{X}_2], \tag{1}$$

where $\underline{e}_1 = [1, 0 \ldots, 0]$. Note that $(m-1) \times n$ matrix $\mathbf{X}_2$ is a clutter-only component of $\mathbf{X}$. The hypotheses H_0 and H_1 correspond to $\Theta_0 = \{\underline{s} = 0, \mathbf{R} > 0\}, \quad \Theta_1 = \{\underline{s} \neq 0, \mathbf{R} > 0\}$. Define the group of transformations $\mathcal{G}$ with group action: $g(\mathbf{X}) = \mathbf{FXH}$, where

- $\mathbf{F}$ invertible $m \times m$
- $\mathbf{H}$ unitary of form: $\mathbf{H} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbf{U} \end{bmatrix}$

Note: $g(\mathbf{X})$ remains multivariate Normal, $\overline{g}(\underline{s}, \mathbf{R}) = \{\mathbf{F}\underline{s}, \mathbf{FRF}^T\}$, and $\overline{g}(\Theta_0) = \Theta_0$ and $\overline{g}(\Theta_1) = \Theta_1$. Hence the probability model is invariant under $\mathcal{G}$. It can be shown that with $\mathbf{X} = [\underline{x}_1, \mathbf{X}_2]$ the **maximal invariant** is given by the scalar:

$$z(\mathbf{X}) = \underline{x}_1^T [\mathbf{X}_2 \mathbf{X}_2]^{-1} \underline{x}_1$$

Furthermore the **induced maximal invariant** is $\delta = \underline{s}^T \mathbf{R}^{-1} \underline{s}$. Written in an alternative but equivalent form in the original coordinates

$$z(\mathbf{X}) = \frac{\underline{b}^T \mathbf{X}^T [\mathbf{X}\mathbf{X}^T]^{-1} \mathbf{X} \underline{b}^T}{\|\underline{b}\|^2}$$

$z(\mathbf{X})$ is seen to be equivalent to calculating the length of the projection of the location vector $\underline{b}$ onto the row space of the measurements $\mathbf{X}$.

Using the reduced data $z(\mathbf{X})$ we have equivalent hypotheses: $\Theta_0 = \{\delta = 0\}, \quad \Theta_1 = \{\delta > 0\}$. The density function of $z(\mathbf{X})$ is non-central F and the most powerful test based on z is a threshold test which is CFAR and uniformly most powerful over Θ_1. The test is equivalent to the Kelly/Reed GLRT test [2],[3].

Case II: Known target signature, unknown clutter:
In this case the target signature is known, and is known to lie within the first subimage, but its position within this subimage is unknown as is the target amplitude a and the clutter covariance $\mathbf{R}$. Let there be p possible positions for the target ($p < m$). Denoting $\mathbf{S}$ the $m \times p$ matrix $[\underline{s}_1, \ldots, \underline{s}_p]$ of these target signatures at different positions, and $\underline{e}$ any of the $p \times 1$ unit vectors $\underline{e}_1, \ldots, \underline{e}_p$ we have the following model $\mathbf{X} = a\mathbf{S}\underline{e} \cdot \underline{b}^T + \mathbf{N}$. Since the target signature and positions are known we can apply a coordinate rotation to put the model in the canonical form

$$\mathbf{X} = a \cdot \|\underline{s}\| \cdot \underline{e} \cdot \underline{e}_1^T + \mathbf{N}$$

Now the hypotheses to be tested are $\Theta_0 = \{a = 0, \mathbf{R} > 0\}$, $\Theta_1 = \{a \neq 0, \mathbf{R} > 0\}$. The group of transformations leaving the model and the decision problem invariant is $\mathcal{G}$ having group action: $g(\mathbf{X}) = \mathbf{FXH}$,

- $\mathbf{F}$ invertible $m \times m$ of form: $\mathbf{F} = \begin{bmatrix} \alpha \mathbf{I}_p & \mathbf{B} \\ O & \mathbf{M} \end{bmatrix}$.
- $\mathbf{H}$ unitary $n \times n$ of form: $\mathbf{H} = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbf{U} \end{bmatrix}$.

Let $\mathbf{X}$ have the partition:

$$\mathbf{X} = \begin{bmatrix} \underline{x}_{11} & \mathbf{X}_{12} \\ \underline{x}_{21} & \mathbf{X}_{22} \end{bmatrix}$$

where $\underline{x}_{11}$ is $p \times 1$, $\mathbf{X}_{12}$ is $p \times (n-1)$, $\underline{x}_{21}$ is $(m-p) \times 1$, and $\mathbf{X}_{22}$ is $(m-p) \times (n-1)$.

Define the $p \times 1$ vector $\underline{u}$ and the $p \times p$ matrix $\mathbf{D}$

$$\begin{aligned} \underline{u} &= \underline{x}_{11} - \mathbf{X}_{12}\mathbf{X}_{22}^T[\mathbf{X}_{22}\mathbf{X}_{22}^T]^{-1}\mathbf{X}_{22}\underline{x}_{21} \\ \mathbf{D} &= \mathbf{X}_{12}[\mathbf{I}_{n-1} - \mathbf{X}_{22}^T[\mathbf{X}_{22}\mathbf{X}_{22}^T]^{-1}\mathbf{X}_{22}]\mathbf{X}_{12}^T. \end{aligned}$$

