Least Squares Arrival Time Estimators for Photons Detected using a Photomultiplier Tube

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ABSTRACT

In many applications employing photodetectors, the determination of the arrival time of individual photons at the surface of the detector can be used to localize the photon source. For the case where the photon intensity is extremely low, the most common type of detector used is the photomultiplier tube. The optimal arrival time estimators for single and multiple photons arriving at the surface of a photomultiplier tube are developed in this study. The optimal timing estimator considered is a weighted non-linear least squares estimate of the detection time for a high gain PMT with gaussian statistics. The least squares estimator is constructed using the mean and covariance function of the photomultiplier output for different arrival times. The RMS error for the least squares arrival time estimator was calculated and compared with the performance of other common timing estimators, including the first photoelectron timing estimators, using a Burle/RCA 8850 PMT.

I. INTRODUCTION

The determination of the arrival time of an optical signal at the surface of a photodetector is a useful technique for localizing the position of the light source. Simple arrival time estimators such as the leading edge or peak detector give good timing resolution when there is a large number of photons in the optical signal and minimal overlap between successive pulses. When either of these conditions are not met, there are other timing estimators, such as the maximum likelihood (ML), that can be employed to obtain better timing performance. The ML estimator has been developed by Tomitani [1] and Hero et al. [2] for the estimation of arrival times from scintillation events. The approach taken in this paper is to develop a weighted least squares (WLS) arrival time estimator for the detection of gamma ray/scintillator interaction times. An important property of the WLS estimator developed here is that it is identical to the ML arrival time estimator when the output statistics from the photodetector are well approximated by a gaussian random process. Another characteristic of this WLS estimator is that it depends only on the mean and covariance of the photodetector's response to the optical signal.

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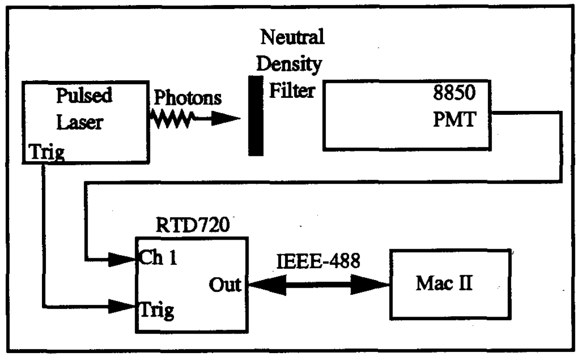

Figure 1: The experimental apparatus for collecting a complete Burle/RCA 8850 PMT response to a single photon.

The mean and covariance are determined numerically from digitized photodetector responses. This paper will concentrate on developing the WLS estimator for weak optical signals containing only a single photon or pair of photons. The optical signals are detected using a Burle/RCA 8850 PMT whose response is digitized using a Tektronix RTD720 real time digitizer. The resulting timing performance for the WLS estimator is then calculated and compared with the timing performance of other common estimators.

II. SINGLE PHOTON WEIGHTED LEAST SQUARES TIMING ESTIMATOR

A. Estimator Structure

Let $\mu(\tau)$ and $\mathbf{K}(\tau)$ denote the mean and covariance respectively, of the vector $\mathbf{X}$ of digitized time samples for a particular gamma ray photon arrival time τ:

$$\mu(\tau) = E\{X(\tau)\} \quad (1)$$
$$\mathbf{K}(\tau) = E\{(X(\tau)-\mu)(X(\tau)-\mu)^T\}. \quad (2)$$

The single photon weighted least squares (WLS) estimator is obtained by minimizing the weighted squared difference

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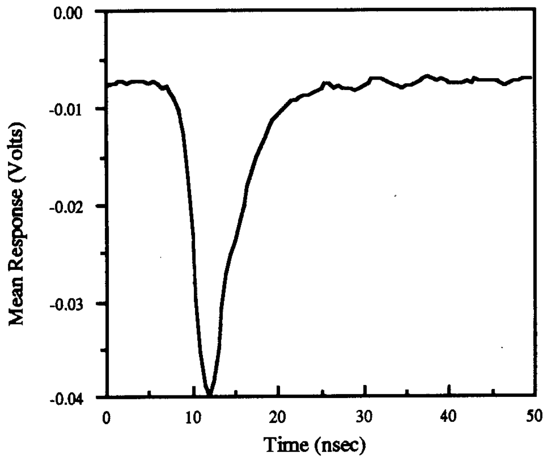

Figure 2: The mean single photon response, $\mu(\tau)$, of a Burle/RCA 8850 PMT.

