Detection of the Number of Signals Using the Benjamini-Hochberg Procedure

Pei-Jung Chung*, Johann F. Böhme, Christoph F. Mecklenbräuker, Alfred O. Hero

***Abstract*— This work presents a novel approach to detect multiple signals embedded in noisy observations from a sensor array. We formulate the detection problem as a multiple hypothesis test. To control the global level of the multiple test, we apply the false discovery rate (FDR) criterion proposed by Benjamini and Hochberg. Compared to the classical familywise error rate (FWE) criterion, the FDR-controlling procedure leads to a significant gain in power for large size problems. In addition, we apply the bootstrap technique to estimate the observed significance level required by the FDR-controlling procedure. Simulations show that the FDR-controlling procedure always provides higher probability of correct detection than the FWE-controlling procedure. Furthermore, the reliability of the proposed test procedure is not affected by the gain in power of the test.**

***Index Terms*— array processing, detection, number of signals, false discovery rate, multiple test, likelihood ratio, bootstrap**

EDICS Category: SAM-SDET, SSP-DETC

I. INTRODUCTION

Determination of the number of signals embedded in noisy sensor outputs is a key issue in array processing and related applications [3]. Many high resolution methods, such as maximum likelihood (ML) approach or MUSIC, assume a known number of signals. Performance of these estimators depend strongly on this knowledge [11]. For example, when the number of signals is incorrectly specified, favorite properties such as consistency and efficiency of the ML estimator may be no longer valid. In radar or geophysics, deciding how many incoming waves is as important as estimating the associated propagation parameters.

Traditional methods based on information theoretic criteria such as Akaike's information criterion (AIC) or Rissanen's minimum description length (MDL) [14], [25], [26], [28] view this problem as model order selection. Another class of methods [5], [27] uses hypothesis tests to decide how many eigenvalues of the sample covariance matrix are equal. The eigenstructure of the spatial correlation matrix is fundamental to all these methods. Consequently, they are often sensitive to signal coherence and low signal to noise ratio (SNR) with the danger of a subspace swap.

For broadband signals, one may extend such narrowband methods by using the focusing technique [19] to transform the sample covariance matrix of array outputs at various frequencies to a common subspace. The aforementioned methods developed for narrowband signals can be applied directly after focusing. An alternative approach in [23] applies an interpolation model suitable for Markov Chain Monte Carlo (MCMC) procedures.

In this work, we consider a detection procedure based on multiple testing with test statistics derived from the generalized likelihood ratio (LR) principle [4], [21]. The proposed approach is suitable for both narrowband and broadband models. When broadband signals are of interest, the combination of information from different frequency bins follows naturally from the asymptotic normality and independence of Fourier transformed data [6]. At each test step, we compute the ML estimate under the assumed number of signals and the corresponding test statistic. In other words, our procedure jointly estimates the number of signals and the parameters of interest. Because the proposed test utilizes the same parameterization as the ML method, it enjoys similar favorite features. The experimental results from seismic measurements reported in [7], [8] demonstrate superior performance of the generalized LR test based approach in scenarios involving small numbers of samples, low SNRs and coherent signals.

In the narrowband model, the ML estimation in the proposed procedure is computationally more costly than the required eigen-decomposition in most MDL type methods [14], [25], [28]. However, in the broadband model,the focusing technique [19] or the MCMC approach [23] requires considerable amount of computation. The computational complexity of our method is moderate compared to [19], [23].

A major concern in multiple testing problems is the control of type one errors. The detection procedure suggested in [4], [7], [21] applied the Bonferroni-Holm procedure [17] to control the classical familywise error-rate (FWE), the probability of erroneously rejecting any of the true hypotheses. As the control of FWE requires each test to be conducted at a significantly lower level, the Bonferroni-Holm procedure often leads to conservative results. For the proposed procedure, this implies that the ability to discover signals is reduced with growing numbers of signals. To overcome this drawback, we adopt the false discovery rate (FDR) criterion suggested by Benjamini and Hochberg [1] to keep balance between type one error control and power[1]. The difference between the FWE and FDR-controlling procedure becomes more dramatic

Pei-Jung Chung is with School of Engineering and Electronics, University of Edinburgh, UK, peijung.chung@gmail.com. Johann F. Böhme is with Signal Theory Group, Ruhr-Universität Bochum, Germany, boehme@sth.rub.de. Christoph F. Mecklenbräuker is with ftw. Telecommunications Research Center Vienna, Austria, cfm@ieee.org. Alfred O. Hero is with Dept. of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, USA, hero@eecs.umich.edu.

[1]The power of a test is the probability of correctly rejecting the null hypothesis.

when the size of problems becomes larger [2]. Therefore, using the FDR criterion can lead to a significant gain in power in situations involving many signals. To ensure the desired FDR level to be controlled by the Benjamini-Hochberg procedure, we shall carefully examine the independence condition on the test statistics as well.

Since the test statistics have no closed form distribution function for broadband signals, we shall apply the bootstrap technique to obtain the observed significance level, the p-value, required by the FDR-controlling procedure.

This paper is organized as follows. We give a brief description about the signal model in the next section. The multiple test procedure for signal detection is developed in section III. Section IV introduces the idea of false discovery rate (FDR) and the Benjamini-Hochberg procedure. In the subsequent section we show that the condition required by the Benjamini-Hochberg procedure is satisfied. The concepts of bootstrap and the procedure for estimating p-values are illustrated in section VI. Simulation results are presented and discussed in section VII. Our concluding remarks are given in section VIII.

II. Problem Formulation

Consider an array of n sensors receiving m broadband signals emitted by far-field sources located at positions described by their angles of arrival $\boldsymbol{\theta}_m = [\,\theta_1, \ldots, \theta_m\,]^T$. We consider n to be known and fixed, whereas m is an unknown non-negative integer which is to be determined from the observed array output data. The n-dimensional sensor array output $\boldsymbol{x}(t)$ is modelled by the time-invariant linear convolution model,

$$\boldsymbol{x}(t) = \int_{-\infty}^{\infty} \boldsymbol{h}_m(t-\tau;\boldsymbol{\theta}_m)\boldsymbol{s}_m(\tau)\,\mathrm{d}\tau + \boldsymbol{u}(t)\ .$$

Here, $\boldsymbol{h}_m(\cdot;\boldsymbol{\theta}_m)$ is the $n \times m$ matrix of impulse responses and its (i,j)th element links the ith element of $\boldsymbol{x}(\cdot)$ to the jth element of $\boldsymbol{s}_m(\cdot)$. The n-dimensional stochastic process $\boldsymbol{u}$ models the additive noise. We assume that $\boldsymbol{x}(t)$ is bandlimited and we sample at the Nyquist rate which we normalize to one. The sensor array outputs $\boldsymbol{x}(t)$, $(t = 0, \ldots, T-1)$ are divided into K snapshots of length $T' = T/K$ where $K \geq n$. The data in the kth observation snapshot is short-time Fourier-transformed

$$\boldsymbol{X}^k(\omega) = \frac{1}{\sqrt{T'}} \sum_{t=0}^{T'-1} \boldsymbol{x}(t)\mathrm{e}^{-\mathrm{j}\omega t}. \tag{1}$$

For large number of samples T', the frequency domain data is described approximately by the regression model [4], [6],

$$\boldsymbol{X}^k(\omega) = \boldsymbol{H}_m(\omega;\boldsymbol{\theta}_m)\boldsymbol{S}_m^k(\omega) + \boldsymbol{U}^k(\omega) \tag{2}$$

where the matrix $\boldsymbol{H}_m(\omega;\boldsymbol{\theta}_m) = [\boldsymbol{d}_1(\omega)\cdots\boldsymbol{d}_i(\omega)\cdots\boldsymbol{d}_m(\omega)] \in \mathbb{C}^{n\times m}$ consists of m plane-wave steering vectors with the ith column $\boldsymbol{d}_i(\omega)$ corresponding to the ith incoming wave arriving from angle θ_i. The short-time Fourier transformed signals and noise are denoted by $\boldsymbol{S}_m^k(\omega)$ and $\boldsymbol{U}^k(\omega)$ in Eq.(2), respectively. In the following analysis, the signal waveform $\boldsymbol{S}_m^k(\omega)$ is considered to be unknown and deterministic. $\boldsymbol{U}^k(\omega)$ consists of spatially uncorrelated sensor noise with cavariance matrix $\boldsymbol{C}_{\boldsymbol{U}}(\omega) = \nu(\omega)\boldsymbol{I}$ where $\nu(\omega)$ is the unknown noise spectral parameter and $\boldsymbol{I}$ is an identity matrix of corresponding dimension[2].