$\underline{u}$ is an estimate of the (canonical) target signature vector $\underline{c}$ formed by subtracting the least squares clutter estimate $\hat{\underline{x}}_{12|22}$ from the target+clutter component $\underline{x}_{11}$. $\underline{u}$ will have independent zero mean elements under H_0, while it will have mean $\underline{c}$ under H_1. $\mathbf{D}$ is the sample covariance matrix of the least squares clutter estimator errors $\underline{x}_{12} - \hat{\underline{x}}_{12|22}$. The **maximal invariant** under $\mathcal{G}$ can be expressed as:

$$\begin{aligned} z_1(\mathbf{X}) &= \underline{u}^T\mathbf{D}^{-1}\underline{u}, && \text{(estimated SNR)} \quad (2) \\ z_2(\mathbf{X}) &= \underline{x}_{21}^T[\mathbf{X}_{22}\mathbf{X}_{22}^T]^{-1}\underline{x}_{21} && \text{(multiple correlation factor)} \\ z_3(\mathbf{X}) &= \frac{\underline{u}\underline{u}^T}{\|\underline{u}\|^2} && \text{(estim. target direction cosines).} \end{aligned}$$

Case III: Target straddling clutter boundary: This is the difficult deep hide scenario where the target parks along a known boundary separating two adjacent clutter regions A and B, e.g. agricultural field vs. forest canopy. In this case $\underline{s}$ is known but at an unknown position within the first subimage, a is unknown, and $\mathbf{R}$ is unknown but has known block diagonal structure. This block diagonal structure arises from the reasonable assumption that the two clutter types are statistically independent. The maximal invariant for the case of known target position ($p = 1$) is identical to the 5 scalar quantities derived in [4] for a related problem of sensor array detection of narrowband propagating signals. For the case of unknown position we can show that the first four components are pairs (z_1, z_2), computed from (2) over region A, and (z_3, z_4), computed from (2) over region B. The component z_5 is a ratio of estimated clutter prediction residuals over regions A and B. This ratio accounts for the fact that the total energy $\|\underline{s}\|^2$ of the target signature is known. The **Induced Max. Invariant** has the form $\delta(\theta) = \{\mathrm{SNR}_1^A, \mathrm{SNR}_p^B \ldots, \mathrm{SNR}_p^A \mathrm{SNR}_p^B, \mathrm{CCR}_{AB}\}$ which are, respectively, the SNR's associated with the p different target positions in regions A and B, and the clutter power ratio of regions A and B. As in the case of $p = 1$ studied in [4], the joint density of $z_1, \ldots, z_5$ is not closed form and, due to the CCR_{AB} term in the induced parameter space, a test based on the maximal invariant is not guaranteed to be CFAR. Here we consider a sub-optimal test (reducing to Bose's test for $p = 1$) which our simulations show to be approximately CFAR for $2p < m$:

$$\max_k \left\{ \frac{\left(q_A\ \underline{s}_{kA}^T\mathbf{D}_A^{-1}\underline{u}_A + q_B\ \underline{s}_{kB}^T\mathbf{D}_B^{-1}\underline{u}_B\right)^2}{q_A\ \underline{u}_A^T\mathbf{D}_A^{-1}\underline{u}_A + q_B\ \underline{u}_B^T\mathbf{D}_B^{-1}\underline{u}_B} \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \quad (3)$$

where $\underline{s}_{kA}$ and $\underline{s}_{kB}$, $k = 1, \ldots, p$, are $p \times 1$ vectors corresponding to the k-th column of the $p \times p$ triangular matrices R_A, R_B obtained from the QR decompositions of $\mathbf{S}_A$ and $\mathbf{S}_B$ in the region-partitioned $m \times p$ target matrix $\mathbf{S} = \begin{bmatrix} \mathbf{S}_A \\ \mathbf{S}_B \end{bmatrix}$, and we have defined $q_A = (1 + z_{1A} + z_{2A})$, $q_B = (1 + z_{1B} + z_{2B})$. We will compare this to the Kelly GLRT test which can be expressed in the form:

$$\max_k \left\{ \frac{\left(\underline{s}_{kAB}^T\mathbf{D}_{AB}^{-1}\underline{u}_{AB}\right)^2}{q_{AB}\underline{u}_{AB}^T\mathbf{D}_{AB}^{-1}\underline{u}_{AB}} \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \quad (4)$$

where the subscript AB indicates that the quantity is computed over the entire image (without boundary information). We will also compare our test (3) to a modified Kelly test which accounts for boundary information by using a block diagonal form $\mathbf{D}_{AB} = \begin{bmatrix} \mathbf{D}_A & O \\ O & \mathbf{D}_B \end{bmatrix}$ in (4).

V. Numerical Example

We simulated an IR Farmland Clutter image containing a road passing through two corn fields under the assumptions that: Corn field clutter covariances are unknown; road defines a linear boundary which has been presegmented from background. 100 subimages were defined by sliding windows of size 10×10 pixels along the boundary. A single known 5×5 synthetic "cross hair" target signature was inserted into the first subimage at an known location in an inscribed 6×4 grid. We implemented three target detectors: the proposed test (3) and two versions of Kelly's GLRT [2], one in original unstructured form and the other modified to account for the presence of the clutter boundary. By varying the thresholds on these tests and gathering empirical statistics on false alarm and detecion rates, we generated ROC curves (shown on the attached figures). These ROC curves indicate the significant gains in detection probability which are achievable through the use of the test (3) relative to classical GLRT methods.

References

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Fig. 1. (a) A segment showing three 10×10 subimages of the 100 simulated subimages. Cross hair target signature is located in first subimage at a SNR of 0dB.

Fig. 2. Cross hair target signature.

Fig. 3. ROC curves relating P_{FA} and P_D for Kelly's GLRT (structured and unstructured) and Proposed Test: SNR = 0dB

Fig. 4. ROC curves relating P_{FA} and P_D for Kelly's GLRT (structured and unstructured) and Proposed Test: SNR = 20dB