Resolution Limits of 20 Questions Search Strategies for Moving Targets

LIN ZHOU† AND ALFRED HERO‡

† School of Cyber Science and Technology, Beihang University, China

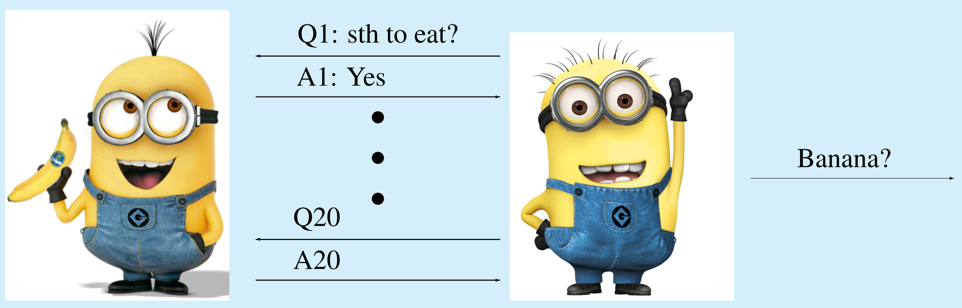
‡ Department of EECS, University of Michigan, Ann Arbor, MI, USA

Emails: lzhou@buaa.edu.cn; hero@eecs.umich.edu

Paper ID: 1204

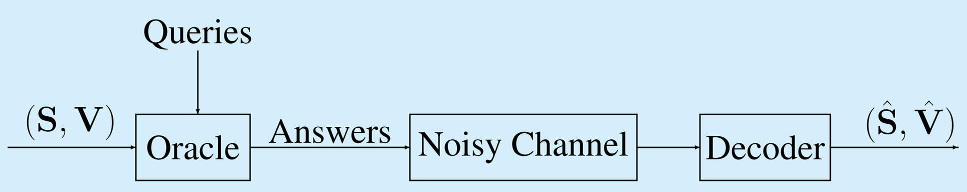
ICASSP 2021 TORONTO Canada June 6-11, 2021

Origin: 20 Questions Game



- Two players: oracle and questioner
- Querying and answering game with a budget (cf. [1]-[3])
- Noiseless: oracle is always honest
- Key: design smart queries to extract the secret as soon as possible

Problem Formulation



- Target: accurately estimate the trajectory of a moving target with unknown initial location $\mathbf{S}=(S_1,\ldots,S_d)\in\mathcal{S}^d$ and moving velocity $\mathbf{V}=(V_1,\ldots,V_d)\in\mathcal{V}^d$
- $\mathcal{S}=[0,1]$, $\mathcal{V}=[-v_+,v_+]$ and $(\mathbf{S},\mathbf{V})\sim f_{\mathbf{SV}}$.
- Task: design queries and decoder (scheme/strategy/policy)
- Motivation: applications such as localizing a moving target using a sensor network

