Optimal Signal Constellations for Fading Space-Time Channels

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Outline

1. Space-time channels
2. Random coding exponent and cutoff rate
3. Discrete K-dimensional constellations
4. Bound on minimum distance
5. Low dimensional constellations
6. Conclusions

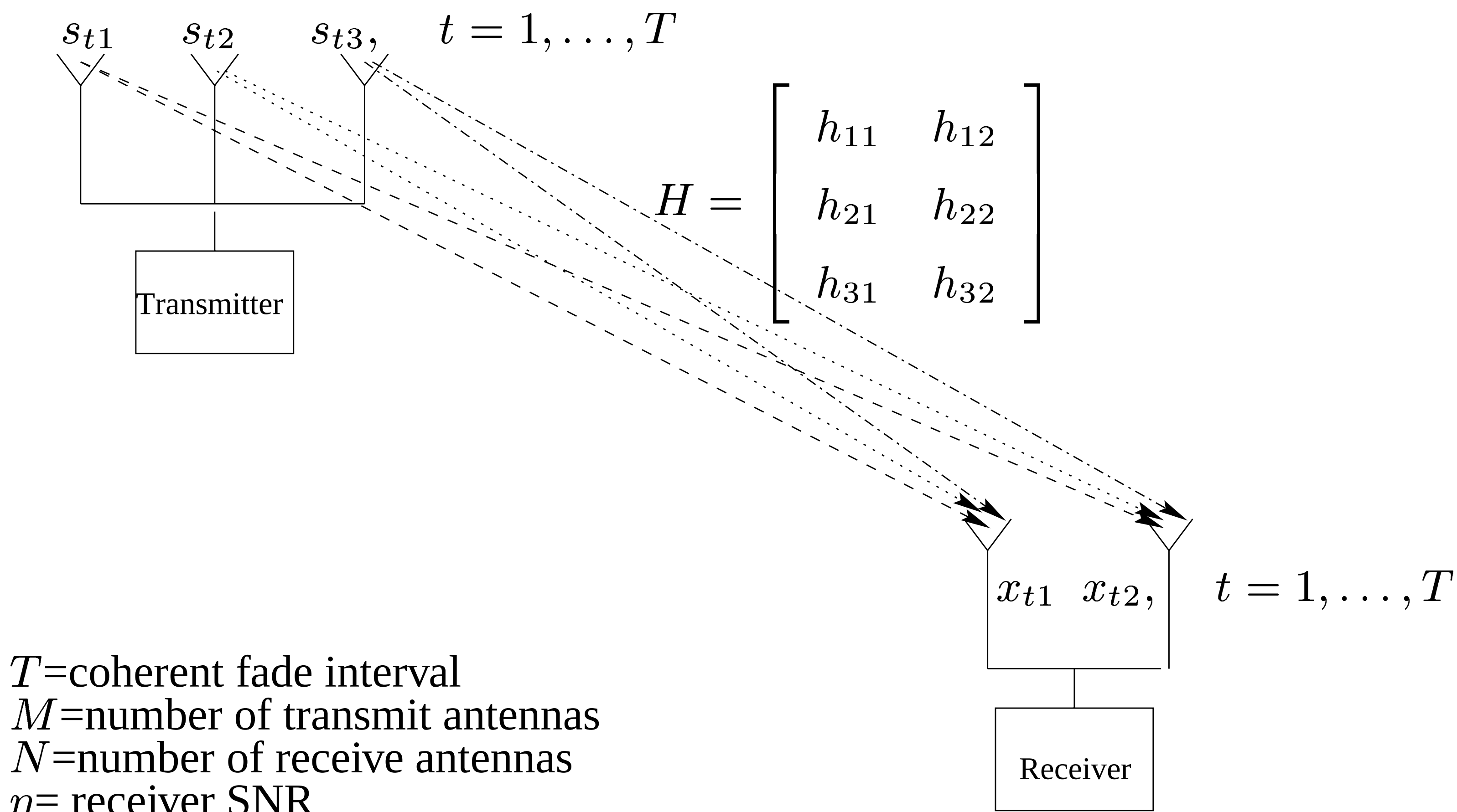


Figure 1. Narrowband space time channel for $M = 3$, $N = 2$

Received signal in l-th frame ($t = 1, \ldots, T$)

$$[x_{t1}^l, \ldots, x_{tn}^l] = \sqrt{\eta}[s_{t1}^l, \ldots, s_{tm}^l] \begin{bmatrix} h_{11}^l & \cdots & h_{1n}^l \\ \vdots & \vdots & \vdots \\ h_{m1}^l & \cdots & h_{mn}^l \end{bmatrix} + [w_{t1}^l, \ldots, w_{tn}^l],$$

or, equivalently

$$X^l = \sqrt{\eta} S^l H^l + W^l$$

- X^l: $T \times N$ received signal matrices
- S^l: $T \times M$ transmitted signal matrices
- H^l: i.i.d. $M \times N$ channel matrices $\sim \mathcal{CN}(0, I_M \bigotimes I_N)$
- W^l: i.i.d. $T \times N$ noise matrices $\sim \mathcal{CN}(0, I_T \bigotimes I_N)$

Block coding over L frames produces blocks of L symbols

$$[S^1, \ldots, S^L]$$

where $S = S^l$ is selected from a symbol alphabet $\mathcal{S}$

Random Block Coding: select S^l at random from $\mathcal{S}$ according to probability distribution $P \in \mathcal{P}$.

- Objective: Find optimal distribution $P(S)$ over $\mathcal{P}$
- Optimality criteria: capacity, outage capacity, random coding error exponent, cut-off rate
- Transmitter constraints:
  - average power constraint: $E[\|S\|^2] = \int \|S\|^2 dP \leq TM$
  - peak power constraint: $\|S\|^2 \leq TM$, for all $S \in \mathcal{S}$

  where

$$\|S\|^2 = \mathrm{tr}\{S^H S\}$$

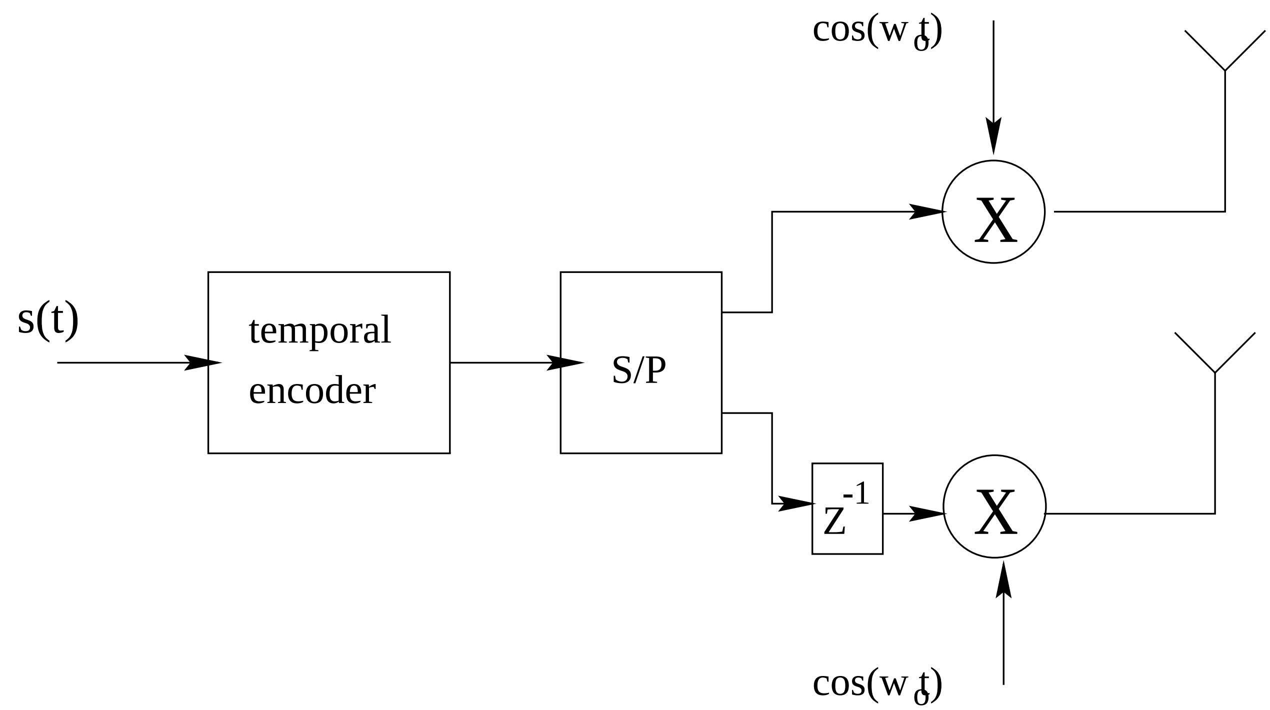


Figure 2. First generation space-time coding: Seshadri&Winters FDD transmitter with delay diversity (1994)