From the stochastic properties of the Fourier transform [6], we know that the Fourier transformed data $\boldsymbol{X}^k(\omega)$ is characterized by asymptotic normality and independence. More precisely, under the regularity conditions formulated in Theorem 4.4.1 in [6], for large T and T', the following properties hold.

1) The Fourier transformed data $\boldsymbol{X}^k(\omega_j), \boldsymbol{X}^{k'}(\omega_j)$, where $\omega_j = \frac{2\pi j}{T}$, $(k, k' = 0, \ldots, K-1)$ are mutually independent for $k \neq k'$ and all $0 \leq j \leq J-1 < T'$.
2) $\boldsymbol{X}^k(\omega_j), \boldsymbol{X}^k(\omega_{j'})$, $(j, j' = 0, \ldots, J-1)$ are mutually independent for $\omega_j \neq \omega_{j'}$ and all k.
3) Given the signal waveform $\boldsymbol{S}_m^k(\omega_j)$, $\boldsymbol{X}^k(\omega_j)$ is complex normally distributed with mean $\boldsymbol{H}_m(\omega_j;\boldsymbol{\theta}_m)\boldsymbol{S}_m^k(\omega_j)$ and covariance matrix $\boldsymbol{C}_{\boldsymbol{U}}(\omega_j) = \nu(\omega)\boldsymbol{I}$.

Properties (1) and (2) provide us a natural way to combine information from different frequency bins and snapshots. Furthermore, property (3) ensures the data's normality without additional assumption on noise. Based on the data set $\{\boldsymbol{X}^k(\omega_j), k = 1, \ldots, K, j = 0, \ldots, J-1\}$, the problem of central interest is to determine the number of signals m embedded in the observations.

III. Signal Detection Using a Multiple Hypothesis Test

We formulate the problem of detecting the number of signals as a multiple hypothesis test. Let $M < n$ denote the maximal number of sources. The following procedure provides an estimate $\hat{m}$ for the number of signals.

For $m = 1$,

H_1 : Data contains only noise.

$$\boldsymbol{X}^k(\omega_j) = \boldsymbol{U}^k(\omega_j)$$

A_1 : Data contains at least 1 signal.

$$\boldsymbol{X}^k(\omega_j) = \boldsymbol{H}_1(\omega_j;\boldsymbol{\theta}_1)\boldsymbol{S}_1^k(\omega_j) + \boldsymbol{U}^k(\omega_j)$$

For $m = 2, \ldots, M$

H_m : Data contains at most $(m-1)$ signals.

$$\boldsymbol{X}^k(\omega_j) = \boldsymbol{H}_{m-1}(\omega_j;\boldsymbol{\theta}_{m-1})\boldsymbol{S}_{m-1}^k(\omega_j) + \boldsymbol{U}^k(\omega_j) \tag{3}$$

A_m : Data contains at least m signals.

$$\boldsymbol{X}^k(\omega_j) = \boldsymbol{H}_m(\omega_j;\boldsymbol{\theta}_m)\boldsymbol{S}_m^k(\omega_j) + \boldsymbol{U}^k(\omega_j) \tag{4}$$

We use the subscript $(m-1)$ or m to emphasize the dimension of the steering matrix and the signal vector under null hypothesis H_m or alternative A_m. Let $\{i_1, i_2, \ldots, i_r\}$ be an arbitrary subset of $\{1, 2, \ldots, M\}$ and suppose that among M hypotheses, r are rejected, namely $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$, then the number of signals is determined by,

$$\hat{m} := \max\{i_1, i_2, \ldots, i_r\}. \tag{5}$$

Which hypotheses are to be rejected depends on the adopted error criterion. In this work, we shall apply the Benjamini-Hochberg procedure to control the false discovery rate.

[2]Extensions to spatially colored noise are discussed in Sec. III-A below.

We apply the generalized likelihood ratio (LR) principle to construct the test statistic T_m , $(m = 1, \dots, M)$

$$T_m = \max_{\boldsymbol{\theta}_m} L(\boldsymbol{\theta}_m) \; - \max_{\boldsymbol{\theta}_{m-1}} L(\boldsymbol{\theta}_{m-1}) \qquad (6)$$

where $L(\boldsymbol{\theta}_m)$ and $L(\boldsymbol{\theta}_{m-1})$ denote the concentrated log-likelihood function under A_m and H_m , respectively. The concentrated likelihoods depend on the non-linear parameters (angles of arrival) only, whereas the linear parameters have been eliminated by a closed-form optimization,

$$L(\boldsymbol{\theta}_m) = \frac{1}{J}\sum_{j=0}^{J-1} \log\left(\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j;\boldsymbol{\theta}_m))\hat{\boldsymbol{R}}(\omega_j)]\right) .$$

Here $\hat{\boldsymbol{R}}(\omega_j) = \frac{1}{K}\sum_{k=1}^{K} \boldsymbol{X}^k(\omega_j)\boldsymbol{X}^k(\omega_j)^H$ represents a non-parametric power spectral estimate of sensor outputs over K snapshots[3] and $\boldsymbol{P}_i(\omega_j;\boldsymbol{\theta}_i)$, $(i = m, (m-1))$ is the projection matrix onto the column space of $\boldsymbol{H}_i(\omega_j;\boldsymbol{\theta}_i)$. For $m = 1$, we define $\boldsymbol{P}_0(\cdot) = \boldsymbol{0}$.

After some manipulations [4], [21], we obtain the following generalized LR test statistic for testing H_m against A_m

$$T_m = \frac{1}{J}\sum_{j=0}^{J-1} T_m(\omega_j), \qquad (7)$$

$$\begin{aligned} T_m(\omega_j) &= \log\left(\frac{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_{m-1}(\omega_j;\hat{\boldsymbol{\theta}}_{m-1}))\hat{\boldsymbol{R}}(\omega_j)]}{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j;\hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)]}\right) \\ &= \log\left(1 + \frac{n_1}{n_2}F_m(\omega_j;\hat{\boldsymbol{\theta}}_m)\right), \qquad (8) \end{aligned}$$

where $\hat{\boldsymbol{\theta}}_i$ is the vector of ML estimates for all angles of arrival assuming that i signals are present.

Under hypothesis H_m, the statistic

$$F_m(\omega_j;\hat{\boldsymbol{\theta}}_m) = \frac{n_2}{n_1}\frac{\mathrm{tr}[(\boldsymbol{P}_m(\omega_j;\hat{\boldsymbol{\theta}}_m) - \boldsymbol{P}_{m-1}(\omega_j;\hat{\boldsymbol{\theta}}_{m-1}))\hat{\boldsymbol{R}}(\omega_j)]}{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j;\hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)]} \qquad (9)$$

is asymptotically F_{n_1,n_2}–distributed with degrees of freedom n_1, n_2, cf. Chapter 26 in [18]. For the single-frequency case, $J = 1$, Eqs. (8) and (9) show the equivalence to the narrowband F-test proposed by Shumway [24]. The statistic (9) can be interpreted as an estimate for the SNR increase induced by the mth signal. The mth signal is declared to be detected if it is strong enough so that the statistic (9) exceeds a given threshold.