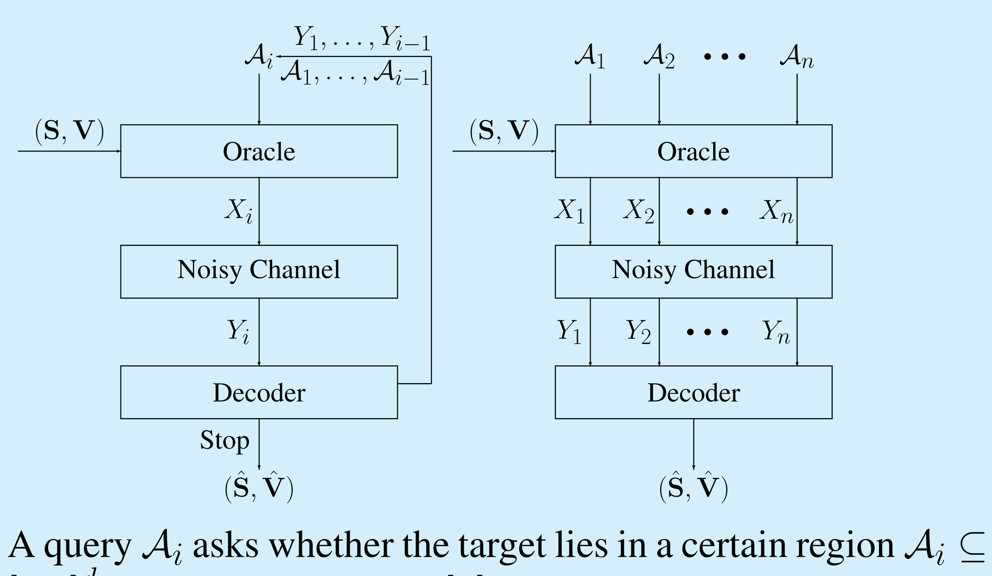
The Torus Model

- Consider a target with initial location $\mathbf{s}=(s_1,\ldots,s_d)\in[0,1]^d$ and velocity $\mathbf{v}=(v_1,\ldots,v_d)\in\mathcal{V}^d$
- Torus model: at each time point $t\in[n]$ and in each dimension $i\in[d]$, the location of the target is

$$\mathrm{l}(s_i,v_i,t) := \begin{cases} 1 & \text{if } \mathrm{mod}(s_i+tv_i,1)=0,\\ s_i+tv_i-\lfloor s_i+tv_i\rfloor & \text{if } s_i+tv_i\in\bigcup_{h\in\mathbb{N}}[2h,2h+1),\\ \lceil s_i+tv_i\rceil-(s_i+tv_i) & \text{otherwise,}\end{cases}$$

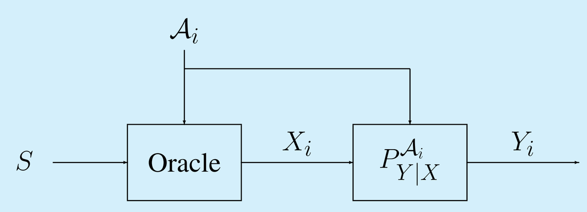
  - the target moves within a unit length interval
  - similar to the unit circle model studied by Kaspi, Shayevitz and Javadi (T-IT 2018)
- The location of the target, denoted as $\mathrm{l}(\mathbf{s},\mathbf{v},t)$, is a d-dimensional vector $(\mathrm{l}(s_1,v_1,t),\ldots,\mathrm{l}(s_d,v_d,t))$. At each dimension,

Adaptive and Non-adaptive Querying



- A query $\mathcal{A}_i$ asks whether the target lies in a certain region $\mathcal{A}_i\subseteq[0,1]^d$ at a discrete time $i\in[n]$ and X_i is the noiseless answer
- *Non-adaptive querying*
  - queries are designed simultaneously offline
  - lower cost, faster execution time due to parallelizability

Measurement Dependent Noisy Channel



- Binary Input alphabet $\mathcal{X}$ and arbitrary output alphabet $\mathcal{Y}$
- For any $\mathcal{A}\subseteq[0,1]^d$, $P^{\mathcal{A}}_{Y|X}$ depends on $\mathcal{A}$ through its size via a bounded Lipschitz continuous function $f:[0,1]\to\mathbb{R}_+$
- Example: for any parameter $\zeta\in(0,1)$ and any $\mathcal{A}\subseteq[0,1]$
  - Measurement dependent binary symmetric channel (BSC)

$P^{\mathcal{A}}_{Y\mid X}$	$X=0$	$X=1$
$Y=0$	$1-\zeta f(\lvert\mathcal{A}\rvert)$	$\zeta f(\lvert\mathcal{A}\rvert)$
$Y=1$	$\zeta f(\lvert\mathcal{A}\rvert)$	$1-\zeta f(\lvert\mathcal{A}\rvert)$

Definition of Query Procedure

Given any $n\in\mathbb{N}$, $\delta\in\mathbb{R}_+$ and $\varepsilon\in[0,1]$, to search for a moving target over a d-dimensional torus, an (n,d,δ,ε)-non-adaptive query procedure consists of