Capacity Results: (avg. power constraint - Telatar, BLTM 95):

1. **Channel H^l Known to Txmt and Rcv**:

Capacity: (*bits/sec/hz*) or ($\frac{bits/channel-use}{T}$)

$$C = \max_{P(S|H)} E[I(S,X|H)] = \max_{P(S|H)} E[\mathcal{H}(X|H) - \mathcal{H}(X|S,H)]$$

α**-Outage Capacity**: $C = \{C_o : P(C(H) > C_o) = \alpha\}$

Since

$$\mathcal{H}(X|H) \leq \ln(|I_N + \eta H^H R_s H|), \quad \text{and} \quad \mathcal{H}(X|S,H) = \mathcal{H}(N)$$

$$\begin{aligned} C(H) &= \max_{R_s \,:\, \mathrm{tr}\{R_s\} \leq M} \ln(|I_N + \eta H R_s H^H|) \\ &= \ln(|I + \eta H R_s^o H|) \end{aligned}$$

where, for $H = UDV$

$$R_s^o = V^H \mathrm{diag}\left(\mu - \frac{1}{\eta d_i^2}\right)^+ V$$

and μ is such that (water-filling)

$$\mathrm{tr}\{R_s^o\} = M$$

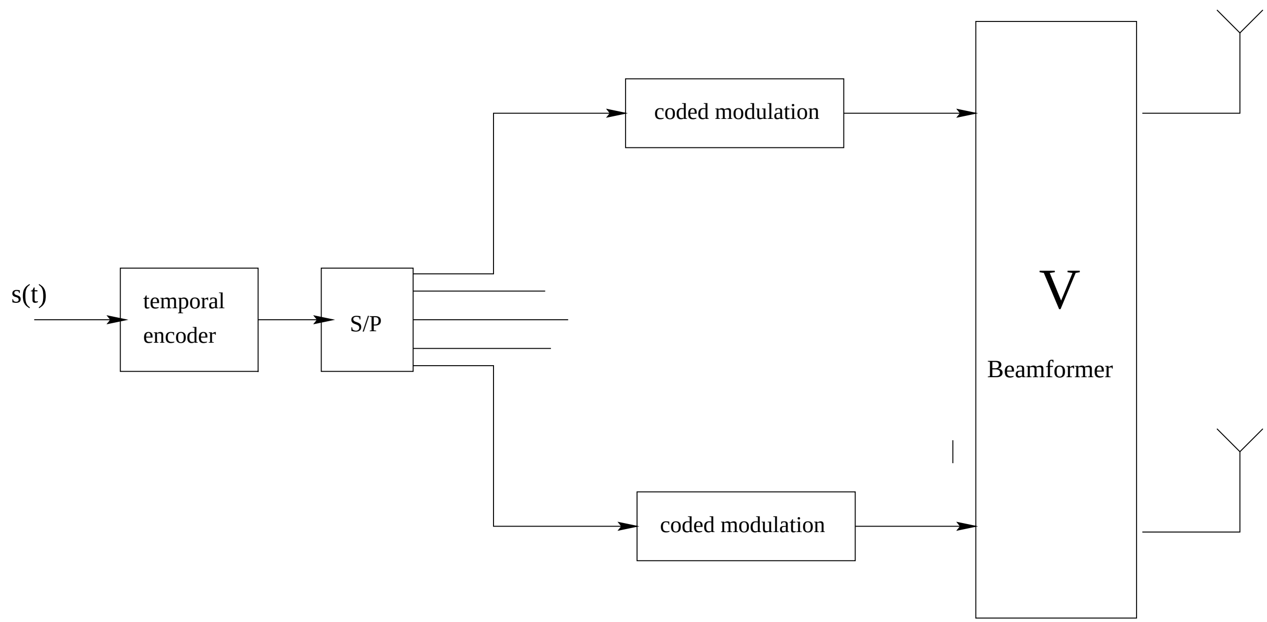


Figure 3. Second generation space-time coding with beamforming

2. **Channel H^l known only to Rcv**:

$$C = \max_{P(S)} E[I(X, S|H)] = \max_{P(S)} E[\mathcal{H}(X|H) - \mathcal{H}(X|S, H)]$$

$\Rightarrow C = E\left[\log \left| I_N + \frac{\eta}{M} \, H^H H \right|\right]$

Capacity achieving distribution:

• S Gaussian with orthogonal rows and columns of identical energy

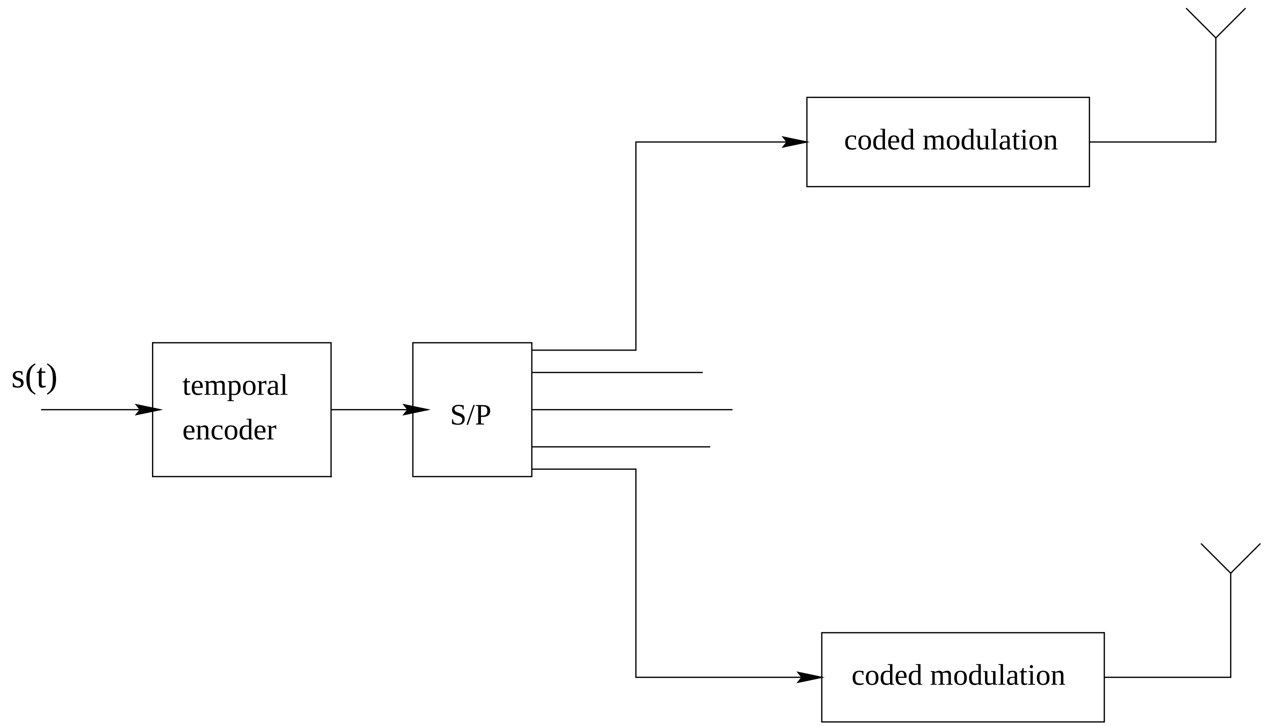
$\Rightarrow$ BLAST (Foschini, BLTJ 1996)

$\Rightarrow$ Space time 4-PSK/4-TCM (Tarokh&etal IT 98, Tarokh&etal COM 99)