If the parameters $\boldsymbol{\theta}_{m-1}$ and $\boldsymbol{\theta}_m$ were known *a priori*, testing H_m against A_m is equivalent to testing the linear model (3) against (4). The degrees of freedom could be obtained as $n_1 = 2K$, $n_2 = 2K(n-m)$. However, $\boldsymbol{\theta}_{m-1}$ and $\boldsymbol{\theta}_m$ are unknown and need to be estimated. Taking the estimated nonlinear parameters into account, the degrees of freedom are given by [21]

$$n_1 = K(2 + r_m), \quad n_2 = K(2n - 2m - r_m) \qquad (10)$$

with $r_m = \dim(\theta_m) = 1$ denoting the dimension of the nonlinear parameter vector θ_m associated with the mth signal.

[3] $(\cdot)^H$ denotes conjugate transpose.

The additional term r_m is obtained through Taylor expansion around the true parameter θ_m.

For certain types of wavefields, r_m may be larger than one. For instance, in seismic applications, the array geometry is usually planar, and the source is often described by two angular parameters, azimuth and elevation. In this setting, $r_m = 2$. In shallow ocean matched field applications with a linear array, the source location is often characterized in cylindrical coordinates. This results in $r_m = 2$ or $r_m = 3$ depending on the array's ambiguity structure.

In the broadband case, a suitable closed-form expression for the distribution of the test statistic T_m under H_m is unknown to the authors. We shall use the bootstrap technique to overcome this difficulty in Section VI.

A. Spatially colored noise

If the noise is spatially correlated, i.e., $\boldsymbol{C}_{\boldsymbol{U}}(\omega) \neq \nu(\omega)\boldsymbol{I}$, we expect a performance degradation if the proposed detection procedure is applied as described in Section III. It is relatively straightforward to extend the proposed method by generalizing to a noise covariance matrix $\nu(\omega)\boldsymbol{J}(\omega)$ where $\boldsymbol{J}(\omega)$ is a known Hermitian positive-definite (and, hence, of full-rank) matrix, suitably normalized to $\mathrm{tr}(\boldsymbol{J}(\omega)) = n$. When the signal model is extended in this way and the log-likelihoods are evaluated then we see that this amounts to extending the test procedure by a pre-whitening step. The corresponding test statistics are computed from the whitened sensor array output. The short-time Fourier transformed sensor array output (2) is linearly transformed via the whitening filter $\boldsymbol{J}(\omega)^{-1/2}$ so that the noise of the transformed data $\tilde{\boldsymbol{X}}^k(\omega) = \boldsymbol{J}(\omega)^{-1/2}\boldsymbol{X}^k(\omega)$ becomes spatially white. The matrix $\boldsymbol{J}(\omega)^{-1/2}$ denotes the inverse of a suitably chosen square root of the positive definite matrix $\boldsymbol{J}(\omega)$.

The generalisation to superposed noise covariance models with several parameters, e.g.

$$\nu_1(\omega)\boldsymbol{I} + \sum_{\ell=2}^{U} \nu_\ell \omega \boldsymbol{J}_\ell(\omega)$$

is possible, but cumbersome.

IV. Control of False Discovery Rate

The control of type one error is an important issue in multiple inferences. A type one error occurs when the null hypothesis H_m is wrongly rejected. The traditional concern in multiple hypothesis problems has been about controlling the probability of committing any type one error in families of simultaneous comparisons. The control of this familywise error-rate (FWE) usually requires each of the M tests to be conducted at a lower level. For example, given a significance level α, the significance level of each test is given by α/M in the classical Bonferroni procedure. When the number of tests increases, the power of the the FWE-controlling procedures such as Bonferroni-type procedures [17] is substantially reduced.

The false discovery rate (FDR), suggested by Benjamini and Hochberg [1], is a completely different point of view for considering the errors in multiple testing. The FDR is defined as the expected proportion of errors among the rejected hypotheses. Suppose that among the M tested hypotheses $\{H_1, H_2, \ldots, H_M\}$, m_0 are true null hypotheses. The number of hypotheses rejected is denoted by R. This observable random variable R can be decomposed $R = V + S$, where V is the number of *incorrectly* rejected null hypotheses and S is the number of *correctly* rejected hypotheses. In terms of these random variables, the FWE is $P(V > 0)$, the probability of making any type one error. The proportion of errors committed by falsely rejecting null hypotheses can be viewed through V/R. Let Q be the unobservable random quotient,

$$Q = \begin{cases} V/R & \text{if } R > 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (11)$$

The FDR is simply $E(Q)$, the expected error rate. The Benjamini-Hochberg Procedure proposed in [1] calls for controlling the FDR at a desired level q, while maximizing $E(R)$. As noted in [1], if all null hypotheses $\{H_1, H_2, \ldots, H_M\}$ are true, the FDR-controlling procedure controls the traditional FWE. But when many hypotheses are rejected, indicating that many hypotheses are not true, an erroneous rejection is not as crucial for drawing conclusion from the whole family of tests. In many applications, it has been argued that the FDR is the more appropriate error rate to control [2]. The difference between FWE- and FDR-controlling procedure is more significant when the size of problems becomes larger. In the proposed detection scheme, *large size problem* means the maximal number of signals, M, is large. Such M's are typical in a wireless multipath propagation environment [12].

Let $\{p_1, p_2, \ldots, p_M\}$ denote the p-values corresponding to the test statistics $\{T_1, T_2, \ldots, T_M\}$ and $p_{(1)} \le p_{(2)} \le \ldots \le p_{(M)}$ denote the ordered p-values corresponding to the hypotheses $\{H_{(1)}, H_{(2)}, \ldots, H_{(M)}\}$ and test statistics $\{T_{(1)}, T_{(2)}, \ldots, T_{(M)}\}$. By definition, $p_m = 1 - P_{H_m}(T_m)$ where P_{H_m} is the distribution function under the null hypothesis H_m. Benjamini and Hochberg showed that when the test statistics *corresponding to the true null hypotheses* are independent, the following procedure controls the FDR at level $q \cdot m_0/M \le q$ [1].

The Benjamini Hochberg Procedure

Define

$$k = \max\left\{m : p_{(m)} \le \frac{m}{M} q\right\} \qquad (12)$$

and reject $H_{(1)} \ldots H_{(k)}$. If no such k exists, reject no hypothesis.

Remark 1 If the test statistics do not ensure dependency or positive dependency, the above procedure is conducted with $q' = q/(\sum_{i=1}^{M} \frac{1}{i})$ instead of q to control the FDR at the same level [2]. Since q' is smaller than q, the modified Benjamini-Hochberg procedure will induce a loss in power. As the independence condition required by the Benjamini-Hochberg procedure is satisfied in the proposed multiple test, we shall use the original version (12) to control the FDR. The resulting test procedure is summarized in Table 1.

Remark 2 In the sequentially rejective Bonferroni-Holm procedure [17], the ordered $p_{(m)}$ is compared with $\alpha/(M-m+1)$ where α is the desired FWE level. Given the same desired FDR and FWE level, i.e. $q = \alpha$, it is easy to verify that $q \cdot (m/M) > \alpha \cdot (1/(M-m+1))$. Thus the FDR-controlling procedure should lead to more powerful results than the FWE-controlling procedure.