- n queries $\mathcal{A}^n$, where for each $i\in[n]$, we query whether the target locates inside a Lebesgue measurable subset $\mathcal{A}_i\in[0,1]^d$
- and a decoder $g:\mathcal{Y}^n\to[0,1]^d\times\mathcal{V}^d$

such that the excess-resolution probability satisfies

$$\mathrm{P_e}(n,d,\delta):=\sup_{f_{\mathbf{SV}}}\Pr\left\{\max_{t\in[0:n]}\|l(\hat{\mathbf{S}},\hat{\mathbf{V}},t)-l(\mathbf{S},\mathbf{V},t)\|_\infty>\delta\right\}\le\varepsilon,$$

Accurate estimation of the trajectory implies accurate estimate of the initial location and velocity, and vice versa

- $|\hat{S}_i-S_i|<\frac{\delta}{2}$ and $n|\hat{V}_i-V_i|<\frac{\delta}{2}$ implies accurate estimation of the trajectory, i.e., $\max_{t\in[0:n]}\|l(\hat{\mathbf{S}},\hat{\mathbf{V}},t)-l(\mathbf{S},\mathbf{V},t)\|_\infty\le\delta$
- $|\hat{S}_i-S_i|>\delta$ or $n|\hat{V}_i-V_i|>2\delta$ implies poor estimate

Fundamental Limit

- Given any number of queries $n\in\mathbb{N}$ and $\varepsilon\in[0,1]$,

$$\delta^*(n,d,\varepsilon):=\inf\{\delta\in\mathbb{R}_+:\exists\text{ an }(n,d,\delta,\varepsilon)\text{−non−adaptive query procedure}\}.$$

  - Best non-asymptotic resolution achievable by any non-adaptive query procedure with n queries and excess-resolution probability ε
- Dual quantity (sample complexity):

$$n^*(d,\delta,\varepsilon)=\inf\{n\in\mathbb{N}:\delta^*(n,d,\varepsilon)\le\delta\}$$

Preliminaries

- $P_X=\mathrm{Bern}(p)$ denotes the Bernoulli distribution
- $P^q_{Y|X}$ denotes $P^{\mathcal{A}}_{Y|X}$ when $f(|\mathcal{A}|)=q$
- $P^{p,q}_Y$ denotes the distribution on $\mathcal{Y}$ induced by P_X and $P^q_{Y|X}$
- For any $(x,y)\in\mathcal{X}\times\mathcal{Y}$, define the mutual information density

$$\imath_{p,q}(x;y):=\log P^q_{Y|X}(y|x)-\log P^{p,q}_Y(y).$$

- "Capacity" of measurement dependent channels $\{P^q_{Y|X}\}_{q\in[0,1]}$

$$C:=\max_{p\in[0,1]}\mathbb{E}[\imath_{p,f(p)}(X;Y)]\,,(X,Y)\sim\mathrm{Bern}(p)\times P^{f(p)}_{Y|X}$$

- "Dispersion" of measurement dependent channels

$$\mathrm{V}_\varepsilon:=\begin{cases}\min_{p\in\mathcal{P}_{\mathrm{ca}}}\mathrm{Var}[\imath_{p,f(p)}(X;Y)] & \text{if }\varepsilon<0.5,\\ \max_{p\in\mathcal{P}_{\mathrm{ca}}}\mathrm{Var}[\imath_{p,f(p)}(X;Y)] & \text{if }\varepsilon>0.5.\end{cases}$$

Main Result

Theorem 1 *For any $\varepsilon\in(0,1)$ and finite $d\in\mathbb{N}$, the minimal achievable resolution $\delta^*(n,d,\varepsilon)$ satisfies the following properties*

- *if $nv_+=O(n^t)$ for $t\in[0.5,1)$,*

$$-2d\log\delta^*(n,d,\varepsilon)=nC+O(nv_+);$$

- *if $nv_+=O(n^t)$ for $t\in[0,0.5)$*

$$-2d\log\delta^*(n,d,\varepsilon)=nC+\sqrt{n\mathrm{V}_\varepsilon}\Phi^{-1}(\varepsilon)+O(\max\{nv_+,\log n\});$$