In practice must transmit training within each frame to learn H

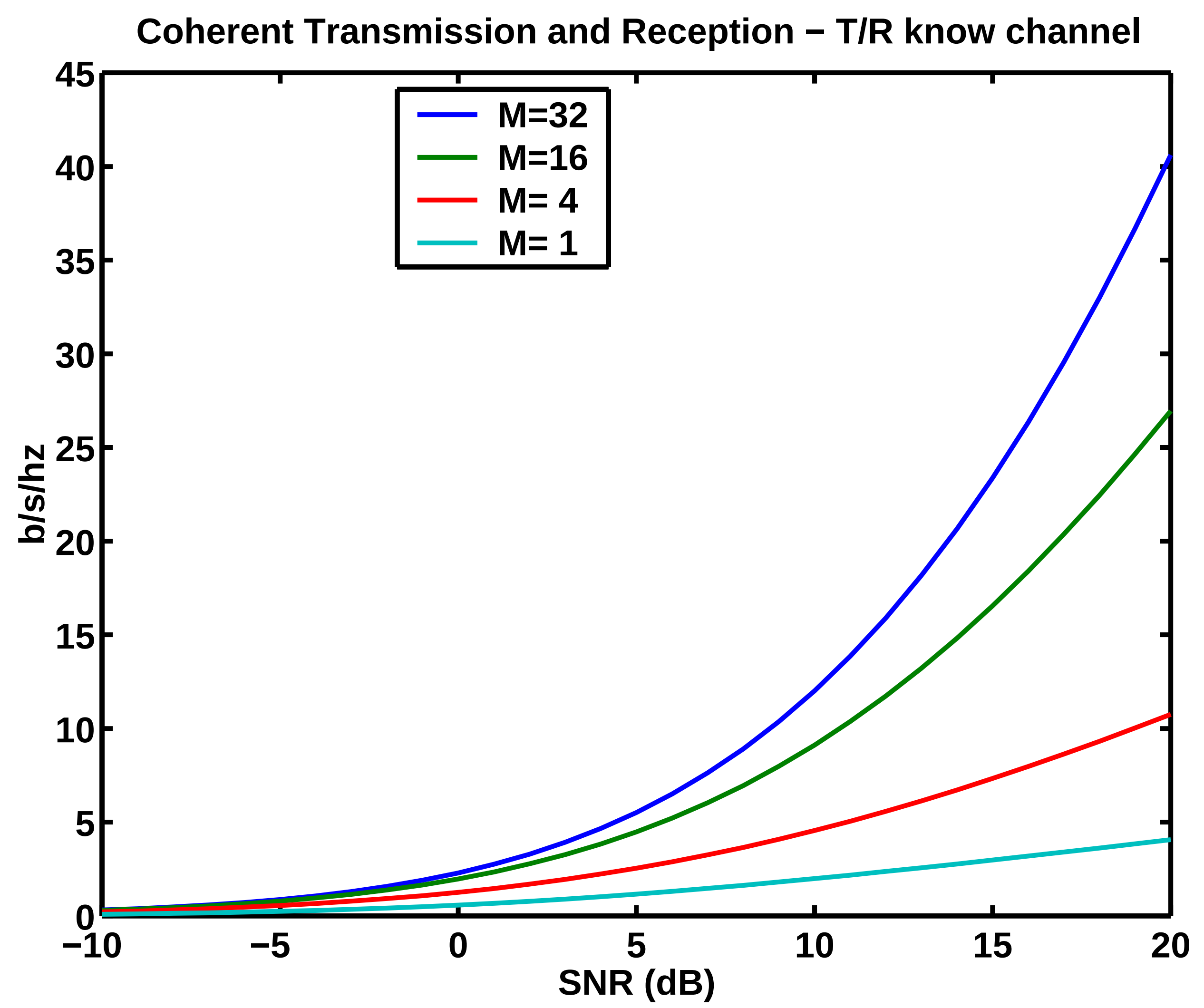
Capacity bounds: (Hochwald&Marzetta SPIE99, Driesen&Foschini COM99)

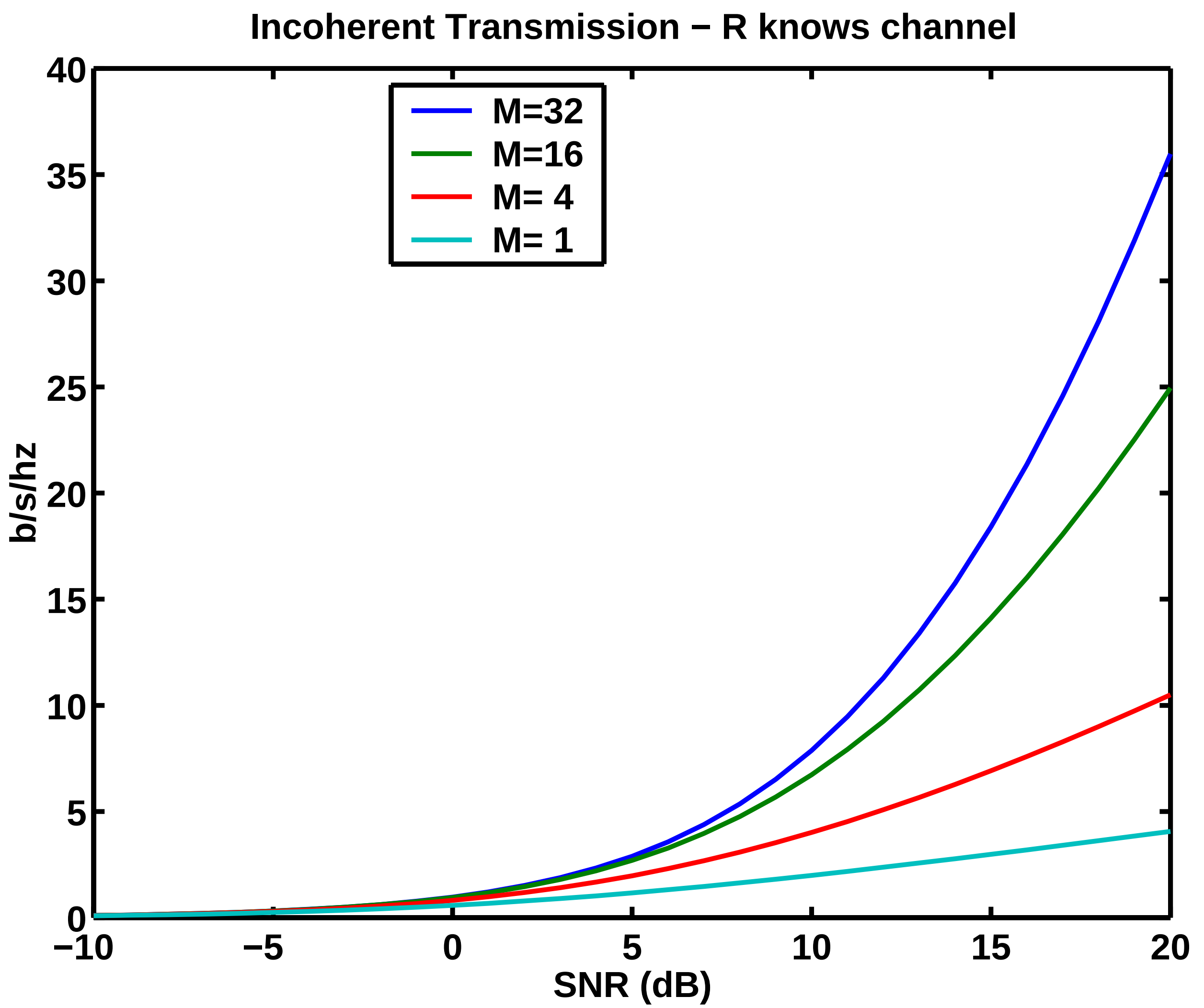
$$\underbrace{\log(1+\eta MN)}_{``=''\text{when rank}(H)=1} \leq C(H) \leq \underbrace{\min(M,N)\log\left(1+\frac{\eta MN}{\min(M,N)}\right)}_{``=''\text{when rank}(H)=\min(M,N)}$$