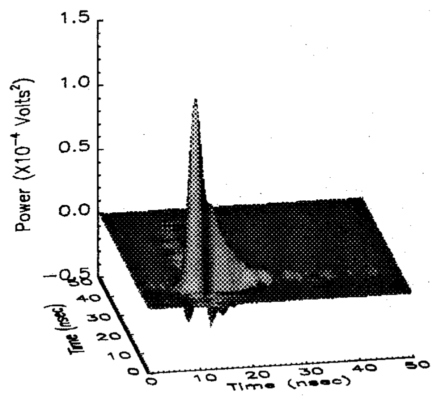

Figure 3: The covariance function, $\mathbf{K}(\tau)$, for a Burle/RCA 8850 PMT.

between $\mathbf{X}$ and its mean $\mu(\tau)$:

$$\tau_{wls}^{s} = \arg\min_{\tau}(\mathbf{X} - \mu(\tau))^T \mathbf{K}^{-1}(\tau)(\mathbf{X} - \mu(\tau)). \quad (3)$$

It is easily shown that τ_{wls}^{s} is identical to the maximum likelihood (ML) estimate of the photon arrival time. In order to apply the arrival time estimator in Equation (3) to single photon data the mean response and covariance function have to be determined for $\mathbf{X}$. Estimates for both of these functions were obtained by sample averaging the digitized response of the phototube using a Tektronix RTD720 real time digitizer. These functions could also be derived from a statistical model of the PMT. The experimental apparatus for digitizing the detector responses is shown in Figure 1. The mean response and covariance function were then calculated based on 100 digitized single photon PMT responses using a 0.5 nsec sampling interval and are shown in Figures 2 and 3 for the case where $\tau = 0$. The mean and covariance, as functions of the arrival time τ, are simply time shifted versions of $\mu(0)$ and $\mathbf{K}(0)$, shifted by the arrival time τ. The WLS estimator given in Equation (3) was implemented using two additional conditions. First, the mean and covariance functions were truncated around the region of interest:

$$\mu(\tau) \rightarrow \mu_T(\tau) \quad (4)$$
$$\mathbf{K}(\tau) \rightarrow \mathbf{K}_T(\tau). \quad (5)$$

This constraint helps increase the computation speed of the estimator. To eliminate outliers, the residual energy given by:

$$E_{norm} = (\mathbf{X} - \mu(\tau))^T(\mathbf{X} - \mu(\tau)) \quad (6)$$

is also required to be small for the estimated value of τ. This restricts the search to only locations where the signal is "close" in Euclidean distance to the mean response. This will again increase the estimator's computation speed and it also reduces the probability of large errors in the arrival time estimates. The WLS arrival time estimator that was implemented on real Burle/RCA 8850 PMT responses is therefore given by the constrained minimization:

$$\tau_{wls}^{s} = \arg\min_{\tau}[(\mathbf{X} - \mu_T(\tau))^T \mathbf{K}_T^{-1}(\tau)(\mathbf{X} - \mu_T(\tau))]. \quad (7)$$

where the residual energy, E_{norm}, is constrained to be small.

B. Calculation of Timing Estimator Performance

The experimental apparatus in Figure 1 was used to digitize the Burle/RCA 8850 PMT's response to a single photon by filtering light from a pulsed laser. The incident light pulse had a 68 psec FWHM. 100 data sets were digitized and stored using a 0.5 nsec sampling interval. The WLS arrival time estimator with a truncation region of 2.5 to 24.5 nsec, see Figure 2, along with a leading edge and peak estimator were applied to the digitized responses using a DEC-5000 workstation. The resulting estimator performances were calculated for these estimators and are shown in Table 1. The standard deviation, $\sqrt{E\{(\hat{\tau} - \mu(\hat{\tau}))^2\}}$, and the RMS error, $\sqrt{E\{(\hat{\tau} - \tau)^2\}}$, were used to evaluate the performances. The RMS error for the leading edge and peak estimator are not given in the table because these estimators are severely biased. The results clearly show that even if such bias can be removed from the other estimators, the WLS estimator has better time resolution.