Discussions

- Theorem 1 is tight under maximal speed constraint v_+
- Refines the result by Kaspi *et al.*, TIT 2018 (Theorem 3):
  - Non-asymptotic, non-vanishing vs asymptotic, vanishing
  - Any measurement dependent channel vs a measurement dependent BSC
  - Multidimensional vs one-dimensional
- Strong converse holds

$$\lim_{n\to\infty}-\frac{1}{n}\log\delta^*(n,d,\varepsilon)=\frac{C}{2d}.$$

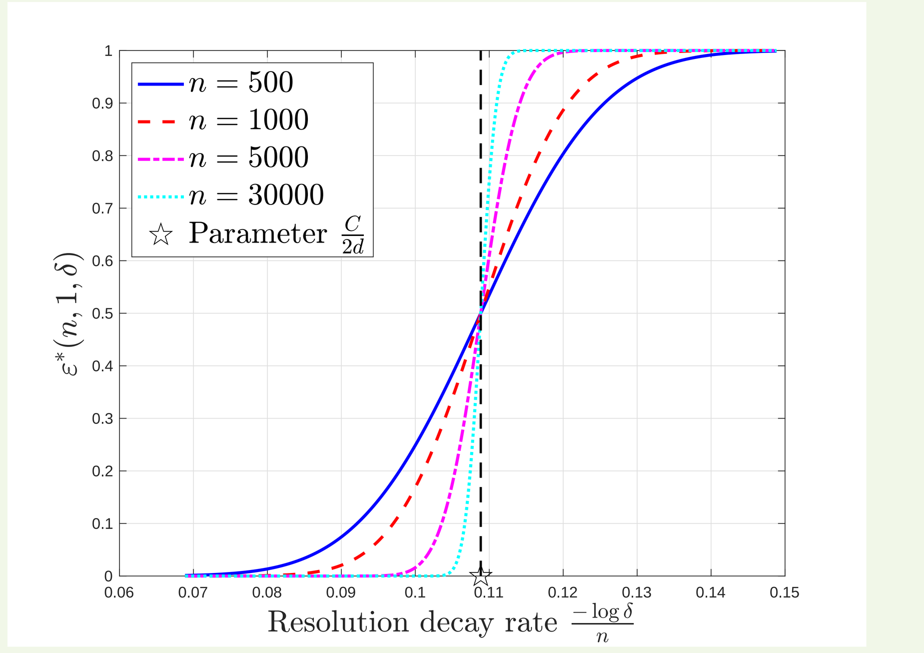
- Proof ideas: finite blocklength channel coding + analysis of the number of quantized trajectories

Phase Transition

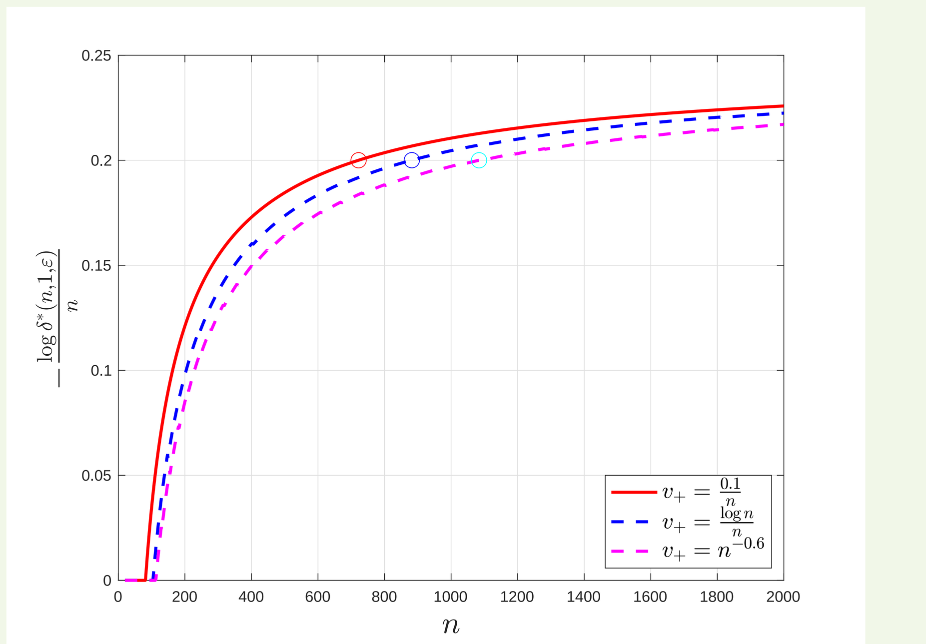
- Minimal excess-resolution probability $\varepsilon^*(n,d,\delta)$
- For maximal speed such that $nv_+=o(\sqrt{n})$

$$\varepsilon^*(n,d,\delta)=\Phi\left(\frac{-d\log\delta-nC}{\sqrt{n\mathrm{V}_\varepsilon}}\right)+o(1)$$