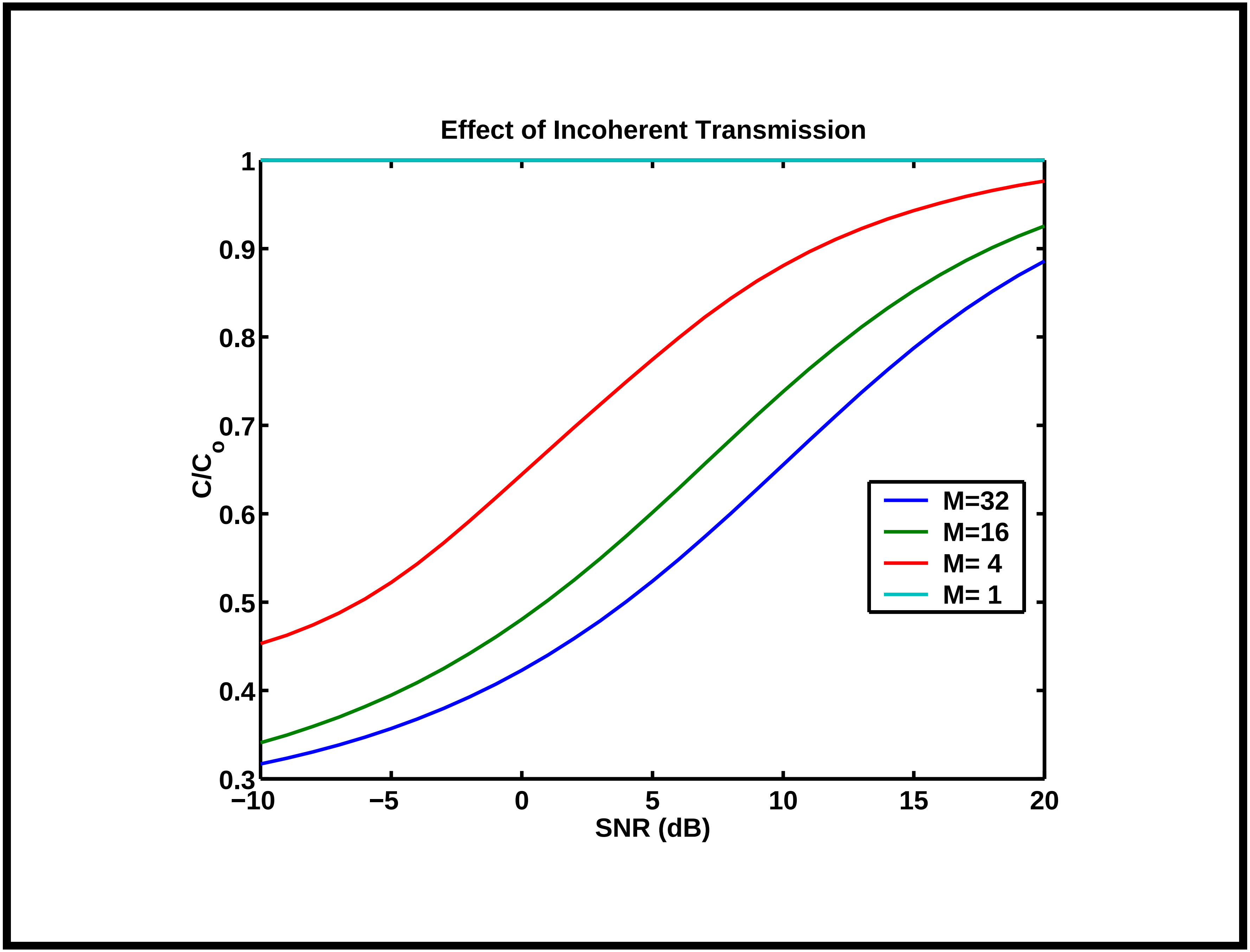


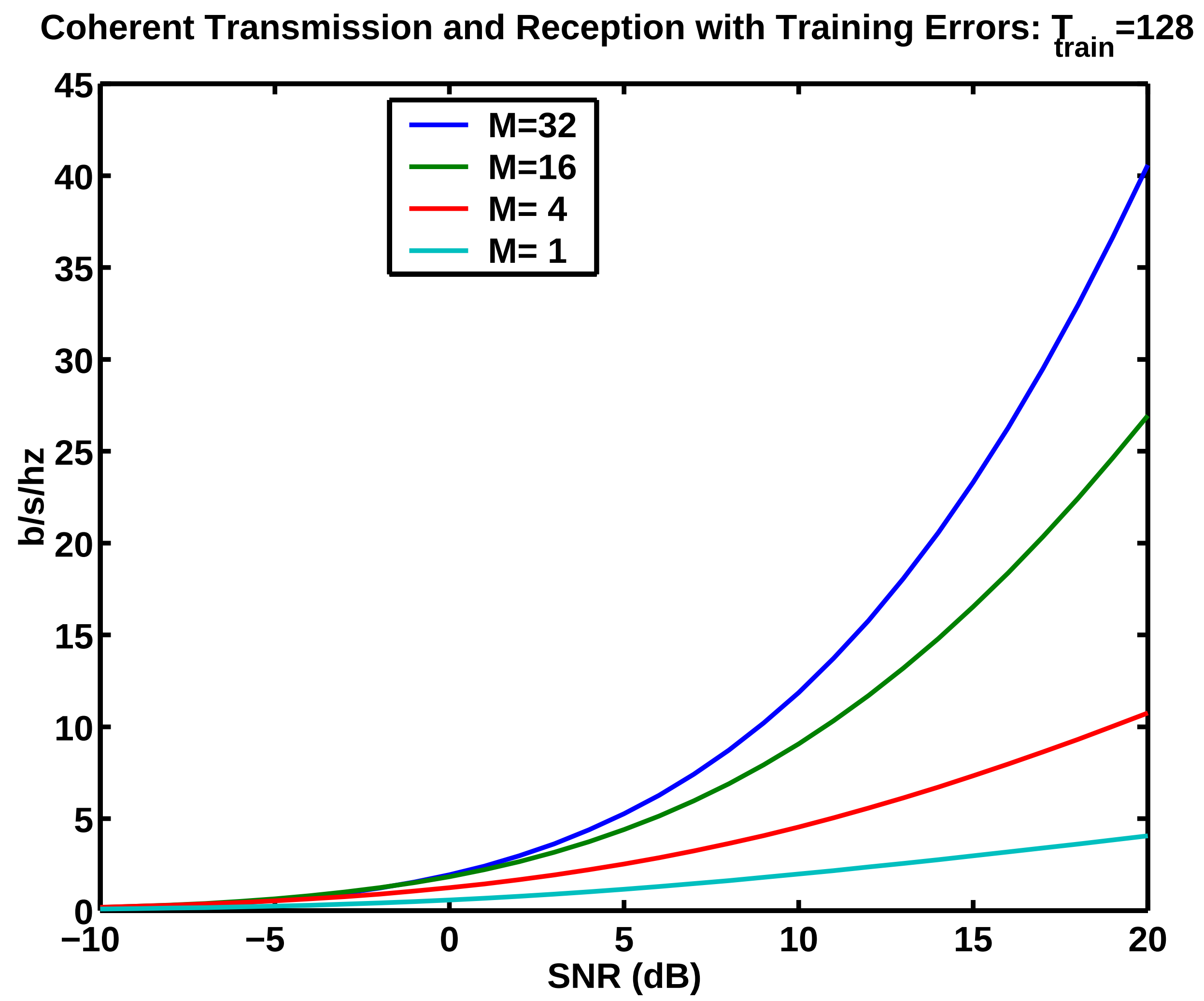
- 4-PSK/4-TCM: 2 bits/sec/Hz (simulation), M=N=2
- BLAST: 1.2 Mbps over 30kHz (40 bits/sec/Hz) in 800MHz band, M=8, N=12