Remark 3 In practice, the proposed multiple test can be implemented in a sequential manner as in [21]. The detection procedure starts with $m = 1$. If H_1 is rejected, one signal is declared to be detected and the procedure goes to $m = 2$. Once H_m is retained, the procedure stops and $(m-1)$ signal are declared to be detected. Such implementations assume implicitly that the p-values are in an ascending order, i.e. $p_1 \le p_2 \le \ldots \le p_m$. However, this assumption is not proved yet and does not always happen with finite samples. Consequently, the sequential implementation results in a lower probability of detection.

Input: Fourier transformed data $\{\boldsymbol{X}^k(\omega_j), k = 1, \ldots, K, j = 0, \ldots, J-1\}$
maximal number of signals M , desired FDR level q .

1) for $m = 1, \ldots, M$
   a) Find the ML estimate $\hat{\boldsymbol{\theta}}_m$
   b) Compute the statistics $T_m(\omega_0), \ldots, T_m(\omega_{J-1})$
   c) Apply the bootstrap procedure (Table 2) to find p_m

   end
2) Apply the Benjamini Hochberg procedure (12) to the sorted p-values
   $$p_{(1)} \le p_{(2)} \le \cdots \le p_{(M)}$$
   and reject hypotheses $H_{(1)} \ldots H_{(k)}$.
3) Determine the number of signals by
   $$\hat{m} = \max\{i_1, i_2, \ldots, i_k\}$$
   where $i_1, i_2, \ldots, i_k$ are the original indices of the rejected hypotheses.

Output: the estimated number of signals $\hat{m}$.

Table 1: Multiple Hypothesis Tests for Detection of the Number of S

V. Independence of Test Statistics

In the following, we shall show that the test statistics under null hypotheses H_m , $(m = 1, \ldots, M)$ are independent. This ensures that the FDR of the proposed test (3) is controlled by the Benjamini-Hochberg procedure. The following result from [18], [20] regarding properties of beta distribution plays a key role in our proof.

Result 1. Let $X_1^2, X_2^2, \ldots X_k^2$ be a sample of mutually independent random variates where X_j^2 follows a $\chi^2_{\nu_j}$ distribution with ν_j degrees of freedom $(j = 1, 2, \ldots, k)$. Then

$$\begin{aligned} V_1^2 &= X_1^2/(X_1^2 + X_2^2) \\ V_2^2 &= (X_1^2 + X_2^2)/(X_1^2 + X_2^2 + X_3^2) \\ &\vdots \\ V_{(k-1)}^2 &= (X_1^2 + \ldots + X_{k-1}^2)/(X_1^2 + \ldots + X_k^2) \end{aligned} \tag{13}$$

are mutually independent random variables, each with a beta distribution with parameters (p, q), denoted by $B(p, q)$. The parameters p, q for V_j^2 are $\frac{1}{2}\sum_{i=1}^{j} \nu_i$, $\frac{1}{2}\nu_{j+1}$ respectively.

Theorem 1. The test statistics T_m, $(m = 1, \ldots, M)$ defined by eq. (7) corresponding to the true null hypotheses are mutually independent.

Proof: The test statistic $T_m = \frac{1}{J}\sum_{j=0}^{J-1} T_m(\omega_j)$ consists of J frequency bins with

$$T_m(\omega_j) = -\log Q_m(\omega_j), \qquad (m = 1, \ldots, M) \tag{14}$$

and

$$Q_m(\omega_j) = \left(\frac{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j; \hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)]}{\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_{m-1}(\omega_j; \hat{\boldsymbol{\theta}}_{m-1}))\hat{\boldsymbol{R}}(\omega_j)]} \right). \tag{15}$$

According to the asymptotic properties of Fourier transformed data, $\boldsymbol{X}^k(\omega_j)$, $(j = 0, \ldots, J-1)$ are independent. Therefore, $T_m(\omega_j)$, $(j = 0, \ldots, J-1)$ are independent for $\omega_i \neq \omega_j$. We need only to consider whether at each frequency bin ω_j, the statistics $T_m(\omega_j)$ under H_m are independent. Furthermore, $T_m(\omega_j)$ and $Q_m(\omega_j)$ are related through a monotone function, independence of the $Q_m(\omega_j)$ implies independence of the $T_m(\omega_j)$.

Now we show that under null hypothesis, $Q_m(\omega_j)$ are independent beta distributed random variables. The term $\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j; \hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)\ ]$ appearing in (15) can be decomposed as

$$\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_m(\omega_j; \hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)\ ] = \nu(\omega_j)[\ Y_M^2(\omega_j) + Y_{M-1}^2(\omega_j) + \ldots + Y_m^2(\omega_j)\ ] \tag{16}$$

where

$$\begin{aligned} Y_m^2(\omega_j) &= \frac{1}{\nu(\omega_j)}\mathrm{tr}[(\boldsymbol{P}_{m+1}(\omega_j; \hat{\boldsymbol{\theta}}_{m+1}) - \boldsymbol{P}_m(\omega_j; \hat{\boldsymbol{\theta}}_m))\hat{\boldsymbol{R}}(\omega_j)\ ] \\ &\qquad (m = 1, \ldots, M-1) \\ Y_M^2(\omega_j) &= \frac{1}{\nu(\omega_j)}\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_M(\omega_j; \hat{\boldsymbol{\theta}}_M))\hat{\boldsymbol{R}}(\omega_j)\ ] \end{aligned}$$

are asymptotically independent and $\chi^2_{\nu_m}$ $(\nu_m \in \mathbb{N})$ distributed under null hypotheses. The denominator $\mathrm{tr}[(\boldsymbol{I} - \boldsymbol{P}_{m-1}(\omega_j; \hat{\boldsymbol{\theta}}_{m-1}))\hat{\boldsymbol{R}}(\omega_j)\ ]$ can be decomposed in a similar manner. From (15) and (16), we have

$$Q_m(\omega_j) = \frac{[\ Y_M^2(\omega_j) + Y_{M-1}^2(\omega_j) + \ldots + Y_m^2(\omega_j)\ ]}{[\ Y_M^2(\omega_j) + Y_{M-1}^2(\omega_j) + \ldots + Y_m^2(\omega_j) + Y_{m-1}^2(\omega_j)\ ]}. \tag{17}$$

According to *Result 1*, for the independent random variables $Y_m^2(\omega_j)$, $(m = 1, \ldots, M)$, each with central $\chi^2_{\nu_m}$ distribution, the random variables $Q_1(\omega_j), \ldots, Q_M(\omega_j)$ are mutually independent, each with beta distribution. The independence of $T_m(\omega_j)$ follows immediately. Because of frequency independence, T_m are also independent under H_m. □

Remark Theorem 1 is valid for $J = 1$. Therefore, the FDR-controlling procedure can be applied to the narrowband signals directly. Since the null hypothesis distribution is completely specified by the F_{n_1,n_2} distribution, the p-values can be determined without bootstrap procedure.

VI. The Bootstrap Test

The bootstrap is a powerful technique for estimating confidence interval or testing hypothesis when conventional methods are not valid. It requires little prior knowledge on the data model [13], [15], [29]. More importantly, it provides accurate estimation of probability distribution when only few data samples are available. The key idea behind bootstrap is that, rather than repeating the experiment, one obtains the *samples* by reassignment of the original data samples. We start with general bootstrap procedures and apply them to the proposed detection procedure.

A. General concept

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_J\}$ be an i.i.d. sample set from a completely unspecified distribution $\mathcal{F}$ with ϑ denoting an unknown parameter, such as the mean or variance, of $\mathcal{F}$. The goal of the following procedure is to construct the distribution of an estimator $\hat{\vartheta}$ derived from $\mathcal{Z}$.

