Image Resolution-Variance Tradeoffs Using the Uniform Cramer-Rao Bound

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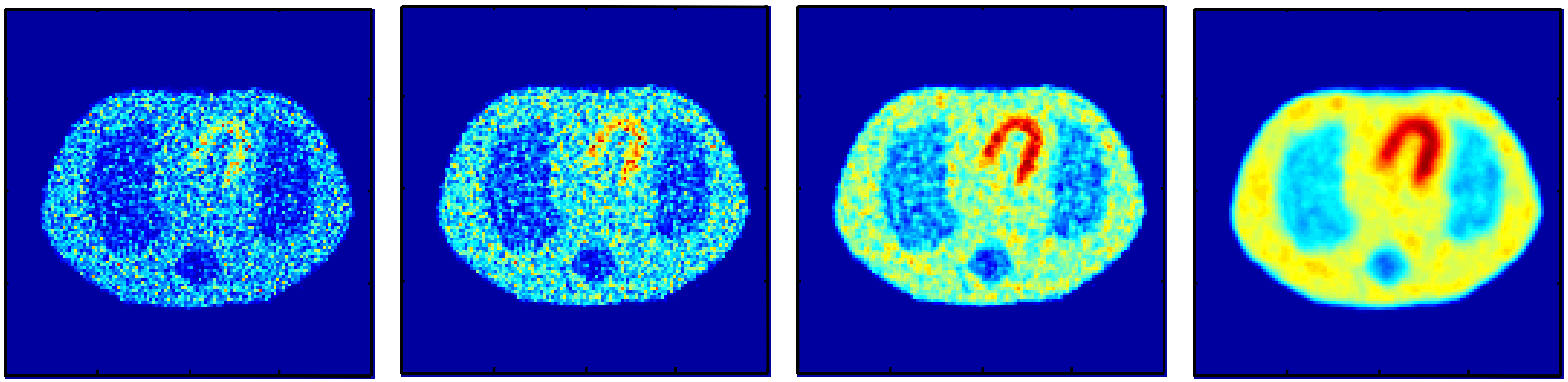
- Most image reconstruction algorithms of interest are *biased*.
- Tradeoff between image bias, resolution, and noise.



Low bias, but noisy images ⟷ Low noise, but biased images

  - For a given amount of bias or resolution, how noisy will the images be?
  - How should one quantify bias, resolution, and noise in a meaningful way?
  - How are these quantities related?

- More specifically
  - what is the fundamental limit of a particular imaging system’s performance, *independent of the choice of estimation algorithm?*

Estimator mean, bias and variance

- Reconstruction at pixel p $$\hat{\theta}_p = \underline{e}_p{}^T \underline{\hat{\theta}}$$
- Mean function $$m_{\underline{\theta}} \equiv E_{\underline{\theta}}\left[\hat{\theta}_p\right]$$
- Bias function $$b_{\underline{\theta}} \equiv E_{\underline{\theta}}\left[\hat{\theta}_p\right] - \theta_p$$
- Variance function $$\mathrm{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \equiv E_{\underline{\theta}}\left(\left[E_{\underline{\theta}}\left[\hat{\theta}_p\right] - \theta_p\right]^2\right)$$

Fundamental Limits on Variance

- CR Bound (biased)

$$\operatorname{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \geq \left[\nabla m_{\underline{\theta}}\right]^T F_Y^{-1} \left[\nabla m_{\underline{\theta}}\right]$$

Estimator Dependent Terms

- Alternative form

$$\operatorname{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \geq \left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]^T F_Y^{-1} \left[\underline{e}_p + \nabla b_{\underline{\theta}}\right]$$

- Uniform CR Bound ($\left\|\nabla b_{\underline{\theta}}\right\|_C^2 \equiv \nabla b_{\underline{\theta}}{}^T C \nabla b_{\underline{\theta}}$)

$$\operatorname{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \geq \underset{\left\|\nabla b_{\underline{\theta}}\right\|_C \leq \delta}{\operatorname{argmin}} \left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)^T F_{\underline{Y}}^{-1} \left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)$$

UCRB Applied to Image Restoration

- Image restoration model: Gaussian blur with additive gaussian noise

- UCRB specifies fundamental limit on restoration variance using *any* deconvolution algorithm

