Scaling Laws and Phase Transitions for Target Detection in MIMO Radar

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***Abstract*—The performance of MIMO radar has been a subject of intense study in the past decades. For such a system, however, the important phenomenon of phase transition has received little attention in the literature. In this paper, we study the phase transition on the target detection probability of a SNR maximizing detector. Such a detector declares a target to be present when the largest eigenvalue of the observed data matrix exceeds a threshold. In particular, we identify a critical value below and above which the limiting detection performance is described by the Tracy-Widom law and the Gaussian law, respectively. Under both laws, the scaling limits and asymptotic expansions of misdetection probability at the vanishing regime are derived using tools from random matrix theory.**

I. Introduction

The SNR-maximizing target detection algorithm is widely used in MIMO radar [1, 2]. For single target detection [3], the SNR maximizing detector is equivalent to the Generalized Likelihood Ratio Test (GLRT) under a Rayleigh fading model [4]. In this single target scenario, the GLRT reduces to a detector that compares the largest eigenvalue of the MIMO radar data matrix to a specified threshold. We call this GLRT detector the largest eigenvalue based detector [5]. In massive MIMO radar, the number of radar antennas (sensors) is very large giving rise to a high dimensional data matrix. In this paper, we show how methods of random matrix theory can be used to specify fundamental limits in the detection performance of the considered GLRT target detector.

Analytical performance of the largest eigenvalue based detector has received substantial attention in different communities. Despite the relatively long history, the behavior of such a detector is still far from being completely understood. In particular, the phase transition phenomenon on the performance of the largest eigenvalue based detector seems to have received little attention. This performance transition is induced from the phase transition of the largest eigenvalue of the spiked model well-known in random matrix theory [6]. The phase transition implies that the performance is governed by more than one distributional laws when the number of sensors and samples approach infinity. In fact, it will be shown that the detection probability is described by two distinct limiting laws depending on the strength of the spikes. In MIMO radar detection, the value of the spike depends on the channel condition and transmit power of the target. When the spike increases the detection probability transitions from the Tracy-Widom distribution to the Gaussian distribution. As a consequence of the phase transition, two different scaling behaviors have been found in the crucial regime when the detection probability approaches one. In this regime, the asymptotic expansions of detection probability under both limiting laws are derived by means of random matrix theory. The phase transition also implies that the target is undetectable if the spike is below the critical value even if the number of sensors and samples is arbitrary large.

The studied phase transition phenomenon and the developed analytical framework in this paper can be applied to other branches of engineering and applied sciences. Due to the duality of MIMO radar and MIMO communications[1], our results are directly applicable to cognitive radio networks [4, 5]. Our results are also applicable to social networks. For example, in community detection a similar phase transition phenomenon has been observed [7], where the characteristics of different communities can be modeled by the spiked model. In inverse covariance estimation [8], which has applications to inference in Gaussian graphical models, the block-sparse covariance matrix assumption is equivalent to the spiked model. The detection probability studied in this paper may be applied to establish detectability of a weak connected component in the spiked Gaussian graphical model.

II. Problem Formulation

A. System Model

Consider the standard model for m-sensor cooperative detection[2] in the presence of a single target

$$\mathbf{x} = \mathbf{h}s + \mathbf{w}, \tag{1}$$

[1]Note that the duality is mainly on the level of systems models. The detailed operational assumptions of the two types of MIMO systems may differ in practice.

[2]For spatially distributed sensors and co-located sensors, the corresponding MIMO radar systems are often referred to as statistical MIMO radars and waveform diverse MIMO radars, respectively. The analysis in this paper is valid for both scenarios.

where the m dimensional complex vector $\mathbf{x} \in \mathbb{C}^m$ is the received data vector. The $m \times 1$ vector $\mathbf{w}$ is the complex Gaussian noise with zero mean and covariance matrix $\sigma^2 \mathbf{I}_m$, where σ^2 denotes the noise power. The $m \times 1$ vector $\mathbf{h} = (h_1, \ldots, h_m)$ represents the channels between the target and the m sensors. The scalar s denotes the transmitted signal of the target, which follows a zero mean Gaussian distribution and is uncorrelated with the noise. The channel vector $\mathbf{h}$ is assumed to be constant during the detection, i.e. we consider deterministic channels. We collect n i.i.d. observations from model (1) to a $m \times n$ $(m \leq n)$ received data matrix $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$. By the above assumptions, the sample covariance matrix $\mathbf{R} = \mathbf{X}\mathbf{X}^\dagger / n$ follows a complex Wishart distribution of dimension m with n degrees of freedom and a population covariance matrix $\mathbf{\Sigma}$. We denote the ordered eigenvalues of $\mathbf{R}$ by $0 \leq \lambda_m \leq \ldots \leq \lambda_1 < \infty$.