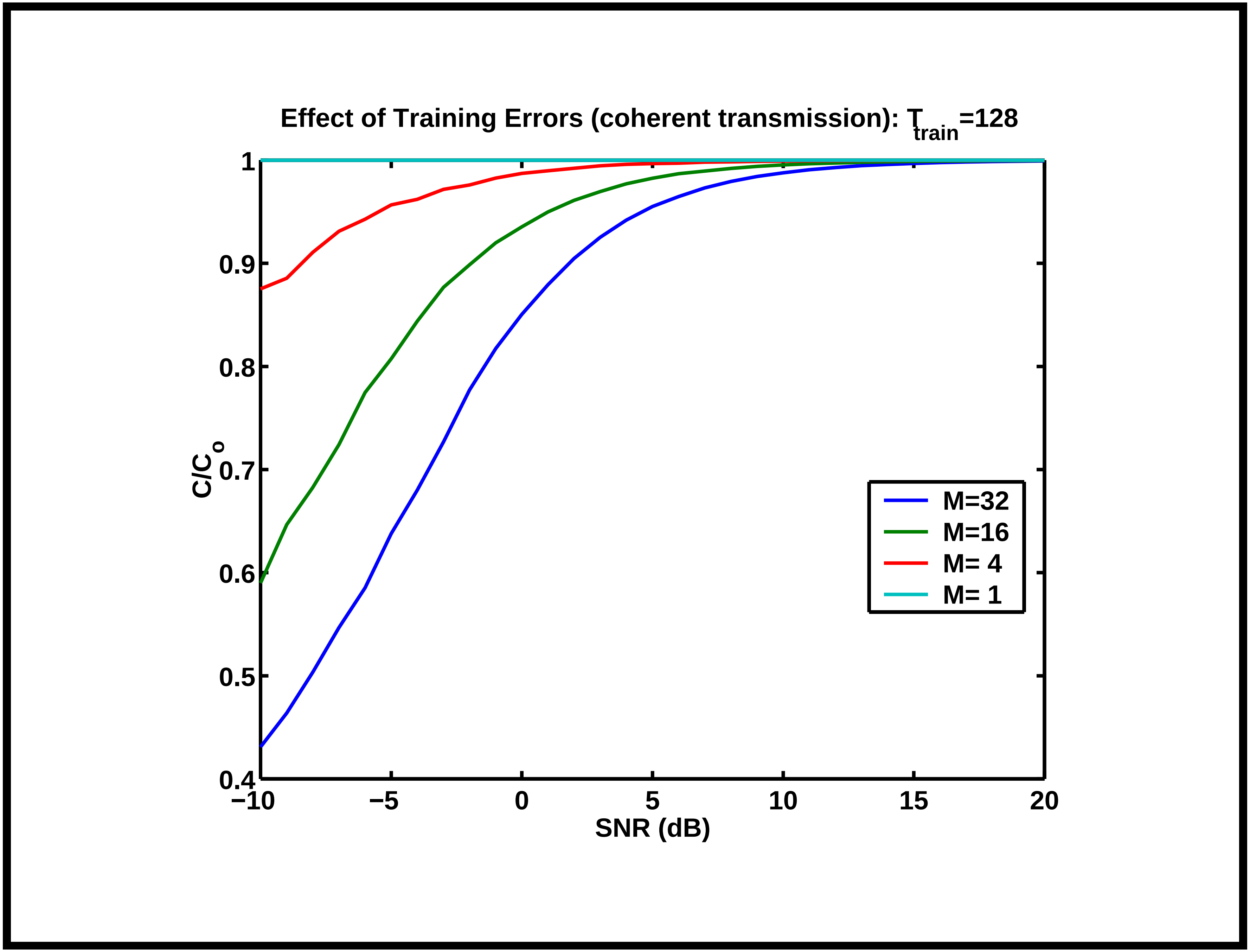
Figure 4. Second generation space-time coding


Coherent Transmission and Reception − T/R know channel
M=32
M=16
M= 4
M= 1
b/s/hz
SNR (dB)
0
5
10
15
20
25
30
35
40
45
−10
−5
0
5
10
15
20


Incoherent Transmission − R knows channel
M=32
M=16
M= 4
M= 1
b/s/hz
0
5
10
15
20
25
30
35
40
−10
−5
0
5
10
15
20
SNR (dB)


Effect of Incoherent Transmission
C/C_o
SNR (dB)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
−10
−5
0
5
10
15
20
M=32
M=16
M= 4
M= 1


Coherent Transmission and Reception with Training Errors: T_train=128
M=32
M=16
M= 4
M= 1
b/s/hz
45
40
35
30
25
20
15
10
5
0
−10
−5
0
5
10
15
20
SNR (dB)


Effect of Training Errors (coherent transmission): T_train=128
C/C_o
SNR (dB)
1
0.9
0.8
0.7
0.6
0.5
0.4
−10
−5
0
5
10
15
20
M=32
M=16
M= 4
M= 1

3. **Slow fading Rayleigh channel:** H **unknown**

Capacity? (Marzetta&Hochwald, BL TM 98, IT 99)

• $C = E\left[\log P(X|S)/P(X)\right]$ (bits/channel-use)

Capacity achieving distribution?

• $S = \Phi V$

where Φ and V are mutually independent matrices

- Φ: $T \times M$ unitary: $\Phi^H \Phi = I_M$
- V: $M \times M$ real diagonal

$$V \to cI_M \text{ as } \eta \to \infty \text{ or } T \to \infty.$$

⇒ Unitary space-time modulation (Hochwald&etal BL TM 1998)

⇒ Differential space-time modulation (Hochwald&Sweldens COM99)

⇒ Space-time group codes (Hughes SAM 00, Hassibi&etal BLTM 00)

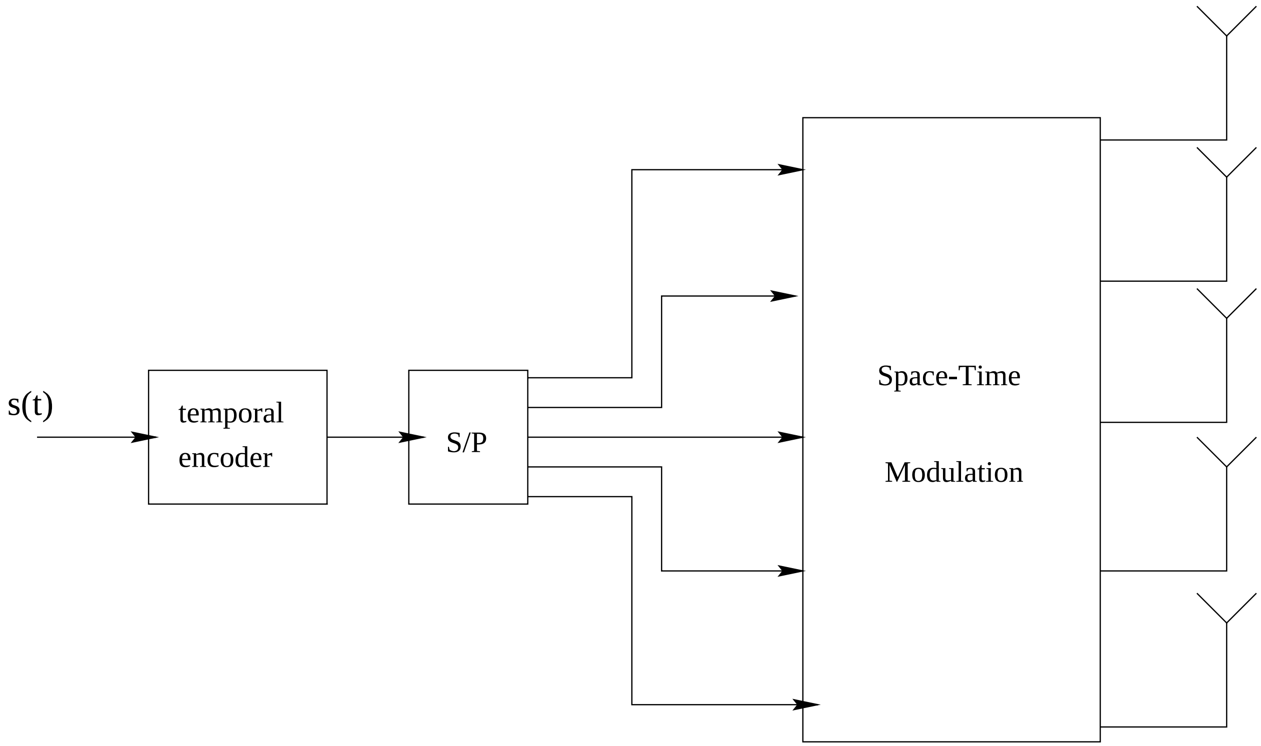


Figure 10. Third generation space-time coding.