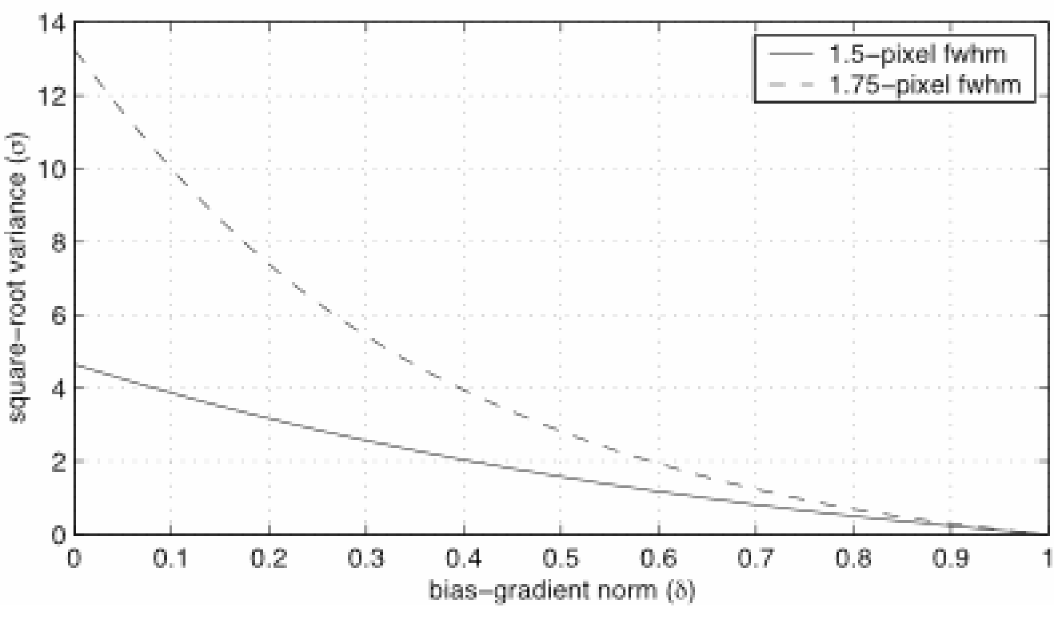
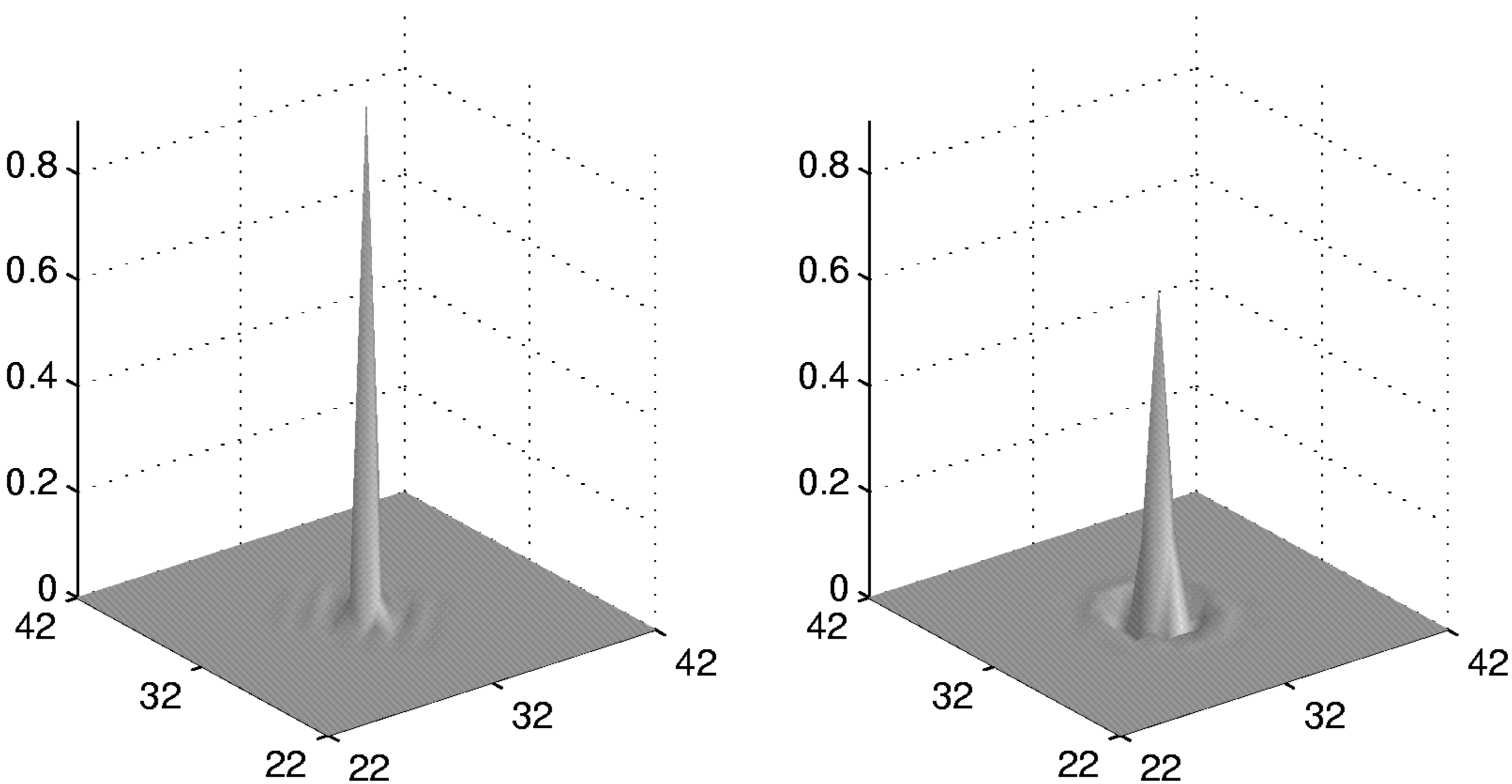


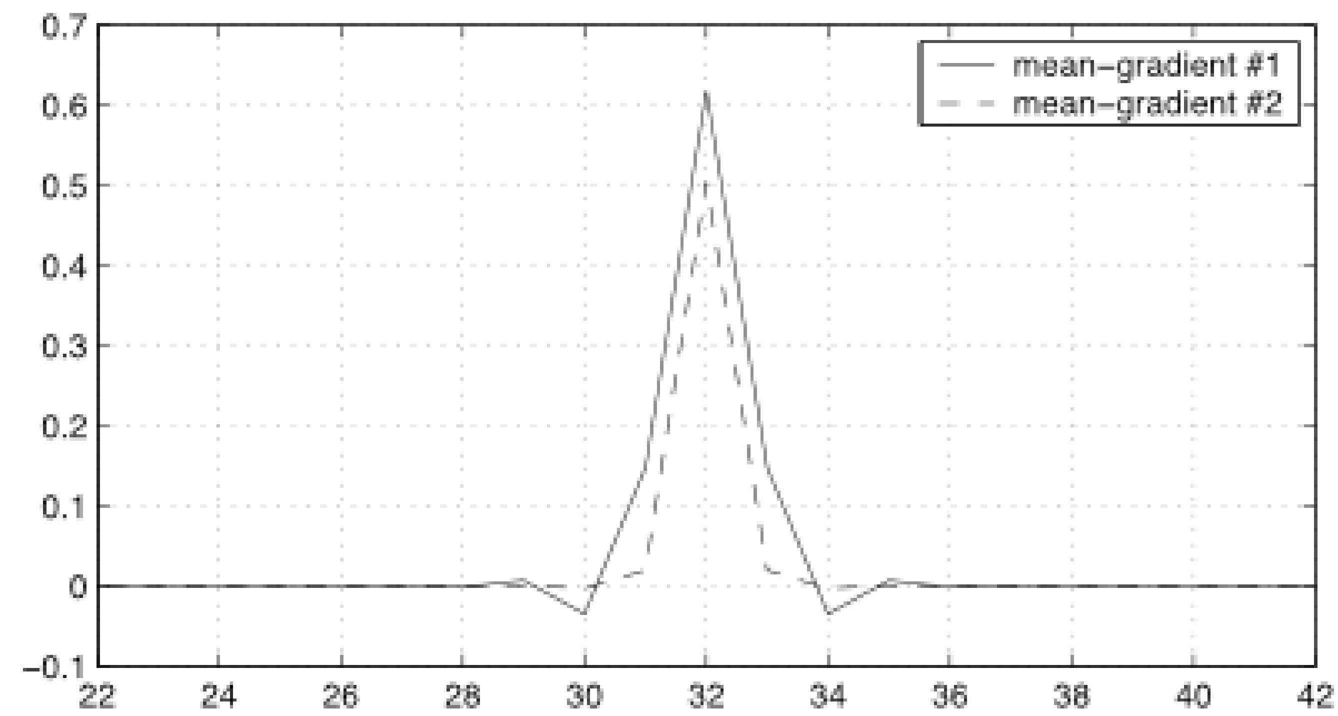
Image Restoration Example: Mean Gradient vs. Point Response