III. DOUBLE PHOTON WEIGHTED LEAST SQUARES TIMING ESTIMATOR

A. Estimator Structure

A WLS arrival time estimator is developed for the detection of an optical signal generated by gamma ray photons arriving at nearly the same time. The goal is to estimate

Estimator Type	Actual Arrival Time	Mean	Standard Deviation	RMS Error
Leading Edge	0.0 nsec	10.0 nsec	0.85 nsec	—
Peak Detector	0.0 nsec	11.9 nsec	0.86 nsec	—
WLS 2.5 - 24.5 nsec	2.5 nsec	2.59 nsec	0.62 nsec	0.63 nsec

Table 1: The performance of the leading edge, peak and WLS arrival time estimators for single photon detection using a Burle/RCA 8850 PMT.

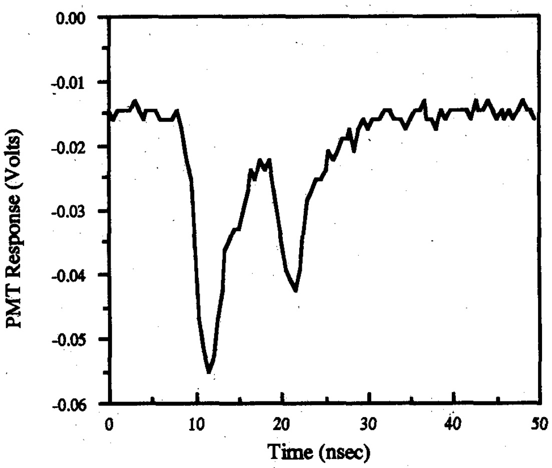

Figure 4: A typical Burle/RCA 8850 PMT response to a pair of photons having 10% overlap.

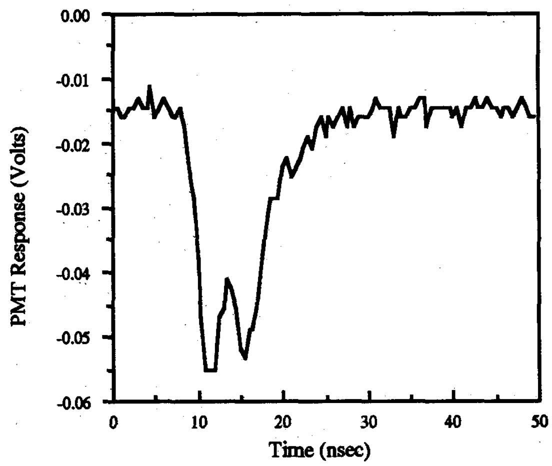

Figure 5: A typical Burle/RCA 8850 PMT response to a pair of photons having 37% overlap.

the arrival time of each photon, τ_1 and τ_2. The WLS estimator derivation is similar to the single photon case except that the minimization is now over both arrival times. The double photon WLS arrival time estimator is given by:

$$\begin{aligned}\tau^d_{wls} &= \arg\min_{\tau_1,\tau_2}[(\mathbf{X} - \mu^d_T(\tau_1,\tau_2))^T \mathbf{K}_T^{-1}(\tau_1,\tau_2) \cdot \\ &\quad (\mathbf{X} - \mu^d_T(\tau_1,\tau_2))] \qquad (8)\end{aligned}$$

where $\mu^d_T(\tau_1,\tau_2)$ and $\mathbf{K}_T(\tau_1,\tau_2)$ are the truncated mean and covariance of the vector $\mathbf{X}$. Again the estimator (8) can be
shown to be equivalent to the ML estimates of (τ_1,τ_2) if $\mathbf{X}$ is gaussian. To implement this estimator in principle, the double photon mean and covariance must be found a priori by simulation or experiment. A simple approximation can be used however, to simplify the implementation:

$$\mu_T(\tau_1,\tau_2) \approx \mu_T(\tau_1) + \mu_T(\tau_2) \qquad \text{(A-1)}$$

$$\mathbf{K}_T^{-1}(\tau_1,\tau_2) \approx (\mathbf{K}_T(\tau_1) + \mathbf{K}_T(\tau_2))^{-1}. \qquad \text{(A-2)}$$