Example: Unitary space-time constellation (Hochwald&etal BLTM 98)

- $T = 8$, $M = 3$, $K = 256$ unitary signal matrices

$$\mathcal{S} = \{\Phi_1, \ldots, \Phi_K\}, \qquad \Phi_k = \Theta^k \Phi_1$$

$$\Phi_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{8}5} & e^{j\frac{2\pi}{8}6} \\ 1 & e^{j\frac{2\pi}{8}2} & e^{j\frac{2\pi}{8}4} \\ 1 & e^{j\frac{2\pi}{8}7} & e^{j\frac{2\pi}{8}2} \\ 1 & e^{j\frac{2\pi}{8}4} & 1 \\ 1 & e^{j\frac{2\pi}{8}1} & e^{j\frac{2\pi}{8}6} \\ 1 & e^{j\frac{2\pi}{8}6} & e^{j\frac{2\pi}{8}4} \\ 1 & e^{j\frac{2\pi}{8}3} & e^{j\frac{2\pi}{8}2} \end{bmatrix}, \qquad \Theta = \begin{bmatrix} e^{j\frac{2\pi}{8}(0)} & 0 & \cdots \\ 0 & \ddots & \\ 0 & \cdots & e^{j\frac{2\pi}{8}(7)} \end{bmatrix}$$

1 Random Coding Error Exponent

The minimum error probability of any decoder of a block code over L frames satisfies (Fano 61)

$$\min P_e \leq e^{-LE_U(R)}, \quad R < C$$

where

- R: symbol rate (nats/symbol)
- C: channel capacity (nats/symbol)
- $E_U(R)$: error exponent

$$E_U(R) =$$

$$\max_{\mu \in [0,1]} \max_{P \in \mathcal{P}} \left\{ -\mu R - \ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} (p(X|S))^{1/(1+\mu)} \, dP(S) \right]^{1+\mu} dX \right\}, \quad \text{nats/syn}$$

$E_U(R)$ has been studied under *avg. power constraint* for

⇒ Known H (Telatar BL TM 96)

⇒ Unknown H (Abou-Faycal & Hochwald BL TM 99)

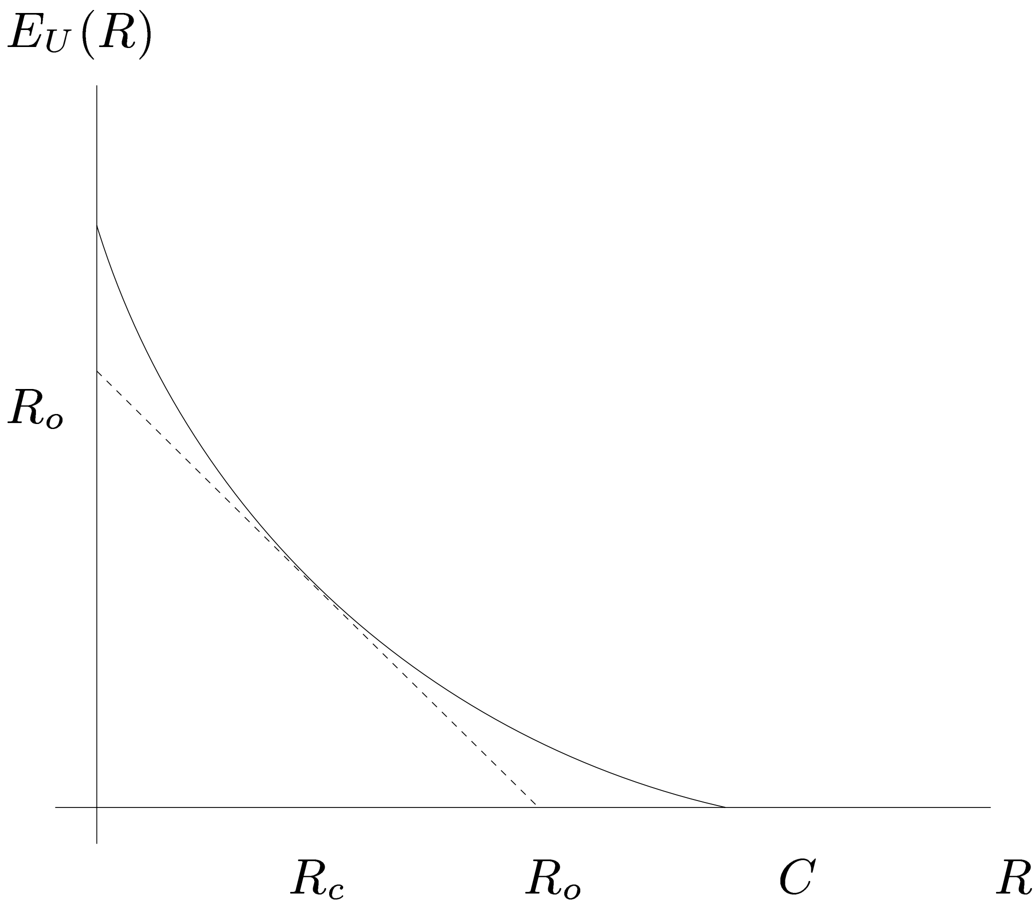


Figure 11. Error exponent $E_U(R)$ and cut-off rate bound $y(R) = R_o - R$.

R_o ***computational cut-off rate*** **lower bound** (Gallager IT 64)

$$E_U(R) \geq R_o - R, \quad R \leq R_o$$

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{X \in \mathcal{X}} \left[\int_{S \in \mathcal{S}} \sqrt{p(X|S)} dP(S) \right]^2 dX, \quad \text{nats/symbol}$$

where $\mathcal{P}$ are suitably constrained distributions over $\mathbb{C}^{T \times M}$

$\Rightarrow$ Cut-off rate analysis has been used to evaluate

- practical coding limits (Wang&Costello COM 95, Hagenauer&etal IT 96)
- different coding and modulation schemes (Massey 74)
- signal design for optical fiber links (Snyder&Rhodes IT 80)
- signaling over multiple access channels (Narayan&Snyder IT81)

FACTS:

- $R_o \leq C$
- R_o is highest practical rate for sequential decoders (Savage 65)
- $E_U(R) \approx R_o - R$ when $R \approx R_c$, the critical rate
- R_o specifies upper bound on optimal decoder error

$$P_e \leq e^{-L(R_o - R)}, \quad R \leq R_o$$

2 Integral Representation for R_o

$$R_o = \max_{P \in \mathcal{P}} - \ln \int_{S_1 \in \mathcal{S}} dP(S_1) \int_{S_2 \in \mathcal{S}} dP(S_2)\, e^{-N D(S_1 \| S_2)}.$$

where

$$D(S_1 \| S_2) \quad \stackrel{\text{def}}{=} \frac{1}{2} \ln \frac{\left| I_T + \frac{\eta}{2}(S_1 S_1^H + S_2 S_2^H) \right|^2}{\left| I_T + \eta S_1 S_1^H \right| \left| I_T + \eta S_2 S_2^H \right|}.$$

Low SNR approximation:

$D(S_1 \| S_2) = \eta^2/8 \| S_1 S_1^H - S_2 S_2^H \|^2 + o(\eta^2)$

The following parallels Theorems 1 and 2 of Marzetta&Hochwald IT 99

Proposition 1 *Assume that the transmitted signal S is constrained to satisfy the peak power constraint $\|S\|^2 \leq MT$. There is no advantage to using $M > T$ transmit antennas. Furthermore, for $M \leq T$ the signal matrices achieving R_o can be expressed as*

$$S = \begin{bmatrix} \\ \Phi \\ \\ \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix}$$

where

- Φ *is* $T \times M$ *unitary matrix* $V^H V = I_M$
- Λ *is* $M \times M$ *non-negative diagonal matrix.*

3 Case of Discrete K-dimensional Constellations

Specialize $\mathcal{P}$ to the discrete distributions over $\mathbb{C}^{T \times M}$

Then $R_o = \tilde{R}_o(K)$ is given by

$$\max_{\{P_i, S_i\}_{i=1}^{K}} -\ln \sum_{i=1}^{K} P_i \sum_{j=1}^{K} P_j \; e^{-ND(S_i \| S_j)} = -\ln \min_{\{P_i, S_i\}_{i=1}^{K}} \underline{P}^T E_K \underline{P}$$

where

- $E_k = ((D(S_i|S_j))_{i,j=1}^{K}$: dissimilarity (distance) matrix
- $\underline{P} = [P_1, \ldots, P_K]^T$

Under peak power constraint, $\|S_i\| \leq TM$,

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K} \left(\min_{\{P_i\}_{i=1}^K} \underline{P}^T E_K \underline{P} \right)$$

Inner maximization:

$$\min_{\underline{P}>0\,:\,\underline{1}_K^T \underline{P}=1} \left\{ \underline{P}^T E_K \underline{P} \right\}$$

Lagrangian

$$J(\underline{P}) = \underline{P}^T E_K \underline{P} - 2c(\underline{1}_K^T \underline{P} - 1)$$

minimized for *equalizer probability* $\underline{P} = \underline{P}^*$

$$E_K \underline{P}^* = c\underline{1}_K \quad \Rightarrow \quad \sum_{j=1}^{K} P_j \, e^{-ND(S_i \| S_j)} = c$$