- Bias-gradient norm $\delta=0.1$ (left) and $\delta=0.5$ (right)



Deficiency of UCRB

- Estimator point response functions can have *identical* bias gradient length, but different resolution properties
  - Identical bias-gradient length $\delta = 0.5$
  - Different spread (or 2nd moment)
- Hence two different systems can have different recoverable resolution, but identical bias gradient length and variance



Extended UCRB

- Perform constrained minimization on variance bound

$$\mathrm{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \geq \underset{\nabla b_{\underline{\theta}}}{\mathrm{argmin}}\left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)^T F_{\underline{Y}}^{-1}\left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)$$

  – Subject to the following two constraints:

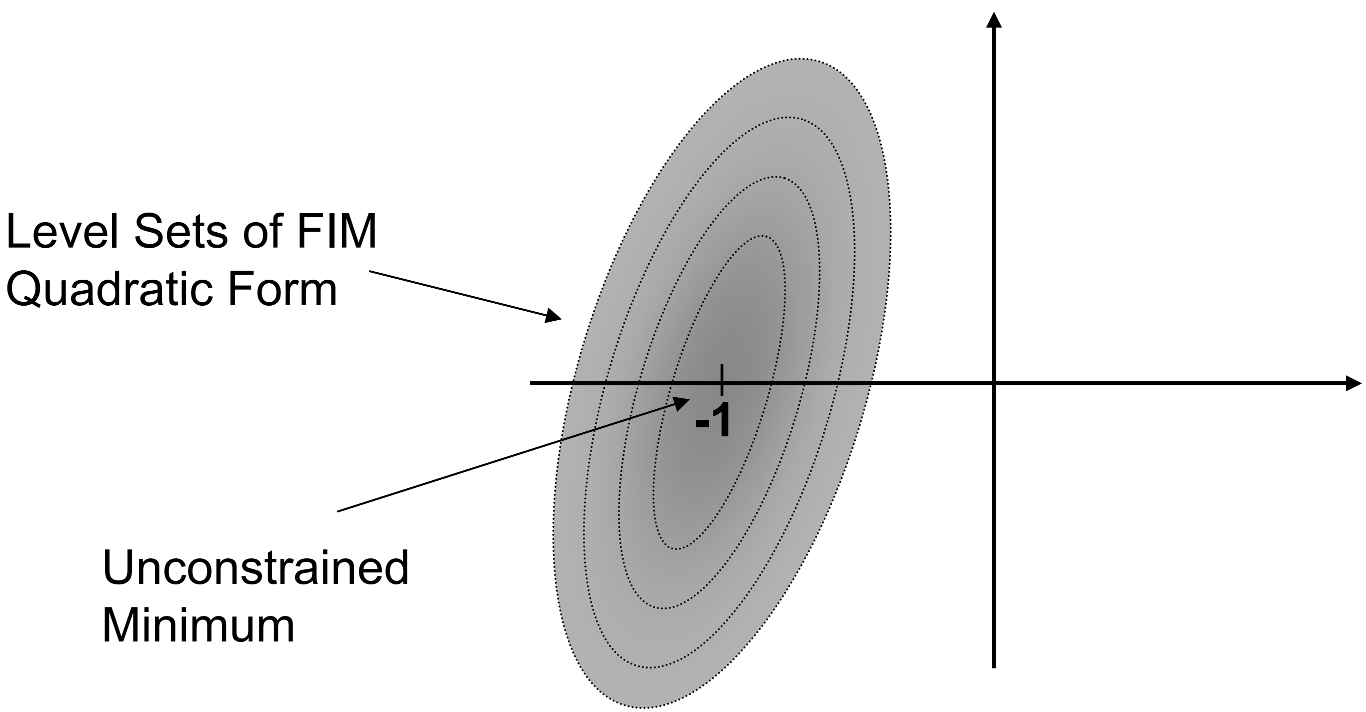
    1) Maximal Bias Variation Constraint $\left\|\nabla b_{\underline{\theta}}\right\|_C^2 \leq \delta^2$

    2) Resolution Constraint $$\frac{\left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)^T M_p \left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)}{\left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)^T \left(\underline{e}_p + \nabla b_{\underline{\theta}}\right)} \leq \gamma^2$$

- Calculate resulting Bias-Resolution-Variance surface

Geometric Interpretation of UCRB with Resolution Constraint

$$\mathrm{var}_{\underline{\theta}}\left(\hat{\theta}_p\right) \geq \underset{\underline{d} \in B \cap R}{\text{arg-min}} \left[\underline{e}_p + \underline{d}\right]^T F_{\underline{Y}}^{-1} \left[\underline{e}_p + \underline{d}\right]$$