- Numerical illustration for a measurement dependent BSC with parameter $\zeta=0.2$ and the Lipschitz continuous function $f(q)=q+0.5$ when $d=1$

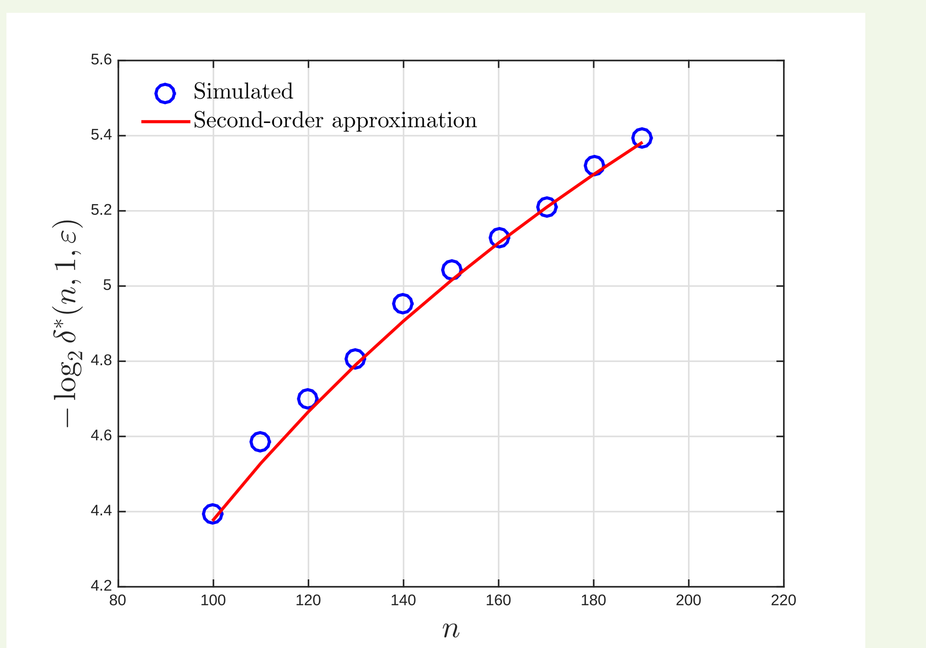


Impact of Maximal Speed



- Measurement dependent BSC with $\zeta=0.05$ and $f(q)=0.5+q$
- The larger the maximal speed, the harder to search

Simulation of Proposed Query Procedure



- Maximal speed $v_+=\frac{0.1}{n}$

References

1. A. Rényi, "On a problem of information theory", *MTA Mat. Kut. Int. Kozl. B*, vol. 6, pp. 505-516, 1961.
2. M. V. Burnashev and K. Zigangirov, An interval estimation problem for controlled observations, Problemy Peredachi Informatsii, vol. 10, no. 3, pp. 51-61, 1974.
3. S. M. Ulam, "Adventures of a Mathematician", *Univ. of California Press*, 1991.
4. B. Jedynak, P. I. Frazier, and R. Sznitman, "Twenty questions with noise: Bayes optimal policies for entropy loss," *Journal of Applied Probability*, vol. 49, no. 1, pp. 114-136, 2012.
5. Y. Kaspi, O. Shayevitz, and T. Javidi, "Searching with measurement dependent noise," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2690-2705, 2018 (also in IEEE ITW, 2014).
6. Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307-2359, 2010.
7. L. Zhou and A. Hero, "Resolution limits of noisy 20 questions estimation", *IEEE Trans. Inf. Theory*, vol. 67, no. 4, pp. 2055–2073, 2021.