Fact: optimal constellation satisfies $E_K^{-1}\underline{1}_K \geq 0$

$$\underline{P}^* = cE_K^{-1}\underline{1}_K, \quad \text{and} \quad c = \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K}$$

and

$$\tilde{R}_o(K) = -\ln \min_{\{S_i\}_{i=1}^K} \frac{1}{\underline{1}_K^T E_K^{-1} \underline{1}_K} = \max_{\{S_i\}_{i=1}^K} \ln \left(\underline{1}_K^T E_K^{-1} \underline{1}_K \right)$$

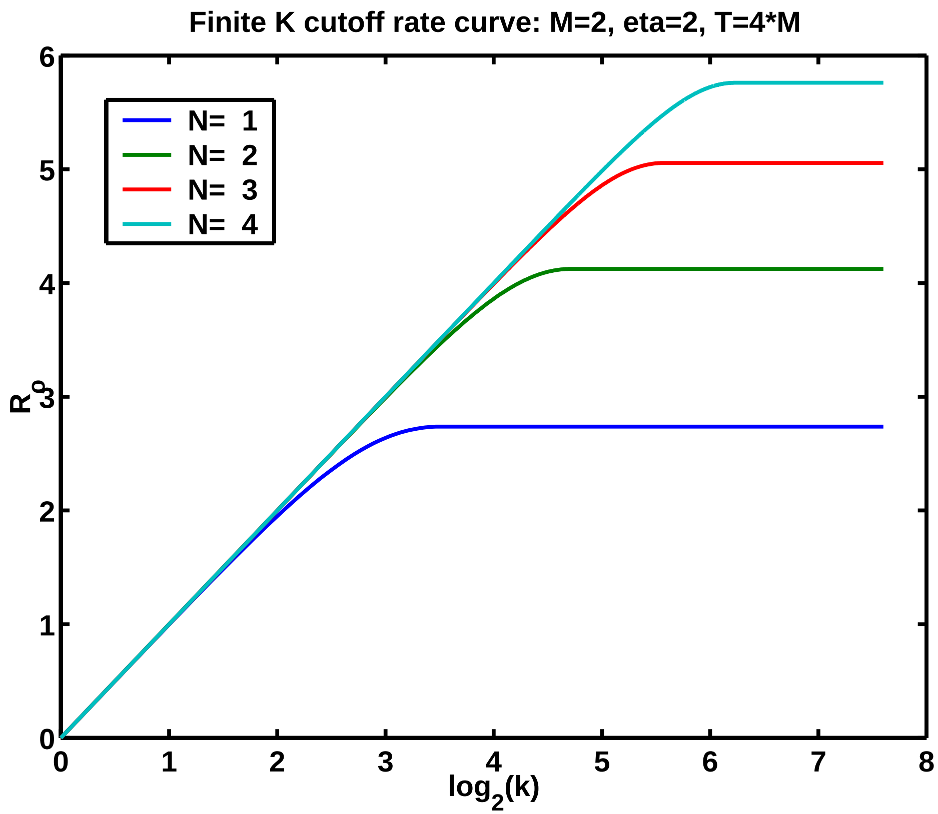
4 Bound on minimum distance

To $o(\eta^2)$ we have bounds

$$D_{\min}^{**} = \max_{\{S_i\}_{i=1}^{K}} \min_{i \neq j} D(S_i \| S_j) \geq \frac{\eta^2}{8} \frac{(TM)^2}{(2^p - 1)^2} > \frac{\eta^2 (TM)^2}{128} K^{-2/T}$$

(a) (b)

Figure 12. *Constellations of signal matrix singular values*


Finite K cutoff rate curve: M=2, eta=2, T=4*M
N= 1
N= 2
N= 3
N= 4
R_0
$log_2(k)$
0
1
2
3
4
5
6
7
8
0
1
2
3
4
5
6

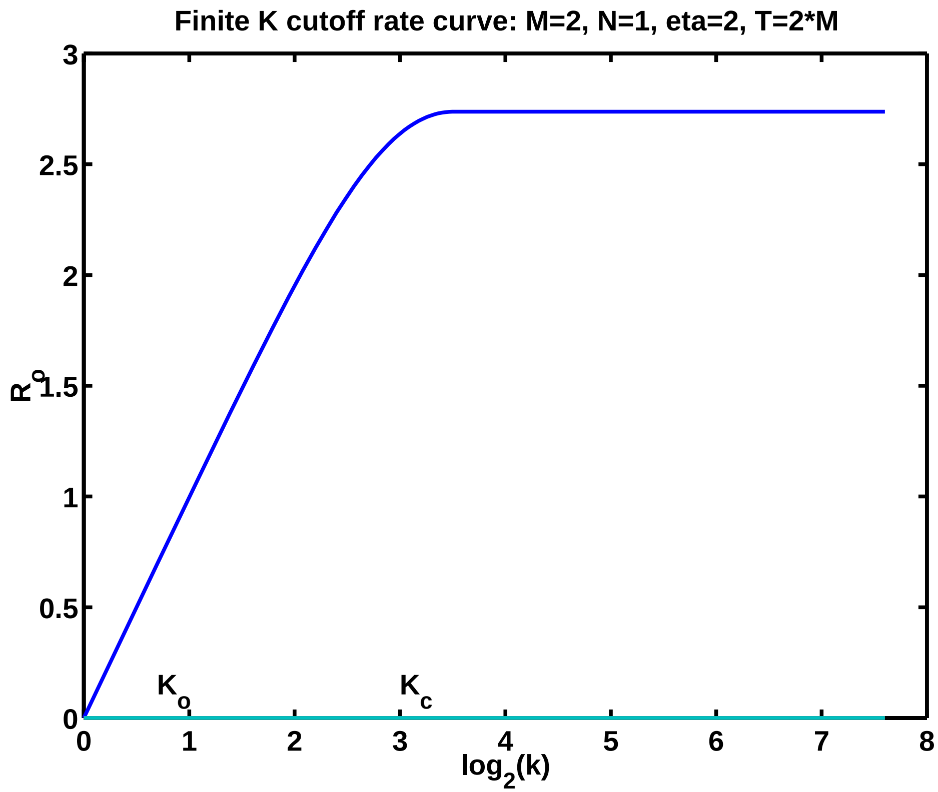


Figure 13. Cutoff-rate curve as function of size K.

Define:

$K_o = \lfloor T/M \rfloor$: "orthogonal size"

= max value K for which closed form expression $\tilde{R}_o$ exists

and

K_c: "logK" transition point

= knee of $\tilde{R}_o$

$\Rightarrow$ diminished returns by increasing K beyond K_c

5 Bound on logK transition point of constellation

Pick "test constellation" $\{S_i\}_{i=1}^K$ for which

$$D_{\min} = \min_{i \neq j} D(S_i \| S_j) > \gamma K^{-2/T}$$

$\gamma = \frac{\eta^2 (TM)^2}{128}$.

$$\begin{aligned}
\tilde{R}_o(K) \quad & \geq \max_{\{P_i\}} \log \left(\frac{1}{\sum_{i,j} P_i P_j e^{-ND(S_i \| S_j)}} \right) \\
& \geq \max_{\{P_i\}} \log \left(\frac{1}{\sum_{i,j} P_i P_j + \sum_{i \neq j} P_i P_j e^{-ND_{\min}}} \right) \\
& = \log \left(\frac{1}{1/K + (K-1)/K \; e^{-ND_{\min}}} \right)
\end{aligned}$$

$$
\begin{aligned}
&> \log\left(\frac{1}{1/K + (K-1)/K\ e^{-N\gamma K^{-2/T}}}\right)\\
&= \log(K) - \log\left(1 + (K-1)\ e^{-N\gamma K^{-2/T}}\right)\\
&\approx \log(K), \qquad (K-1)\ e^{-N\gamma K^{-2/T}} \leq 1
\end{aligned}
$$