The bootstrap principle

1. Given a sample set $\mathcal{Z} = \{z_1, z_2, \ldots, z_J\}$
2. Draw a bootstrap sample $\mathcal{Z}^* = \{z_1^*, z_2^*, \ldots, z_J^*\}$ from $\mathcal{Z}$ by resampling with replacement.
3. Compute the bootstrap estimate $\hat{\vartheta}^*$ from $\mathcal{Z}^*$.
4. Repeat 2. and 3. to obtain B bootstrap estimates
$$\hat{\vartheta}_1^*,\ \hat{\vartheta}_2^*, \ldots,\ \hat{\vartheta}_B^*.$$
5. Approximate the distribution of $\hat{\vartheta}$ by that of $\hat{\vartheta}^*$.

In step 2., a pseudo random number generator is used to draw a random sample of J values, with replacement, from $\mathcal{Z}$. A possible bootstrap sample might look like $\mathcal{Z}^* = \{z_{10}, z_8, z_8, \ldots, z_2\}$. Given the sample set $\mathcal{Z}$, the bootstrap procedure can be easily adapted to calculate a confidence interval of $\hat{\vartheta}$ or construct a hypothesis test.

For the problem testing the hypothesis $H_0 : \vartheta = \vartheta_0$ against $A_0 : \vartheta \neq \vartheta_0$, we define the test statistic as

$$\hat{\mathcal{T}} = \frac{|\hat{\vartheta} - \vartheta_0|}{\hat{\sigma}} \tag{18}$$

where $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ and $\hat{\sigma}^2$ denotes an estimator of the variance of $\hat{\vartheta}$. The inclusion of $\hat{\sigma}$ guarantees $\hat{\mathcal{T}}$ is asymptotically pivotal. Given a significance level α, the bootstrap test computes the threshold t_α based on the bootstrap approximation for the distribution of $\hat{\mathcal{T}}$ under H_0.

In the Benjamini-Hochberg procedure, the observed significance level, denoted by p , rather than the threshold t_α is needed. We use bootstrap samples to estimate the p-value through the following relation [15],

$$\hat{p} = P\left(\frac{|\hat{\vartheta}^* - \hat{\vartheta}|}{\hat{\sigma}^*} \geq \frac{|\hat{\vartheta} - \vartheta_0|}{\hat{\sigma}} \right). \quad (19)$$

where $P(\cdot)$ represents the probability that the bootstrap estimates larger than the normalized test statistic $\hat{\mathcal{T}}$. The square root of variance, σ^*, can be obtained through direct computation or nested bootstrap [29]. In the bootstrap sample $\hat{\mathcal{T}}^* = \frac{|\hat{\vartheta}^* - \hat{\vartheta}|}{\hat{\sigma}^*}$, we use $\hat{\vartheta}$ instead of ϑ_0 to have good power properties. Originally, eq. (19) is used to construct confidence interval. Here we apply it to obtain p-values.

The bootstrap procedure for estimating p-values

1. *Resampling*: Draw a bootstrap sample $\mathcal{Z}^*$.
2. Compute the bootstrap statistic
$$\hat{\mathcal{T}}^* = \frac{|\hat{\vartheta}^* - \hat{\vartheta}|}{\hat{\sigma}^*}.$$
3. Repeat 1. and 2. to obtain B bootstrap statistics.
4. *Ranking*: $\hat{\mathcal{T}}^*_{(1)} \leq \hat{\mathcal{T}}^*_{(2)} \leq \ldots \leq \hat{\mathcal{T}}^*_{(B)}$
5. Choose L so that
$$\hat{\mathcal{T}}^*_{(L-1)} \leq \hat{\mathcal{T}} \leq \hat{\mathcal{T}}^*_{(L)} \ldots \leq \hat{\mathcal{T}}^*_{(B)}.$$
Estimate the observed p-value by $\hat{p} = L/B$.

B. Application to multiple signal detection

To apply the bootstrap principle, we recall that the test statistic T_m in (7) is the sample mean of J samples

$$z_j = \log\left(1 + \frac{n_1}{n_2} F_m(\omega_j; \hat{\boldsymbol{\theta}}_m)\right), \quad (j = 0, \ldots, J-1). \quad (20)$$

Because of asymptotic independence between various frequency bins, the random variables $F_m(\omega_j; \hat{\boldsymbol{\theta}}_m)$ in (20) are asymptotically independent, identically F_{n_1,n_2}-distributed. Therefore, $\{z_0, \ldots, z_{J-1}\}$ are i.i.d. samples from the random variable

$$Z_m = \log\left(1 + \frac{n_1}{n_2} F_m\right), \quad (21)$$

with F_m being F_{n_1,n_2}-distributed. Note that n_1, n_2 are functions of m and must be computed for each H_m by the formula (10).

Furthermore, under H_m , Z_m is beta-distributed $B(p, q)$ with parameters $p = n_2/2$, $q = n_1/2$. The mean and variance of Z_m, denoted by μ_m, σ_m^2, respectively, are determined by n_1, n_2 through the following equation [22],

$$\mu_m = \Psi(\frac{n_1}{2} + \frac{n_2}{2}) - \Psi(\frac{n_2}{2}), \quad (22)$$

$$\sigma_m^2 = \Psi'(\frac{n_2}{2}) - \Psi'(\frac{n_1}{2} + \frac{n_2}{2}) \quad (23)$$

where $\Psi(s) = (\log\Gamma(s))'$ and $\Psi'(s)$ represent the first derivative of logarithm of the gamma function and $\Psi(s)$, respectively[4].

Based on the above observations, we reformulate the hypothesis test (3) as a two-sided test

$$H_m: \quad E[Z_m] = \mu_m$$
$$A_m: \quad E[Z_m] \neq \mu_m \;.$$

Clearly, the test statistic $T_m = \frac{1}{J}\sum_{j=0}^{J-1} z_j$ is a natural estimator for $E[Z_m]$. For each hypothesis H_m, we apply **the bootstrap procedure for estimating p-values** to obtain p_m.

Input:	T_m , $\mathcal{Z} = \{ T_m(\omega_0), T_m(\omega_1), \ldots, T_m(\omega_J$ degrees of freedom n_1, n_2.
Initialization:	Compute μ_m, σ_m^2 by (22), (23). $\hat{\mu}_m = T_m$, $\hat{\mathcal{T}} = \frac{|\hat{\mu}_m - \mu_m|}{\sigma_m}$.
Bootstrap:	1) *Resampling*: Draw a bootstrap sample 2) Compute the bootstrap statistic $\hat{\mathcal{T}}^* = \frac{|\hat{\mu}^*_m - \hat{\mu}_m|}{\hat{\sigma}^*_m}$. 3) Repeat 1) and 2) to obtain B bootstrap 4) *Ranking*: $\hat{\mathcal{T}}^*_{(1)} \leq \hat{\mathcal{T}}^*_{(2)} \leq \ldots \leq \hat{\mathcal{T}}^*_{(B)}$ 5) Choose L so that $\hat{\mathcal{T}}^*_{(L-1)} \leq \hat{\mathcal{T}} \leq \hat{\mathcal{T}}^*_{(L)} \ldots \leq \hat{\mathcal{T}}^*_{(B)}$. Estimate the observed p-value by $\hat{p} = L/$
Output:	$p_m = \hat{p}$

Table 2: The bootstrap procedure for estimating p_m corresponding to $\hat{\sigma}^*_m$ can be obtained by direct computation or nested bootstrap sampling.