B. Detection Problem

We consider a binary hypothesis test

$$\mathcal{H}_0 \quad : \quad \mathbf{\Sigma} = \sigma^2 \mathbf{I}_m \tag{2}$$

$$\mathcal{H}_1 \quad : \quad \mathbf{\Sigma} = \sigma^2 \mathbf{I}_m + \gamma \mathbf{h}\mathbf{h}^\dagger, \tag{3}$$

where hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ denote the absence and presence of the target[3]. Here, $\gamma = \mathbb{E}[ss^\dagger]$ is the transmit power[4] of the target and the received SNR is given by

$$\text{SNR} = \gamma ||\mathbf{h}||^2 / \sigma^2. \tag{4}$$

Under this hypothesis test, we further assume that the noise power σ^2 is known, which without loss of generality is set at $\sigma^2 = 1$. In this case, the largest eigenvalue based detector

$$T_{\text{LE}} = \lambda_1 \tag{5}$$

is optimal under the generalized likelihood ratio criterion [4]. Comparing T_{LE} with a predetermined threshold z,

$$T_{\text{LE}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} z \tag{6}$$

the presence or absence of the target is decided.

The ordered eigenvalues of the population covariance matrix (3) under the hypothesis $\mathcal{H}_1$ are given by

$$\sigma_i = \begin{cases} 1 + \gamma ||\mathbf{h}||^2, & i = 1, \\ 1, & i = 2, \ldots, m, \end{cases} \tag{7}$$

where $\sigma_i, i = 1, \ldots, m$ are also referred to as spikes in random matrix theory. Since the entries of the received data matrix $\mathbf{X}$ are i.i.d. Gaussian, we can assume without loss of generality that $\mathbf{\Sigma}$ under $\mathcal{H}_1$ is given by

$$\mathbf{\Sigma} = \begin{pmatrix} 1 + \gamma ||\mathbf{h}||^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m-1} \end{pmatrix}. \tag{8}$$

[3]For hypothesis $\mathcal{H}_1$, we assume to have infinite time-bandwidth product, i.e., the ambiguity function is a delta function.

[4]We assume perfect time-frequency synchronization at radar receiver unit.

We are interested in the asymptotic regime when the numbers of sensors and samples approach infinity with their ratio being fixed, i.e.,

$$m, n \to \infty, \ c^2 = \frac{n}{m} \in [1, \infty). \tag{9}$$

The focus of this paper is to study the detection probability

$$P_{\text{d}}(z) = 1 - \mathbb{P}(\lambda_1 \leq z) \tag{10}$$

in the asymptotic regime (9), which relies on the limiting distribution of the largest eigenvalue λ_1. In particular, from a detection theoretical viewpoint, it is important to understand the behavior when the misdetection probability vanishes, i.e.,

$$1 - P_{\text{d}}(z) = P_{\text{m}}(z) \to 0. \tag{11}$$

As will be shown, depending on the strength of the spike σ_1, the asymptotic detection performance behaves differently, which is induced from the phase transition of the largest eigenvalue.

Note that the covariance matrix $\mathbf{\Sigma}$ under $\mathcal{H}_1$ can be also mapped to presence of a strong target return. There, the magnitude of the spike will be dependent on target reflectivity, fading in the medium, and target location.

III. Phase Transition of the Largest Eigenvalue

The covariance matrix $\mathbf{\Sigma}$ of the form (8) is also known as the spiked model in literature. The essential feature is that as the dimension m approaches infinity, the number of eigenvalues of $\mathbf{\Sigma}$ that are greater than one is finite[5]. In the asymptotic regime (9), it is known that there exists a critical value such that the limiting laws of λ_1 are different for spikes below and above the critical value.