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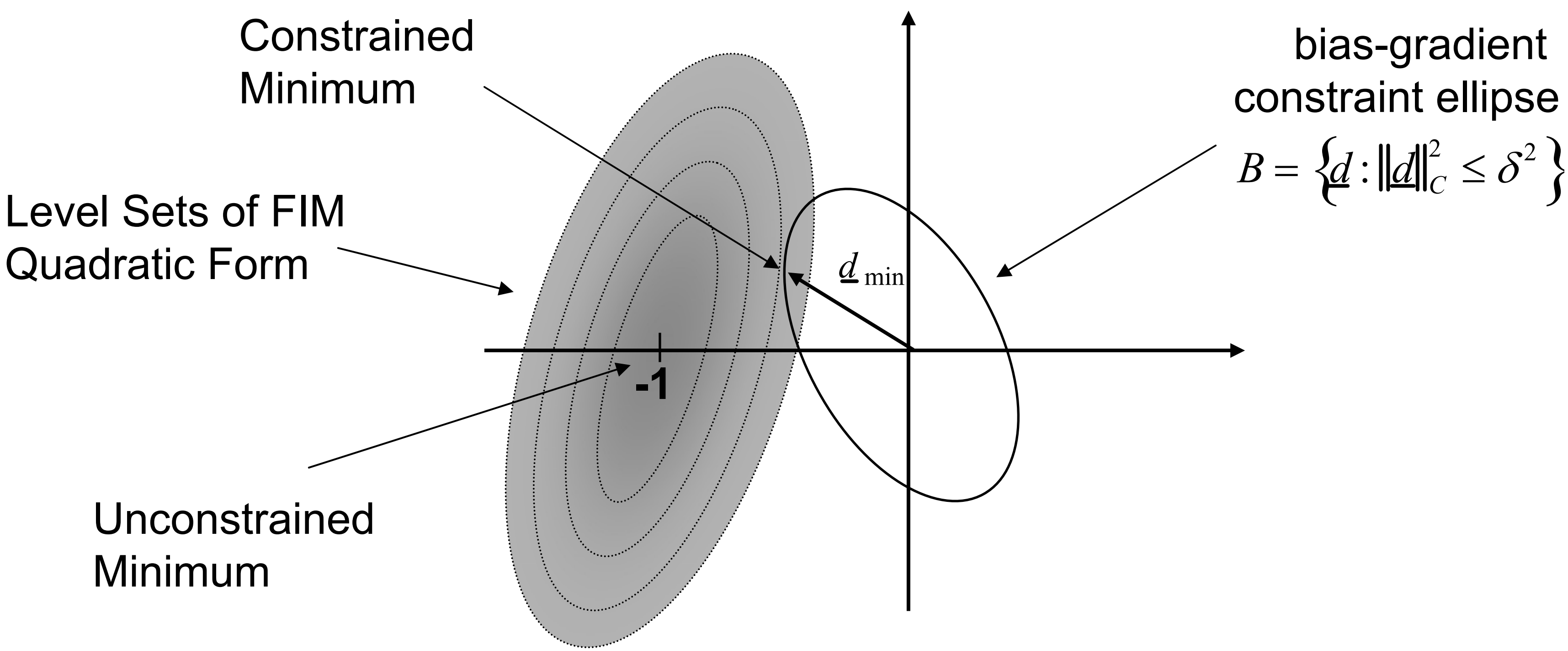

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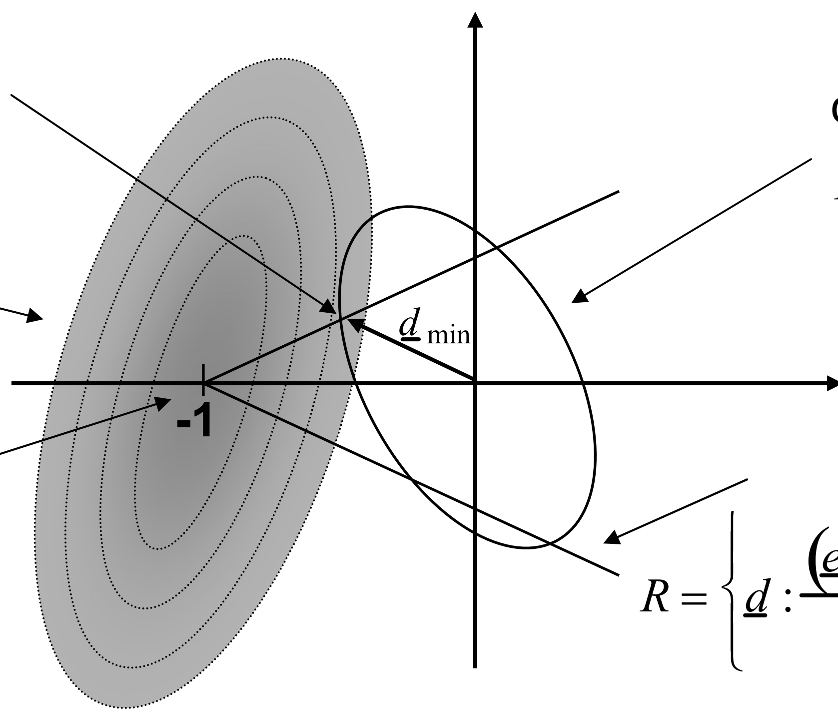


