Image Resolution-Variance Tradeoffs Using the Uniform Cramer-Rao Bound

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• Most image reconstruction algorithms of interest are *biased*.
• Tradeoff between image bias, resolution, and noise.

- For a given amount of bias or resolution, how noisy will the images be?
- How should one quantify bias, resolution, and noise in a meaningful way?
- How are these quantities related?

• More specifically
  - what is the fundamental limit of a particular imaging system’s performance, *independent of the choice of estimation algorithm*?
Estimator mean, bias and variance

• Reconstruction at pixel $p$ 
  \[ \hat{\theta}_p = e_p^T \hat{\theta} \]

• Mean function 
  \[ m_{\hat{\theta}} \equiv E_{\hat{\theta}} \left[ \hat{\theta}_p \right] \]

• Bias function 
  \[ b_{\hat{\theta}} \equiv E_{\hat{\theta}} \left[ \hat{\theta}_p \right] - \theta_p \]

• Variance function 
  \[ \text{var}_{\hat{\theta}} \left( \hat{\theta}_p \right) \equiv E_{\hat{\theta}} \left( \left[ E_{\hat{\theta}} \left[ \hat{\theta}_p \right] - \theta_p \right]^2 \right) \]
Fundamental Limits on Variance

- CR Bound (biased)

$$\text{var}_\theta(\hat{\theta}_p) \geq \left[ \nabla m_\theta \right]^T F_Y^{-1} \left[ \nabla m_\theta \right]$$

- Alternative form

$$\text{var}_\theta(\hat{\theta}_p) \geq \left[ e_p + \nabla b_\theta \right]^T F_Y^{-1} \left[ e_p + \nabla b_\theta \right]$$

- Uniform CR Bound ($\|\nabla b_\theta\|_C^2 \equiv \nabla b_\theta^T C \nabla b_\theta$)

$$\text{var}_\theta(\hat{\theta}_p) \geq \arg\min \left( e_p + \nabla b_\theta \right)^T F_Y^{-1} \left( e_p + \nabla b_\theta \right)$$

$$\|\nabla b_\theta\|_C \leq \delta$$
UCRB Applied to Image Restoration

• Image restoration model: Gaussian blur with additive gaussian noise

• UCRB specifies fundamental limit on restoration variance using *any* deconvolution algorithm
Image Restoration Example: Mean Gradient vs. Point Response

- Bias-gradient norm $\delta=0.1$ (left) and $\delta=0.5$ (right)
Deficiency of UCRB

- Estimator point response functions can have *identical* bias gradient length, but different resolution properties
  - Identical bias-gradient length $\delta = 0.5$
  - Different spread (or 2nd moment)
- Hence two different systems can have different recoverable resolution, but identical bias gradient length and variance
Extended UCRB

• Perform constrained minimization on variance bound

\[
\text{var}_\theta\left(\hat{\theta}_p\right) \geq \arg\min_{\nabla b_{\theta}} \left( e_p + \nabla b_{\theta} \right)^T F_{\bar{Y}}^{-1} \left( e_p + \nabla b_{\theta} \right)
\]

– Subject to the following two constraints:

1) Maximal Bias Variation Constraint

\[
\left\| \nabla b_{\theta} \right\|_C^2 \leq \delta^2
\]

2) Resolution Constraint

\[
\frac{\left( e_p + \nabla b_{\theta} \right)^T M_p \left( e_p + \nabla b_{\theta} \right)}{\left( e_p + \nabla b_{\theta} \right)^T \left( e_p + \nabla b_{\theta} \right)} \leq \gamma^2
\]

• Calculate resulting Bias-Resolution-Variance surface
Geometric Interpretation of UCRB with Resolution Constraint

\[ \text{var}_\theta (\hat{\theta}_p) \geq \arg \min_{d \in B \cap R} \left[ \epsilon_p + d \right]^T F^{-1}_Y \left[ \epsilon_p + d \right] \]
Geometric Interpretation of UCRB with Resolution Constraint

$$\text{var}_q(\theta_p) \geq \arg\min_{d \in B \cap R} \left[ (\varepsilon_p + d)^T F^{-1}_{\theta_p} [\varepsilon_p + d] \right]$$