These are good approximations when the overlap of the single photon responses at times τ_1 and τ_2 is not overly severe. Using these assumptions an approximate double photon WLS estimator is defined as:

$$\begin{aligned}\tau^d_{wls} &= \arg\min_{\tau_1,\tau_2}[(\mathbf{X} - \mu_T(\tau_1) - \mu_T(\tau_2))^T \cdot \\ &\quad (\mathbf{K}_T(\tau_1) + \mathbf{K}_T(\tau_2))^{-1} \cdot \\ &\quad (\mathbf{X} - \mu_T(\tau_1) - \mu_T(\tau_2))] \qquad (9)\end{aligned}$$

where again, we constrain the residual energy, $(\mathbf{X} - \mu(\tau_1) - \mu(\tau_2))^T(\mathbf{X} - \mu(\tau_1) - \mu(\tau_2))$, to be small.

B. Calculation of Timing Estimator Performance

The double photon data was created by adding together a pair of shifted single photon PMT responses. The shift between the responses controls the amount of overlap in the pulse. The WLS arrival time estimator with a truncation region of 2.5 to 27 nsec was applied to double photon responses having pulse overlaps of 10% and 37%. A typical double photon response for 10% pulse overlap is shown in Figure 4 while Figure 5 contains a typical response for 37% overlap. The RMS timing error, given in Table 2, was calculated for this data using the DEC-5000 and is

Estimator Type	Percentage Overlap	Actual Arrival Time	Means	Standard Deviation	RMS Error
WLS		2.5 nsec	2.73 nsec	0.85 nsec	
2.5 - 27 nsec	10%	12.5 nsec	12.43 nsec	0.91 nsec	1.27 nsec
WLS		2.5 nsec	2.81 nsec	0.91 nsec	
2.5 - 27 nsec	37%	7.5 nsec	7.36 nsec	1.09 nsec	1.46 nsec

Table 2: The performance of the WLS arrival time estimator for a pair of photons detected by a Burle/RCA 8850 PMT.

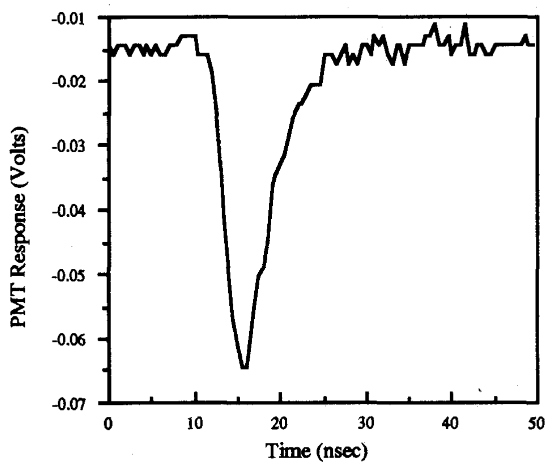

Figure 6: A double photon pulse with 37% overlap forming only a single peak.

based on 100 realizations. The timing performance in the double photon case degrades considerably from single photon arrival time estimation but still maintains advantages over the other estimators. The problem with the leading edge and peak detectors is due to the confounding effect of the overlapping single photon responses. When there is significant pulse overlap, the leading edge threshold is not capable of detecting both peaks. The peak detector also has problems with the double photon data when the pulse overlap is such that only a single peak is formed. This case can be seen in Figure 6 where overlap of 37% only forms a single photopeak.

IV. Conclusion

The WLS arrival time estimator has been shown to out perform the leading edge and peak timing estimators. This estimator's major limitation is implementation speed. The estimator is quite slow when compared to leading edge estimators because it must invert a covariance matrix for each candidate arrival time. A possible sub-optimal estimator based on the WLS approach may be to replace the covariance matrix with a less complicated matrix. This would simplify the inversion making the estimator faster but perhaps at the expense of timing resolution. The WLS estimator was also found to be effective in separating overlapping pulses. This property becomes quite important when pulse pileup is significant.

References

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