Specifically, for the considered model (8) the critical value was found in [6] as

$$\sigma_{\text{crit}} = 1 + \frac{1}{c}. \tag{12}$$

In the case $\sigma_1 < \sigma_{\text{crit}}$ the limiting law of the largest eigenvalue is governed by the Tracy-Widom distribution [6],

$$\mathbb{P}\left(\frac{\lambda_1 - a_{\text{TW}}}{b_{\text{TW}}} \leq z\right) \xrightarrow{(9)} F_{\text{TW}}(z), \tag{13}$$

where the respective center and scale sequences are given by

$$a_{\text{TW}} = \left(1 + \frac{1}{c}\right)^2, \tag{14}$$

$$b_{\text{TW}} = \frac{(1+c)^{4/3}}{c} n^{-2/3}. \tag{15}$$

On the other hand, when $\sigma_1 > \sigma_{\text{crit}}$ the limiting law of the largest eigenvalue is governed by the standard Gaussian distribution [6, 9],

$$\mathbb{P}\left(\frac{\lambda_1 - a_{\text{G}}}{b_{\text{G}}} \leq z\right) \xrightarrow{(9)} F_{\text{G}}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\frac{1}{2}t^2} \, \mathrm{d}t, \tag{16}$$

[5]In our case, only σ_1 is greater than one.

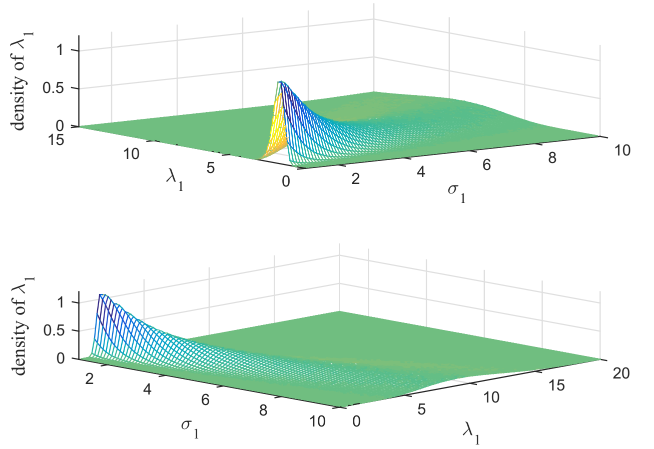

Fig. 1. Phase transition of the largest eigenvalue: $n = 10$, $m = 4$, and $\sigma_{\text{crit}} = 1.6325$.

where the respective center and scale sequences are given by

$$a_{\mathrm{G}} = \sigma_1 + \frac{\sigma_1 c^{-2}}{\sigma_1 - 1}, \tag{17}$$

$$b_{\mathrm{G}} = \left(\sigma_1^2 - \frac{\sigma_1^2 c^{-2}}{(\sigma_1 - 1)^2}\right)^{1/2} n^{-1/2}. \tag{18}$$

The above transition is often referred to as Baik-Ben Arous-Péché phase transition [6]. To illustrate this phase transition phenomenon, we plot in Fig. 1 the density of λ_1 as a function of the largest spike $\sigma_1 = 1 + \gamma||\mathbf{h}||^2$. We consider a scenario of $m = 4$ sensors with $n = 10$ samples per sensor. As a result, the critical value (12) equals $\sigma_{\text{crit}} = 1.6325$. We observe that as σ_1 increases the shape of the density changes significantly, cf. [10, Fig. 2]. In particular, the mean and fluctuation of λ_1 increase notably as predicted by (14), (15), (17), and (18).

Under hypothesis $\mathcal{H}_0$, the corresponding largest eigenvalue also converges to the Tracy-Widom distribution [11] with the same center (14) and scale (15) sequences. Thus, the result (13) implies that if the spike is below the critical value, the largest eigenvalue behaves asymptotically as if σ_1 equals one. In this case, the hypothesis $\mathcal{H}_1$ in (3) degenerates to hypothesis $\mathcal{H}_0$ in (2), which corresponds to the undetectable scenario. This can also be seen from the fact that the sequences (14) and (15) do not depend on σ_1. To see the impact of this behavior on the vanishing rate of misdetection probability, we first reformulate the results (13) and (16) by the definitions (10) and (11) as

$$P_{\mathrm{m}}(z) = \mathbb{P}(\lambda_1 \leq z) \xrightarrow{(9)} \begin{cases} F_{\mathrm{TW}}\left(\frac{z - a_{\mathrm{TW}}}{b_{\mathrm{TW}}}\right), & \sigma_1 < \sigma_{\text{crit}}, \\ F_{\mathrm{G}}\left(\frac{z - a_{\mathrm{G}}}{b_{\mathrm{G}}}\right), & \sigma_1 > \sigma_{\text{crit}}. \end{cases} \tag{19}$$