Bias-Resolution-Variance Tradeoff Surface

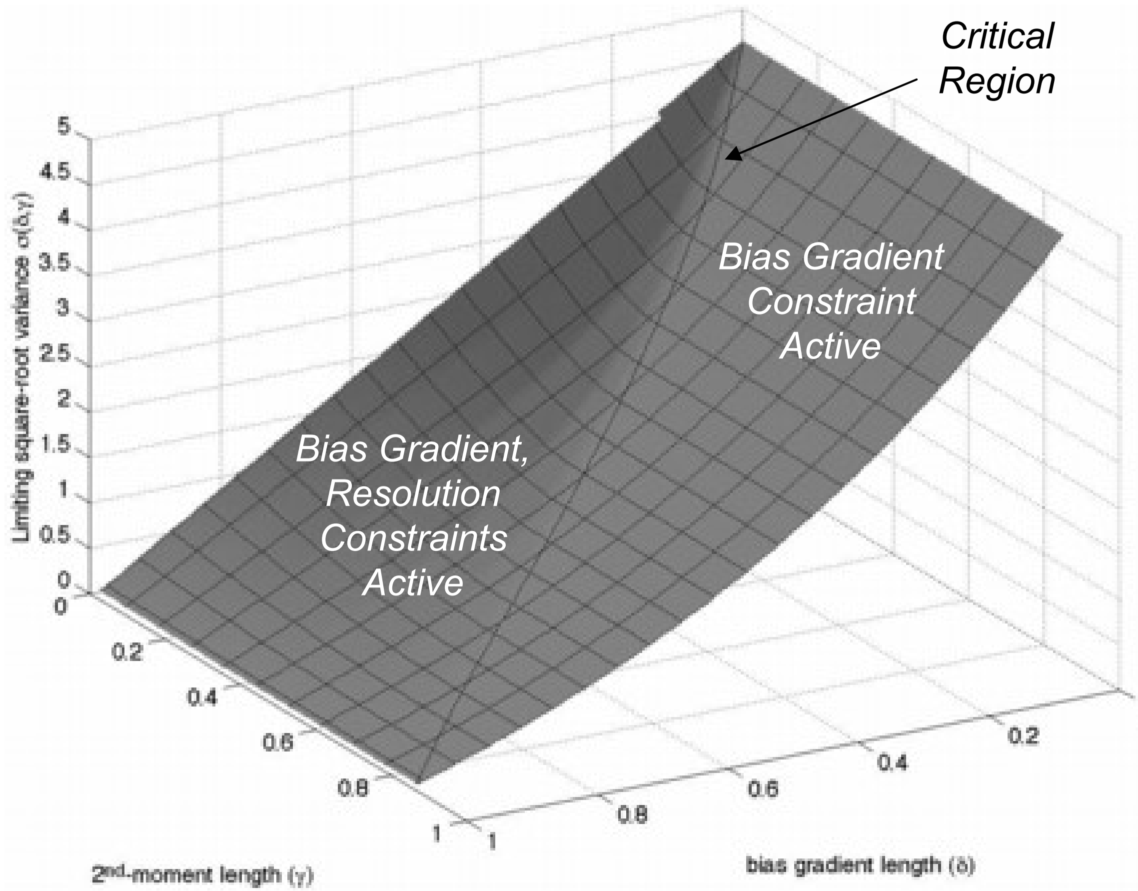
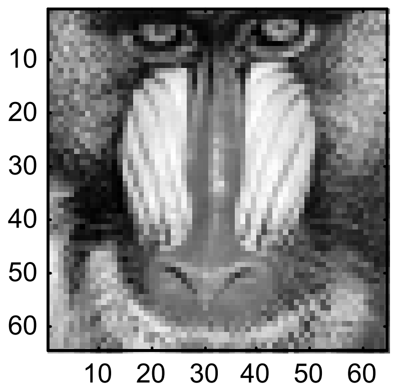


Image Restoration Example

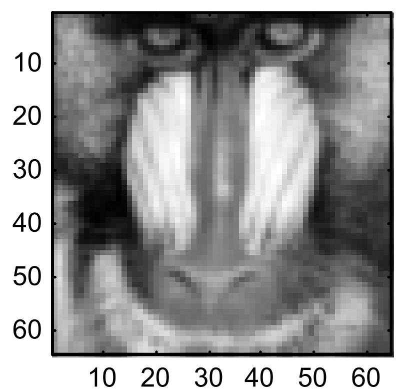
64x64 pixel image



Noise and blur degraded measurement with IID additive Gaussian Noise

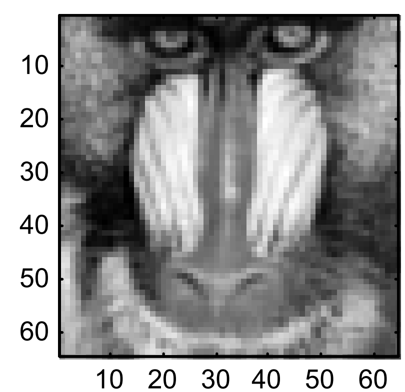
$$\underline{y} = A\underline{x} + \underline{n}$$