VII. Simulations

We demonstrate performance of the proposed algorithm by numerical experiments. A uniform linear array of 15 sensors with inter-element spacings of half a wavelength $\lambda/2$ is used. The wavelength is defined by $\lambda = v/f_0$ where v represents the propagation velocity and f_0 is a pre-selected reference frequency. In the following, we apply the proposed multiple test to narrowband and broadband data generated pseudo-randomly by $m = 3$, 8, and 12 uncorrelated signal sources. The noise is complex normally distributed with zero mean and covariance matrix $\nu(\omega)\boldsymbol{I}$. In additional to the FDR criterion, we apply the sequentially rejective Bonferroni-Holm procedure [17] to control the FWE level. The FDR and FWE

[4] Ψ, Ψ' are also known as polygamma functions.

are controlled at level $q = 0.1$ and $\alpha = 0.1$, respectively. Each experiment performs 100 trials.

A. narrowband signals

1) Comparison with the MDL approach: In the narrowband case, we use the MDL criterion [26] as a benchmark. Similar to the proposed method, the performance of [26] is not affected by fully correlated signals. The MDL criterion derived in [26] differs from other information theoretic approaches [14], [25], [28] in that it exploits the nonzero eigenvalues $\lambda_1 \geq \ldots \geq \lambda_m$ of the $n \times n$ noise covariance matrix $\hat{\boldsymbol{R}}_N(\hat{\boldsymbol{\theta}}_i) = (\boldsymbol{I} - \boldsymbol{P}_i(\omega;\hat{\boldsymbol{\theta}}_i))\hat{\boldsymbol{R}}(\omega)(\boldsymbol{I} - \boldsymbol{P}_i(\omega;\hat{\boldsymbol{\theta}}_i))$ rather than the m smallest eigenvalues of the sample covariance matrix $\hat{\boldsymbol{R}}(\omega)$. This is the key to its robustness against signal coherence. The number of signals is determined by minimizing the following function of m ([26], Eq. (22.b))

$$MDL(m) = K(n-m)\log\left(\frac{\frac{1}{n-m}\sum_{i=1}^{n-m}\lambda_i}{(\Pi_{i=1}^{n-m}\lambda_i)^{\frac{1}{n-m}}}\right) + \frac{1}{2}m(2n-m+1)\log K. \tag{24}$$

The first term is a measure for the spherical equality of the eigenvalues. The second term is a penalty function that avoids overestimation of model order. The test statistic (9) of the proposed multiple testing procedure relies on the estimated increase in SNR while (24) depends on the equality of the estimated noise eigenvalues. Since the ML estimate $\hat{\boldsymbol{\theta}}_i$ is required to compute $\hat{\boldsymbol{R}}_N(\hat{\boldsymbol{\theta}}_i)$, the computational cost associated with the criterion $MDL(m)$ is comparable to that associated with the test statistic (8).

2) Signals of equal strengths: In the first experiment, the narrowband signals are generates by $m = 3$ sources located at $\boldsymbol{\theta} = [-30° \ 20° \ 24°]$ relative to the broadside of the array. All signal sources are of equal strengths. The signal to noise ratio (SNR), defined as $10\log\left(|S_m(\omega)|^2/\nu(\omega)\right)$, varies from -10 to 6 dB in 1 dB step. The number of snapshots $K = 30$. Note that two sources are located closely to each other. The maximum number of signals M is set to be 4.

Fig. 1 shows the empirical probability of correct detection versus SNR. By *correct detection* we mean that the estimated number of signals equals the true number of signals, i.e. $\hat{m} = m$. All three curves go to 100% as SNR increases. The FDR-controlling procedure performs slightly better than the FWE-controlling procedure in the threshold region -10 to -4 dB. From -4 dB on, both methods achieve almost 100% probability of correct detection. At -4 dB, the MDL approach has only 25% probability of correct detection. It requires 4 dB more than the other two procedures to provide 100% probability of correct detection.

In the second experiment, the number of signals is increased to $m = 8$. The maximum number of signals M is set to be 9. All signal sources are well separated except two located at $[\ 20° \ 24°]$. The results are depicted in Fig. 2. Because the number of signals is increased, all three methods require higher SNR to achieve the same performance. Since larger m implies more hypotheses, the difference between the FDR- and FWE-controlling procedures becomes more significant in the region from -10 to -4 dB. At -8 dB, one can observe a difference as large as 10%. Although the gap between the MDL approach and the multiple test based approaches is reduced in the second experiment, its SNR threshold still remains much higher than the other procedures. For 90% of correct detection, both multiple test based methods require -5 dB SNR, but the MDL approach requires -1 dB SNR.

In the third experiment, we increase the number of signals to $m = 12$ which is slightly smaller than the number of sensors $n = 15$. The maximum number of signals M is set to be 13. Two sources remain closely located at $[\ 20° \ 24°]$. From Fig. 3, we observe that this relatively large number of signals leads to a significantly higher SNR threshold. Clearly, the FDR-controlling procedure leads to the best performance. As expected, the difference between the FDR- and FWE-controlling procedure is most significant among these three experiments as the number of hypotheses is the largest. Although the threshold region comes closer to those of the multiple test based procedures, the MDL approach has an overall lower probability of correct detection. In particular, the FDR-controlling procedure has more than 30% higher probability of correct detection at SNR= 14 dB.

3) Signals of various strengths: The simulations discussed previously are carried out with signals of equal strengths. We repeat these experiments with signals of various strengths. For $m = 3$, the SNR difference of the signals is $[-2 \ 1 \ 0]$ dB where 0 dB corresponds to the reference signal. For $m = 8$, three sources differ from the reference signal by -2, -1, 1 dB, respectively. For $m = 12$, four signals from the reference signal by -2, -1, 1, 2 dB, respectively.

Numerical results show that all three algorithms behave similarly to Figs. 1, 2 and 3. Table I summaries samples taken in the threshold regions. Compared to signals of equal strengths, the probability of correct detection is slightly reduced. In all experiments, the FDR-controlling procedure has the highest probability of correct detection. The gain in power of the FDR criterion becomes more significant with increasing m. In the considered scenarios, the MDL approach always needs higher SNRs to achieve the same performance as the multiple testing based methods. The gap between them is largest for $m = 3$ and decreases when m increases. The cause for this phenomena may be that the test statistic (8) is more sensitive to the presence of a new signal when the number of signals is small. In other words, when m is large, a new signal will not cause so much change in SNR as when m is small.

B. Broadband signals

In the broadband case, we choose $J = 10$ frequency bins equally spaced between $(17/32)f_0$ and f_0 for processing. The number of snapshots $K = 10$. Each experiment uses the same number of signals and source locations as in the narrowband case. We consider two scenarios: (1) signals of equal strengths and (2) signals of various strengths.

1) Signals of equal strengths: The first experiment considers $m = 3$. From Fig. 4, we observe that the FDR-controlling procedure performs slightly better than the FWE-controlling procedure. Both procedures achieve 100% of correct detection at -3 dB. This is 1 dB higher than in the narrowband case.

In the second experiment, the number of signals is increased to $m = 8$. The results presented in Fig. 5 show that the FDR-controlling procedure has a significant gain in power compared to the FWE-controlling procedure. In the region from -2 to 2 dB, the difference between two curves is more than 15% . At 0 dB, FDR-controlling procedure has 77% probability of correct detection, while FWE-controlling procedure has only 45% probability of correct detection. For SNR as high as 4 dB, the probability of correct detection of the FWE-controlling procedure is still below that of the FDR-controlling procedure and does not achieve 100% probability of correct detection.