This gives lower bound on K_o

$$
K_o \geq \left\{K : K^{2/T} \ln K = \gamma N\right\}
$$

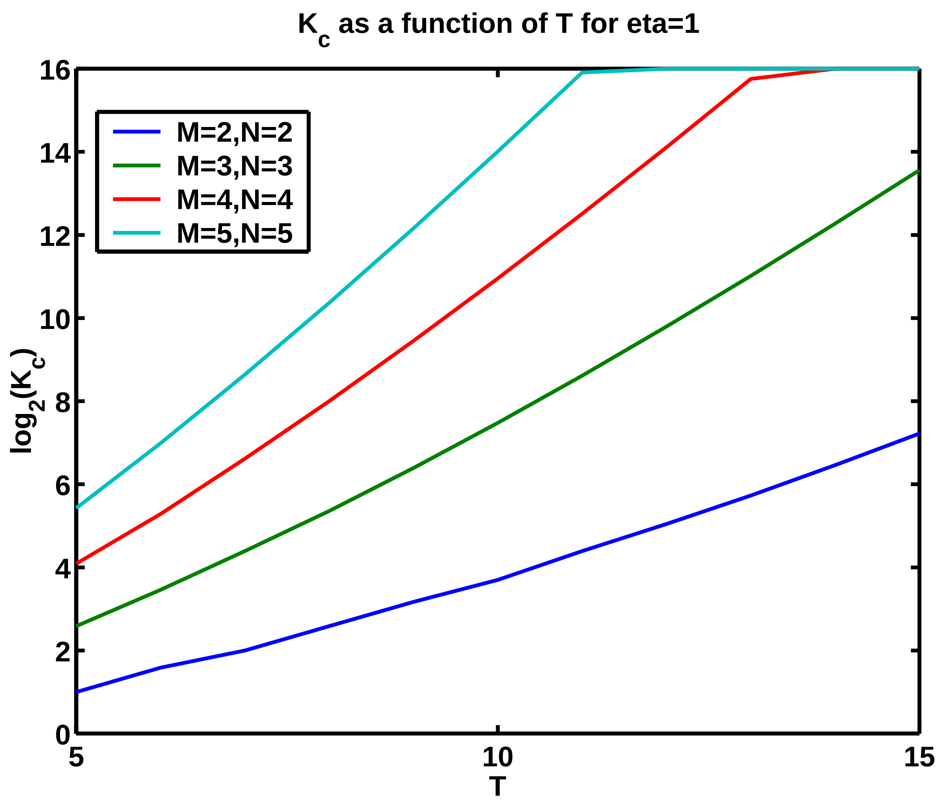


Figure 14. Corner sizes for equal M and N.

6 Low Dimensional Constellations $K \leq K_o$

For given η, T and M define the integer M_o

$$M_o = \text{argmax}_{m \in \{1,\ldots,M\}} \left\{ m \ln \frac{(1+\eta TM/(2m))^2}{1+\eta TM/m} \right\}.$$

First a result on max attainable distance under peak power constraint

Proposition 2 *Let $2M \leq T$. Then*

$$D_{\max} \stackrel{\text{def}}{=} \max_{S_1, S_2 \in \mathcal{S}^K_{\text{peak}}} D(S_1 \| S_2) = M_o \ln \frac{(1+\eta TM/(2M_o))^2}{1+\eta TM/M_o}. \qquad (1)$$

Furthermore, the optimal signal matrices which attain $D_{\max}$ can be taken as scaled rank M_o mutually orthogonal unitary $T \times M$ matrices of the form

$$S_1 = \eta \text{TM}\ \Phi_1, \qquad S_2 = \eta \text{TM}\ \Phi_2$$

where, for $j = 1, 2$,

$$\Phi_j^H \Phi_j = I_{M_o}, \quad \text{and} \quad \Phi_i^H \Phi_j = 0, \;\; i \neq j.$$

Proof is based on alternative representation for $D(S_1 \| S_2)$

$$D(S_1 \| S_2) = \quad \tfrac{1}{2} \ln \frac{\left|I_M + \frac{\eta}{2} S_1^H S_1\right|^2 \left|I_M + \frac{\eta}{2} S_2^H S_2\right|^2}{\left|I_M + \eta S_1^H S_1\right| \left|I_M + \eta S_2^H S_2\right|} \left|I_M - \kappa^H \kappa\right|^2,$$

where κ is a $M \times M$ multiple signal correlation matrix

$$\kappa = \tilde{S}_2^H \tilde{S}_1$$

$$\tilde{S}_i = \frac{\eta}{2} S_i \left[I_M + \frac{\eta}{2} S_i^H S_i\right]^{-1}$$

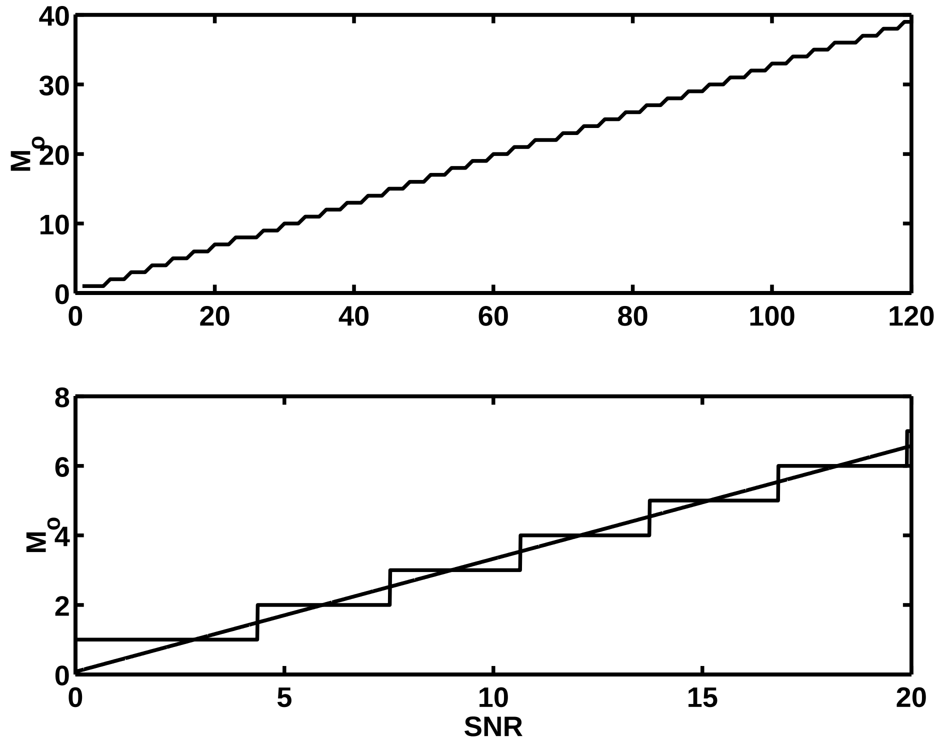


Figure 15. Top panel: M_o as a function of the SNR parameter ηTM. Bottom panel: blow up of first panel over a reduced range of SNR.

Proposition 3 *Let* $2M \leq T$ *and let* M_o *be as defined in (1). Suppose that* $M_o \leq \min\{M, T/K\}$. *Then the peak constrained* K *dimensional cut-off rate is*

$$\tilde{R}_o(K) = \ln\left(\frac{K}{1 + (K-1)e^{-ND_{\max}}}\right)$$

and $D_{\max}$ *is given by (1). Furthermore, the optimal constellation attaining* $\tilde{R}_o(K)$ *is the set of* K *rank* M_o *mutually orthogonal unitary matrices and the optimal probability assignment is uniform:* $P_i^* = 1/K$*,* $i = 1, \ldots, K$.

Example constellations for $T \times M = 4 \times 2$

- $M_o = 1$, $K = 4$: ($\eta^2 TM < 4.8$)

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

- $M_o = 2$, $K = 2$: ($\eta^2 TM \geq 4.8$)

$$\{S_i\}_{i=1}^{K} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

7 Conclusions

- Peak power contrained cut-off rate reduces to minimizing Q-form
- optimal constellation equalizes the decoder error rates
- Average distance for optimal K-dim constellation decreases at most by $K^{-2/T}$
- Optimal low rate constellation is a set of scaled mutually orthogonal unitary matrices.
- Rank of the unitary signal matrices decreases in SNR
- For very low SNR, no diversity advantage: apply power to a single antenna element at a time.

References

[1] A. O. Hero and T. L. Marzetta, "On computational cut-off rate for space time coding," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 2000.

[2] I. Abou-Faycal and B. M. Hochwald, "Coding requirements for multiple-antenna channels with unknown Rayleigh fading," Technical Memorandum, Bell Laboratories, Lucent Technologies, Murray Hill, NJ, 1999.

[3] T. L. Marzetta and B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh fading," *IEEE Trans. on Inform. Theory*, vol. IT-45, pp. 139–158, Jan. 1999.