$$E[\underline{n}\,\underline{n}^T] = K = \sigma I$$

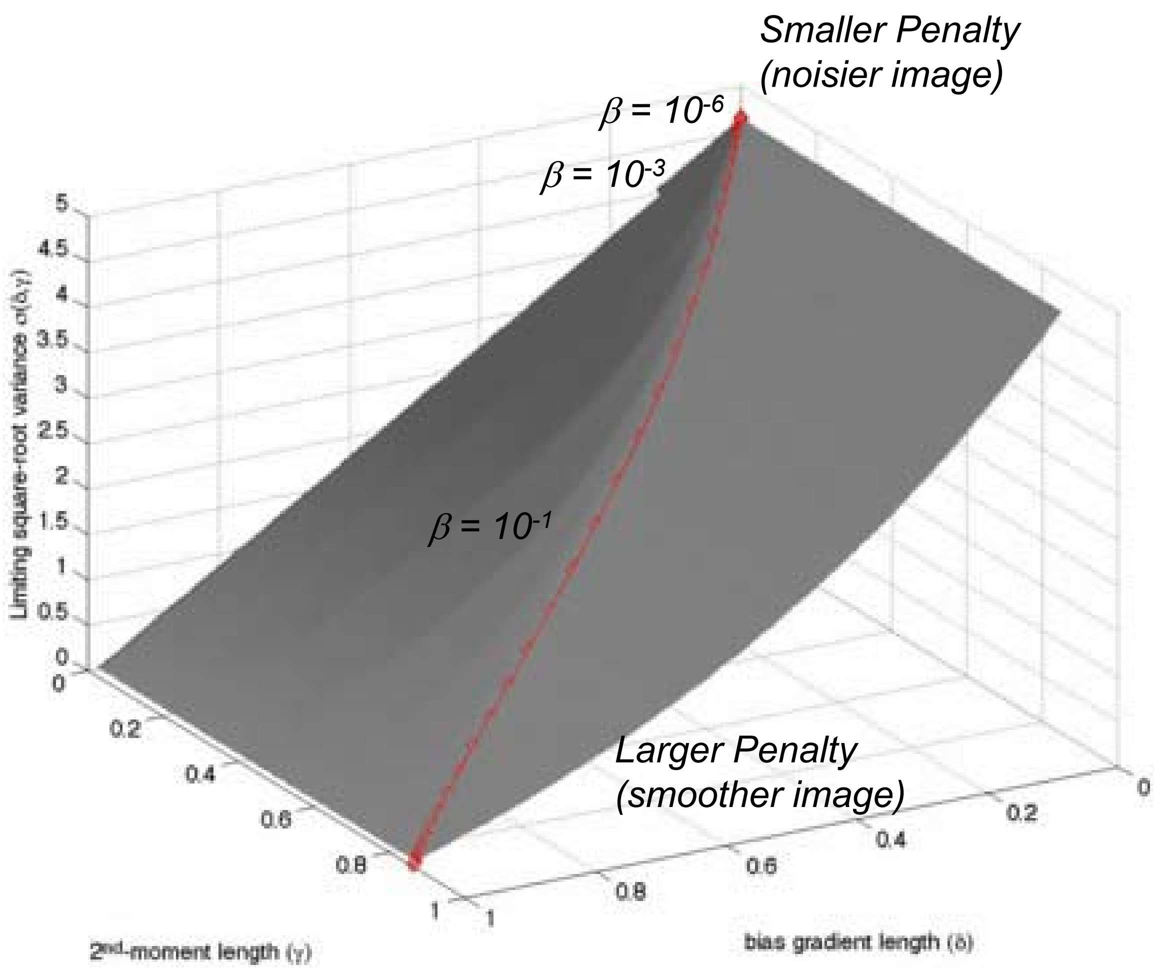


Penalized Weighted Least-Squares Estimator

$$\hat{\underline{x}} = \left[A'K^{-1}A + \beta P\right]^{-1} A'K^{-1}\underline{y}$$

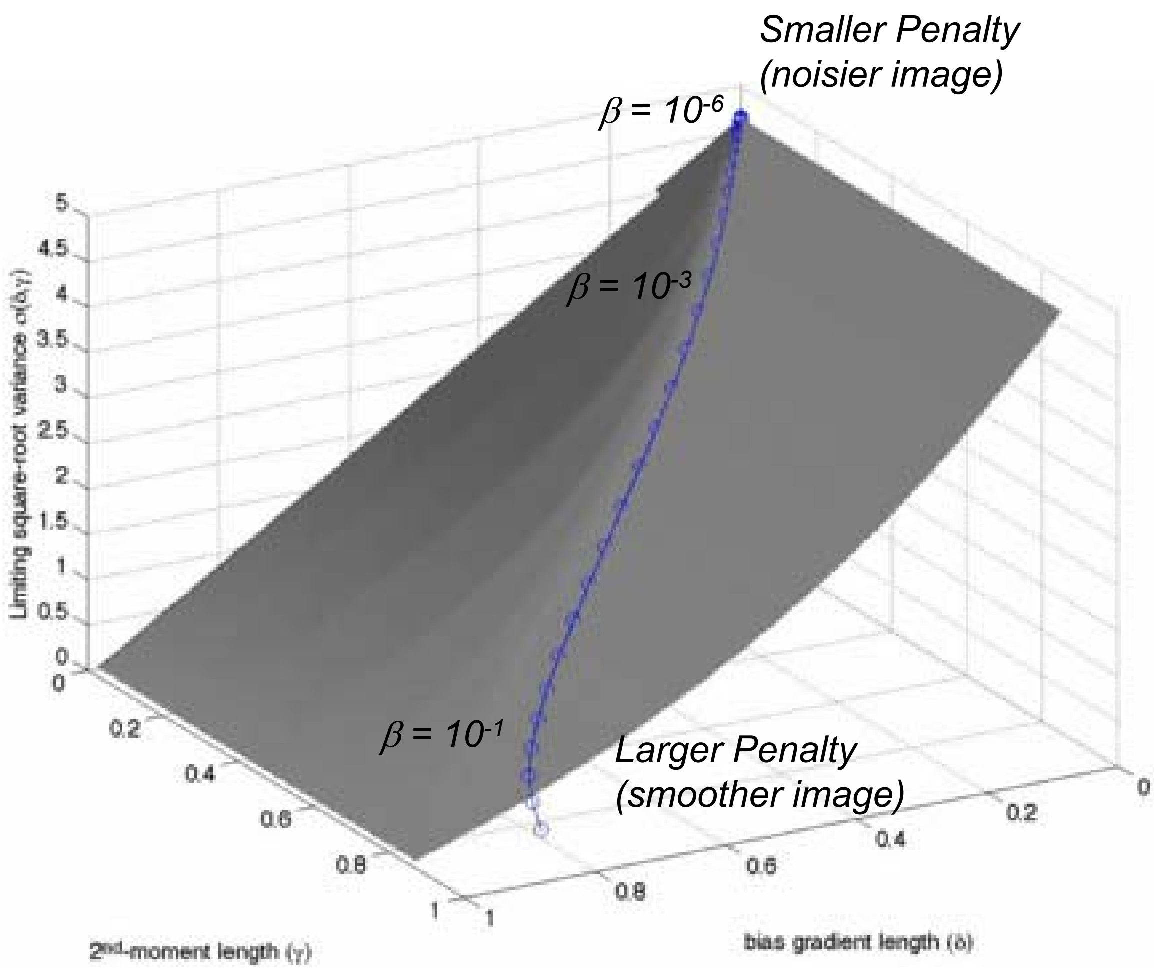


Tikonov PWLS Estimator: Trajectory #1



- Penalized Weighted Least-Squares estimator regularized with Identity-matrix penalty P
  - Penalize squared-magnitude (energy) of individual pixels