- Constrained Minimum
- Level Sets of FIM Quadratic Form
- Unconstrained Minimum

Bias-gradient constraint ellipse

$$B = \{ d : \|d\|^2_C \leq \delta^2 \}$$
Geometric Interpretation of UCRB with Resolution Constraint

\[
\text{var}_d(\hat{\theta}_p) \geq \arg\min_{d \in B \cap R} \left[ (\varepsilon_p + d)^T F_{\hat{\theta}}^{-1} (\varepsilon_p + d) \right]
\]

- **Constrained Minimum**
- **Level Sets of FIM Quadratic Form**
- **Unconstrained Minimum**
- **bias-gradient constraint ellipse** 
  \[ B = \{ d : \|d\|^2 \leq \delta^2 \} \]
- **mean-gradient constraint cone** 
  \[ R = \left\{ d : \frac{(\varepsilon_p + d)^T M_p (\varepsilon_p + d)}{(\varepsilon_p + d)^T (\varepsilon_p + d)} \leq \gamma^2 \right\} \]
Bias-Resolution-Variance Tradeoff Surface
Image Restoration Example

64x64 pixel image

Noise and blur degraded measurement with IID additive Gaussian Noise

\[ y = Ax + n \]
\[ E[n n^T] = K = \sigma I \]

Penalized Weighted Least-Squares Estimator

\[ \hat{x} = \left( A'K^{-1}A + \beta P \right)^{-1} A'K^{-1}y \]
Tikonov PWLS Estimator: Trajectory #1

- Penalized Weighted Least-Squares estimator regularized with Identity-matrix penalty $P$
  - Penalize squared-magnitude (energy) of individual pixels

$eta = 10^{-6}$

$eta = 10^{-3}$

$eta = 10^{-1}$

Smaller Penalty (noisier image)

Larger Penalty (smoother image)
Roughness PWLS Estimator: Trajectory #2

- Penalized Weighted Least-Squares estimator regularized with roughness penalty $P$
  - First Order Pixel Neighborhood
  - Penalize differences between neighboring pixels

\[
\beta = 10^{-6}
\]

\[
\beta = 10^{-3}
\]

\[
\beta = 10^{-1}
\]

\[
\begin{bmatrix}
-1 & 4 & -1 \\
-1 & & -1
\end{bmatrix}
\]
PWLS Estimator Trajectories #1, #2

- Roughness penalty (Laplace) estimator lies slightly above bound surface
- Bound achieved by Identity-regularized estimator

[Diagram showing trajectories with Smaller Penalty (noisier image) and Larger Penalty (smoother image)]
MM T-PWLS Estimator Trajectory #3

- Penalized Weighted Least-Squares estimator regularized with Identity penalty
  - Mis-matched Estimator
    - 1.5 pixel FWHM blur
    - Estimator assumes a 1.75 pixel FWHM blur
    - Estimator is over-compensating

- Smaller Penalty (noisier image)
- Larger Penalty (smoother image)
- Additional variance due to mis-matched estimator
MM T-PWLS Estimator: Trajectory #4

- Penalized Weighted Least-Squares estimator regularized with Identity penalty

- Mis-matched Estimator
  - 1.75pixel FWHM blur
  - Estimator assumes a 1.5pixel FWHM blur
  - Estimator is *under-compensating*
Conclusions

• Resolution constraint prescribes estimator-independent CR bound
• Bound can be used to:
  – assess optimality of a given reconstruction/restoration algorithm in terms of bias-res-var tradeoff
  – Perform optimal system design
• Bound is achieved by PWLS estimator with penalty matched to the bias gradient norm matrix
• For unknown point spread response bound cannot be attained for any estimator