In the third experiment, $m = 12$. Both procedures need higher SNRs to achieve reliable estimation. As shown in Fig. 6, the SNR threshold region covers a wider range than those in Figs. 4 and 5. Compared to experiments with $m = 3$ and $m = 8$, the advantage of the FDR criterion becomes even more remarkable. From SNR= 7 to 12 dB, the FDR-controlling procedure performs significantly better than the FWE-controlling procedure. At SNR= 9 dB, the probability of correct detection of the FWE-controlling procedure is improved by the FDR-controlling procedure from 38% to 78%. To reach 100% probability of correct detection, the FDR-controlling procedure requires 12 dB, while the FDR-controlling procedure requires 17 dB.

2) Signals of various strengths: The above experiments are repeated with signals of various strengths. The relative signal strnegths are the same as in the narrowband case. We obtain similar results as Figs. 4, 5 and 6. Table II includes relevant values in the threshold regions. The FDR-controlling procedure outperforms the FWE-controlling procedure in all experiments. The increase in the power gain of the FDR-controlling procedure becomes more significant with increasing m.

C. Noise only

To test the reliability of the proposed test, we simulate data that contains only noise. The maximum number of signals is chosen to be 4. The number of snapshots K varies from 10 to 40 in a $\Delta K = 5$ step. Since the number of signals is zero, *correct detection* occurs when $\hat{m} = 0$. From Fig. 7 one can observe that both procedures have probability of correct decision higher than 96% for all K's. This implies that although q and α are chosen to be 0.1, the false alarm rate is lower than 4%. Using broadband signals, we can observe similar results.

In summary, the FDR-controlling procedure provides more powerful results than the FWE-controlling procedure in all experiments. When narrowband signals are applied, both multiple test based procedures outperform the MDL approach in the considered settings. The advantage of using the FDR criterion becomes more significant for large numbers of signals and broadband signals. In the broadband case, the FDR-controlling procedure leads to a gain as high as 40% in probability of correct detection. Furthermore, the gain in power does not affect reliability of the proposed test.

VIII. Conclusion

We discussed broadband signal detection using a multiple hypothesis test under an FDR consideration of Benjamini and Hochberg. Compared to the classical FWE criterion, the FDR criterion leads to more powerful tests and controls the errors at a reasonable level. We proved that the independence condition required by the Benjamini-Hochberg procedure is satisfied in the proposed detection scheme. Since the test statistics have no closed form distribution function, we applied the bootstrap technique to determine the p-values numerically. Simulation results show that the FDR-controlling procedure has always a higher probability of correct detection than the FWE-controlling procedure. As expected, the advantage of using the FDR criterion becomes more significant when the number of signals increases. More importantly, the false alarm rate remains low despite a potential gain in power of the FDR-controlling procedure.

Acknowledgements

The authors wish to thank the anonymous reviewers for their comments which helped significantly to improve the manuscript. The work of C.F. Mecklenbräuker was supported by Kplus, Infineon Technologies, and Siemens AG Austria through the project C10.

References

[1] Y. Benjamini and Y. Hochberg. Controlling the false discovery rate: a practical and powerful approach to multiple testing. *J. Roy. Statist. Soc. Ser. B*, (57):289-300, 1995.

[2] Y. Benjamini and D. Yekutieli. The control of the false discovery rate in multiple testing under dependency. *An. Statist.*, (4):1165-1188, 2001.

[3] J. F. Böhme. Array processing. In *Advances in spectrum analysis and array processing*, Editor S. Haykin, Prentice Hall, Englewood Cliffs, N.J. 1991, pages 1-63.

[4] J. F. Böhme. Statistical array signal processing of measured sonar and seismic data. In *Proc. SPIE 2563 Advanced Signal Processing Algorithms*, pp. 2-20, San Diego, USA, Jul 1995.

[5] R. F. Brcich and A. M. Zoubir and P.Perlin. Detection of sources using bootstrap techniques. *IEEE Trans. Signal Processing*, 50(2), pp. 206-215, February 2002.

[6] D. R. Brillinger. Time series analysis: data analysis and theory, expanded edition, Holden-Day, San Francisco, 1981.

[7] P.-J. Chung and A. B. Gershman, Alex B. and J. F. Böhme. Comparative Study of Two-Dimensional Maximum Likelihood and Interpolated Root-MUSIC with Application to Teleseismic Source Localization. *Proc. IEEE Signal Processing Workshop on Statistical Signal and Array Processing.* Pocono Manor, Pennsylvania, USA, August 14–16, 2000.

[8] P.-J. Chung and M. L. Jost, Alex B. and J. F. Böhme. Seismic Wave Parameter Estimation and Signal Detection Using Broadband Maximum Likelihood Methods. *Computer and Geosciences*, 27(2), pp. 147-156, March 2001.

[9] P.-J. Chung and J. F. Böhme and A. O. Hero and C. F. Mecklenbräuker. Signal detection using a multiple hypothesis test. *Proc. Third IEEE sensor array and multichannel signal processing workshop*, Barcelona, Spain, 2004.

[10] P.-J. Chung and J. F. Böhme and C. F. Mecklenbräuker and A. O. Hero. On signal detection using the Benjamini-Hochberg procedure. *Proc. IEEE workshop on statistical signal processing*, Bordeaux, France, 2005.

[11] P.-J. Chung: *ML Estimation under Misspecified Number of Signals*, Proc. Asilomar Conference on Signals, Systems, and Computers, October 30 - November 2, 2005.

[12] P.-J. Chung and N. Czink and C. F. Mecklenbräuker. Model Order Selection for Multipath MIMO Channels Using the Benjamini-Hochberg Procedure, ITG-/IEEE-Workshop on Smart Antennas, March 13-14, 2006, Ulm Germany.

[13] E. Eforn. Bootstrap method. Another look at Jacknife. *The Annals of Statistics*, 7:1-26, 1979.

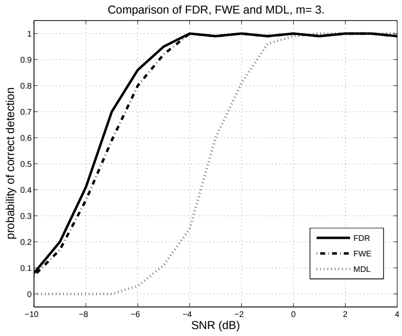

Fig. 1. Empirical probability of correct detection vs. SNR. Number of signals $m = 3$, number of snapshots $K = 30$. Two sources are located closely to each other. All signals are of equal strengths. '–': FDR-controlling procedure, '-.': FWE-controlling procedure, '- -': Minimum description length.

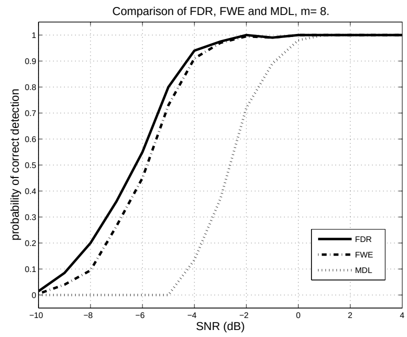

Fig. 2. Empirical probability of correct detection vs. SNR. Number of signals $m = 8$, number of snapshots $K = 30$. Two sources are located closely to each other. All signals are of equal strengths.

[14] E. Fishler and H. Messer. On the use of order statistics for improved detection of signals by the MDL criterion. *IEEE Trans. Signal Processing*, 49(5), pp. 2242-2247, August 2000.

[15] P. Hall. *The bootstrap and Edgeworth expansion*. Springer Verlag, New York, 1992.

[16] Y. Hochberg and A. Tamhane. *M*ultiple Comparison Procedures. Wiley, New York, 1987.

[17] S. Holm. A simple sequentially rejective multiple test procedure. *Scand. J. Statist.*, 6:65-70, 1979.