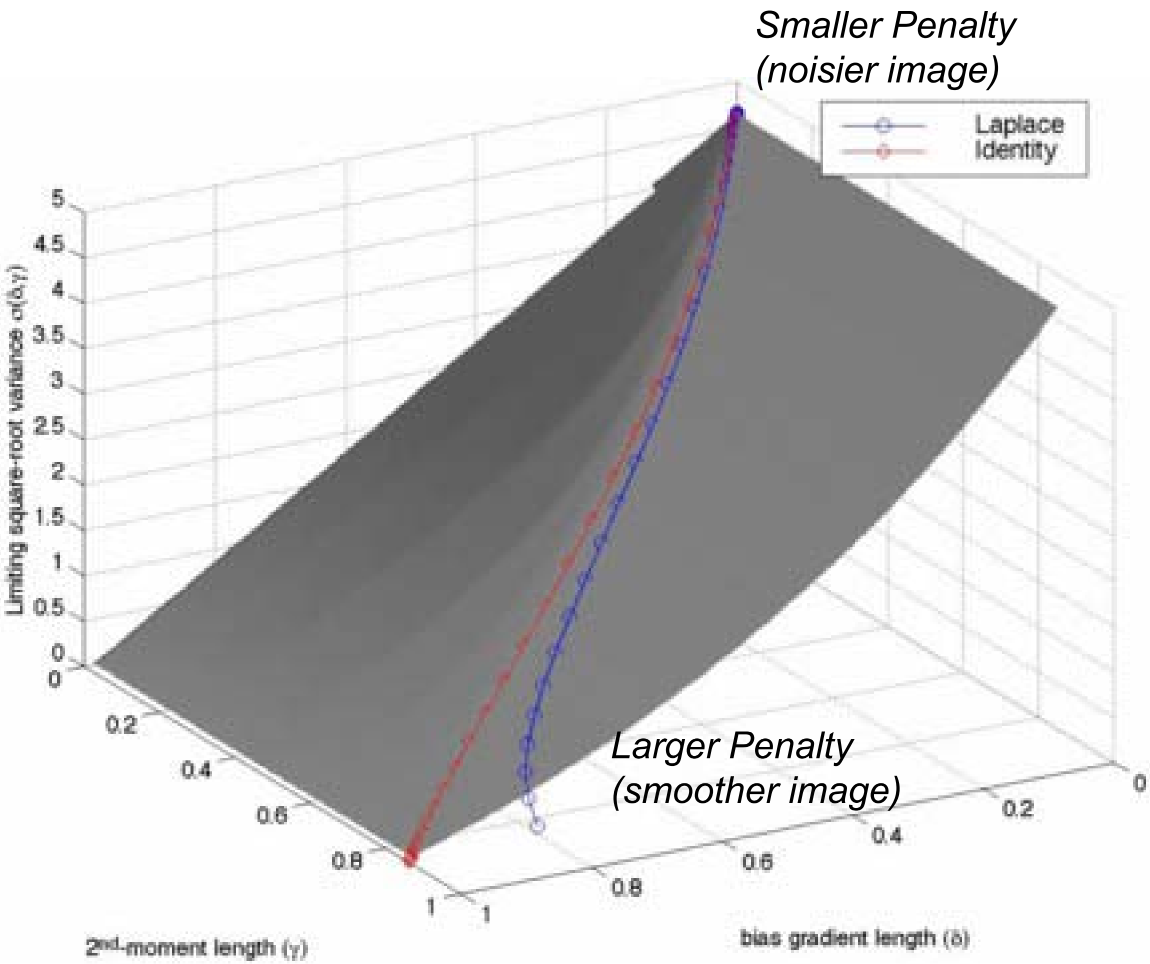
Roughness PWLS Estimator: Trajectory #2



- Penalized Weighted Least-Squares estimator regularized with roughness penalty P
  - First Order Pixel Neighborhood
  - Penalize differences between neighboring pixels