[18] N. L. Johnson and S. Kotz. *Continuous univariate distributions-2*. John Wiley & Sons, 1970.

[19] H. Wang and M. Kaveh. Coherent signal-subspace processing for detection and estimation of angles of arrival of multiple wide-band sources. *IEEE Trans. Acoustics, Speech, Signal Processing*, 33(8), pp. 823-831, August 1985.

[20] E. Lukacs. A characterization of the gamma distribution. *Ann. Statis.*, 26(2):319-324, Jun 1955.

[21] D. Maiwald, *Breitbandverfahren zur Signalentdeckung und -ortung mit Sensorgruppen in Seismik- und Sonaranwendungen*. Dr.-Ing. Dissertation, Dept. of Electrical Engineering, Ruhr-Universität Bochum, Germany. Shaker Verlag, 1995.

[22] C. F. Mecklenbräuker and P. Gerstoft and J. F. Böhme and P.-J. Chung. Hypothesis testing for geoacoustic environmental model using likelihood ratio. *J. Acoust. Soc. Am.*, 105(3):1738-1748, March 1999.

[23] W. Ng and J. P. Reilly and T. Kirubarajan and J.-R. Larocque. Wideband array signal processing using MCMC methods. *IEEE Trans. Signal Processing*, 53(2), pp. 411-426, February 2005.

[24] R. H. Shumway. Replicated time-series regression: an approach to signal estimation and detection. In *Handbook of Statistics, Vol. 3*, Editor: D. R. Brillinger and P. R. Krishnaiah, Elsevier Science Publishers B.V., pp. 383-408, 1983.

[25] M. Wax and T. Kailath. Detection of signals by information theoretic criteria. *IEEE Trans. Acoust., Speech, Signal Processing*, 33(2), pp. 387-392, April 1985.

[26] M. Wax and I. Ziskind. Detection of the number of coherent signals by the MDL Principle. *IEEE Trans. Acoust., Speech, Signal Processing*, 37(8), pp. 1190-1196, August 1989.

[27] D. B. Williams and D. H. Johnson. Using the sphericity test for source detection with narrow-band passive arrays. *IEEE Trans. Acoust., Speech, Signal Processing*, 38(11), pp. 2008-2014, November 1990.

[28] K. M. Wong and Q.T. Zou and J. P. Reilly and P. C. Yip. On information theoretic criteria for Determining the number of signals in high resolution array processing. *IEEE Trans. Acoust., Speech, Signal Processing*, 38(11), pp. 1959-1971, November 1990.

[29] A. M. Zoubir and D. R. Iskander. *Bootstrap techniques for signal processing.* Cambridge Press, 2004.

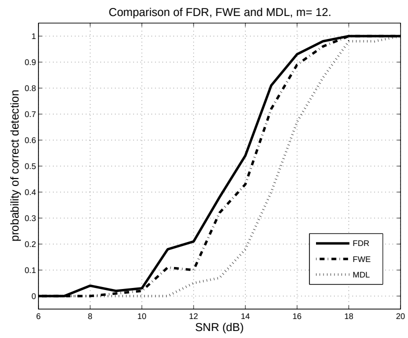

Fig. 3. Empirical probability of correct detection vs. SNR. Number of signals $m = 12$, number of snapshots $K = 30$. Two sources are located closely to each other. All signals are of equal strengths.

		Probability of correct detection					
		-8 dB	-6dB	-4 dB	-2 dB	12 dB	14 d
$m = 3$	FDR	0.49	0.93	0.99	0.98	-	-
	FWE	0.42	0.89	0.97	0.98	-	-
	MDL	0	0	0.33	0.96	-	-
$m = 8$	FDR	0.13	0.59	0.97	1.00	-	-
	FWE	0.10	0.49	0.97	1.00	-	-
	MDL	0	0	0.06	0.83	-	-
$m = 12$	FDR	-	-	-	-	0.12	0.5
	FWE	-	-	-	-	0.09	0.4
	MDL	-	-	-	-	0.03	0.1

TABLE I

EMPIRICAL PROBABILITY OF CORRECT DETECTION VS. SNR. NARROWBAND SIGNALS WITH VARIOUS STRENGTHS.

		Probability of correct detection					
		-6 dB	-2dB	0 dB	6 dB	8 dB	10 d
$m = 3$	FDR	0.61	1.00	1.00	1.00	-	-
	FWE	0.45	1.00	1.00	1.00	-	-
$m = 8$	FDR	0.00	0.24	0.78	1.00	-	-
	FWE	0.00	0.06	0.54	0.98	-	-
$m = 12$	FDR	-	-	-	-	0.08	0.3
	FWE	-	-	-	-	0.00	0.1

TABLE II

EMPIRICAL PROBABILITY OF CORRECT DETECTION VS. SNR. BROADBAND SIGNALS WITH VARIOUS STRENGTHS.

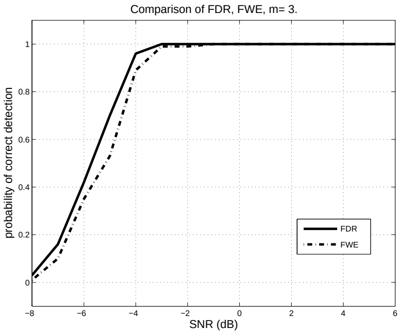

Fig. 4. Empirical probability of correct detection vs. SNR. Number of signals $m = 3$, number of frequency bins $J = 10$, number of snapshots $K = 10$. Two sources are located closely to each other. All signals are of equal strengths. '$-$': FDR-controlling procedure, '$-\cdot$': FWE-controlling procedure.

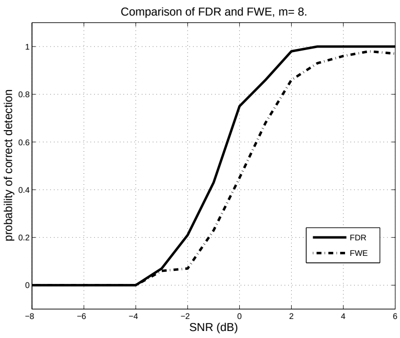

Fig. 5. Empirical probability of correct detection vs. SNR. Number of signals $m = 8$, number of frequency bins $J = 10$, number of snapshots $K = 10$. Two sources are located closely to each other. All signals are of equal strengths.

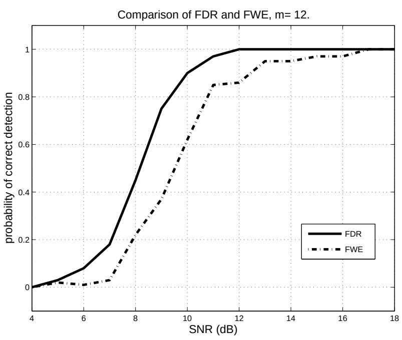

Fig. 6. Empirical probability of correct detection vs. SNR. Number of signals $m = 12$, number of frequency bins $J = 10$, number of snapshots $K = 10$. Two sources are located closely to each other. All signals are of equal strengths.

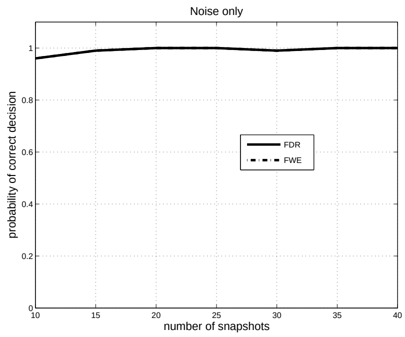

Fig. 7. Empirical probability of correct detection vs. number of snapshots. Array data contains noise only ($m = 0$). Number of frequency bins $J = 1$.