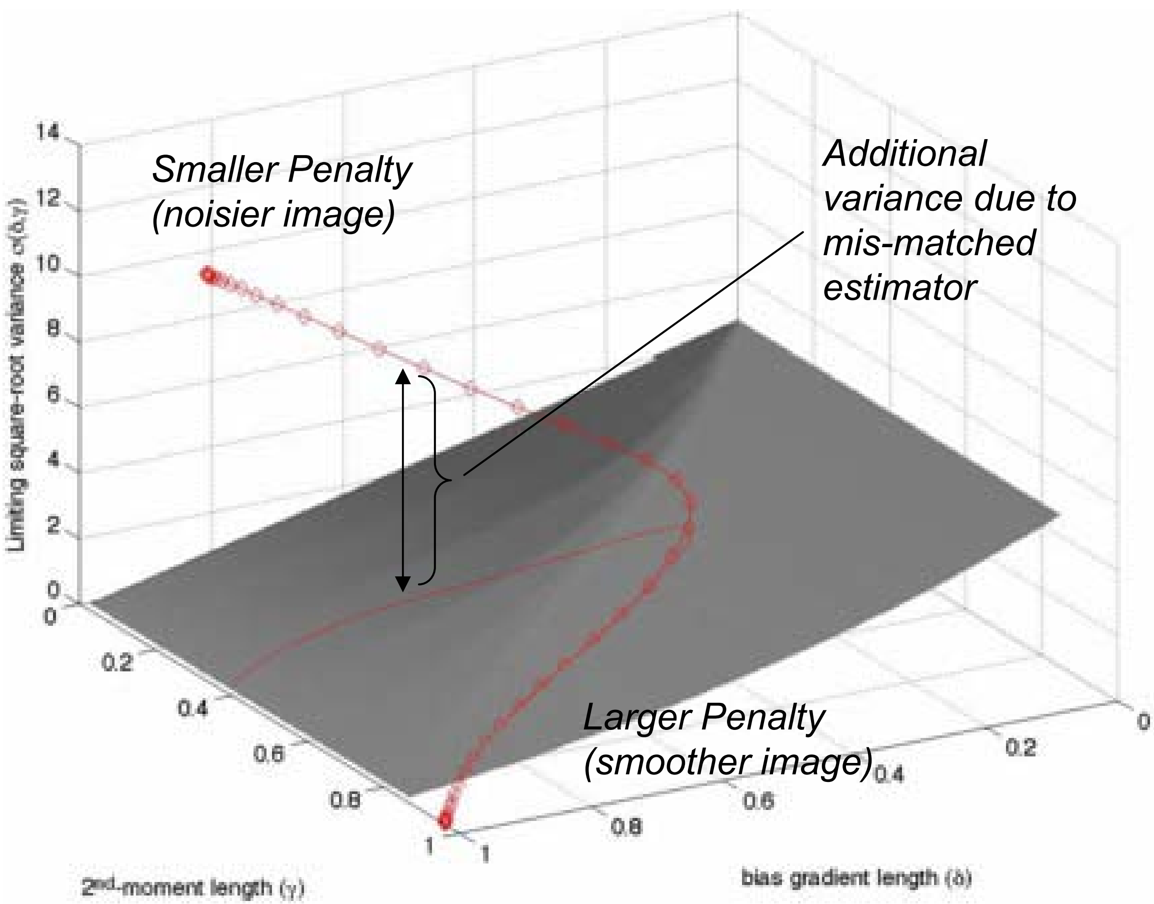
	-1	
-1	4	-1
	-1	

PWLS Estimator Trajectories #1, #2



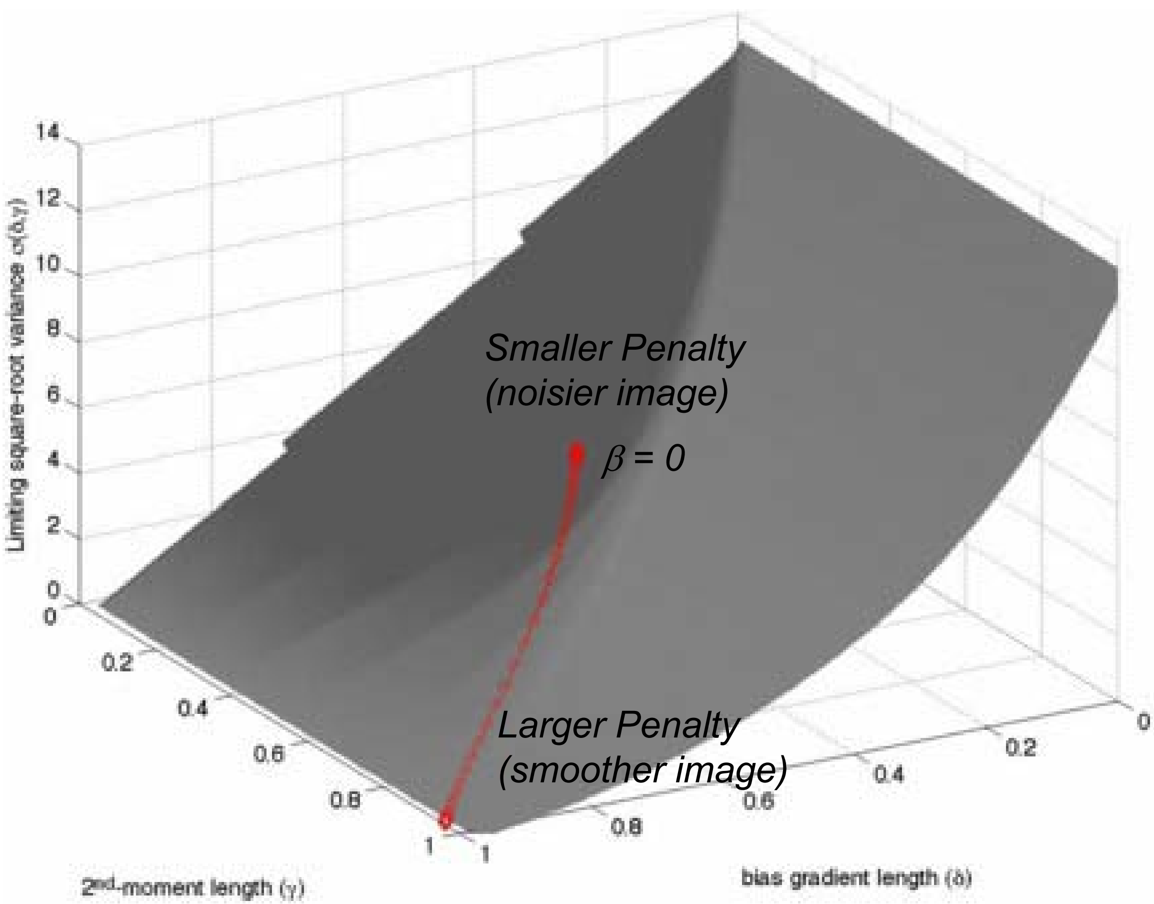
- Roughness penalty (Laplace) estimator lies slightly above bound surface
- Bound achieved by Identity-regularized estimator

MM T-PWLS Estimator Trajectory #3



- Penalized Weighted Least-Squares estimator regularized with Identity penalty
- Mis-matched Estimator
  - 1.5pixel FWHM blur
  - Estimator assumes a 1.75pixel FWHM blur
  - Estimator is *over-compensating*

MM T-PWLS Estimator: Trajectory #4



- Penalized Weighted Least-Squares estimator regularized with Identity penalty

- Mis-matched Estimator
  - 1.75pixel FWHM blur
  - Estimator assumes a 1.5pixel FWHM blur
  - Estimator is *under-compensating*

Conclusions

- Resolution constraint prescibes estimator-independent CR bound
- Bound can be used to:
  - assess optimality of a given reconstruction/restoration algorithm in terms of bias-res-var tradeoff
  - Perform optimal system design
- Bound is achieved by PWLS estimator with penalty matched to the bias gradient norm matrix
- For unknown point spread response bound cannot be attained for any estimator