Since both the Tracy-Widom and Gaussian distributions are supported on $(-\infty, \infty)$, the rate $P_{\mathrm{m}}(z) \to 0$ is determined by the rate $-a_{\mathrm{TW}}/b_{\mathrm{TW}} \to -\infty$ in the Tracy-Widom regime and the rate $-a_{\mathrm{G}}/b_{\mathrm{G}} \to -\infty$ in the Gaussian regime. These rates are obtained respectively as

$$-\frac{a_{\mathrm{TW}}}{b_{\mathrm{TW}}} = -\frac{c\left(1 + \frac{1}{c}\right)^2}{(1 + c)^{4/3}} n^{2/3} \sim -n^{2/3}, \tag{20}$$

$$-\frac{a_{\mathrm{G}}}{b_{\mathrm{G}}} = -\frac{\left(\sigma_1 + \frac{\sigma_1 c^{-2}}{\sigma_1 - 1}\right)}{\left(\sigma_1^2 - \frac{\sigma_1^2 c^{-2}}{(\sigma_1 - 1)^2}\right)^{1/2}} n^{1/2} \sim -n^{1/2}. \tag{21}$$

Thus, the rate that the misdetection probability tends to zero scales with the number of samples[6] as $n^{2/3}$ and $n^{1/2}$ when the spike is below and above the critical value, respectively.

IV. Asymptotics of Misdetection Probability at Vanishing Regime

A detailed analysis beyond the scaling limits requires the left tail behavior of the Tracy-Widom and Gaussian distribution functions. In this section, we derive explicit asymptotic expansions of misdetection probability in the vanishing regime by using recent advances in random matrix theory.

A. Tracy-Widom Regime

To define the Tracy-Widom distribution we need the following notations. Let A be the operator on $L^2(z, \infty)$ with the kernel given by the Airy kernel [10, 11],

$$A(x, y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}, \tag{22}$$

where $\mathrm{Ai}(x)$ denotes the Airy function. The corresponding Fredholm determinant [11] of the operator A in terms of classical matrix determinants is written as

$$\det(I - A)|_{L^2(z,\infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(z,\infty)^k} \det\left(A(x_i, x_j)\right)_{i=1,j=1}^{k} \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

The Tracy-Widom distribution (13) can be then defined as

$$F_{\mathrm{TW}}(z) = \det(I - A)|_{L^2(z,\infty)}. \tag{23}$$

The asymptotics of the Airy-kernel Fredholm determinant as $z \to -\infty$ was derived in [12, 13] as

$$\det(I - A)|_{L^2(z,\infty)} = \tau \frac{\mathrm{e}^{-\frac{1}{12}|z|^3}}{|z|^{1/8}} \left(1 + \frac{3}{2^6 |z|^3} + \mathcal{O}\left(|z|^{-6}\right)\right), \tag{24}$$

where the constant

$$\tau = 2^{1/24} \mathrm{e}^{\zeta'(-1)} \approx 0.87237 \tag{25}$$

and $\zeta(x)$ denotes the Riemann zeta function. The constant is also known as Dyson's constant, which was first conjectured to be the form (25) in [10]. This was rigorously proven in [12, 13]. Finally, by the relation (19), the misdetection probability at the vanishing regime $P_{\mathrm{m}}(z) \to 0$ in the case $\sigma_1 < \sigma_{\text{crit}}$ is approximated by

$$P_{\mathrm{m}}(z) \approx \tau b_{\mathrm{TW}}^{1/8} \frac{\mathrm{e}^{-\frac{|z - a_{\mathrm{TW}}|^3}{12 b_{\mathrm{TW}}^3}}}{|z - a_{\mathrm{TW}}|^{1/8}} \left(1 + \frac{3 b_{\mathrm{TW}}^3}{2^6 |z - a_{\mathrm{TW}}|^3}\right). \tag{26}$$

[6]The scaling with number of sensors would be the same since we assumed that $c = \sqrt{n/m}$ is a constant.

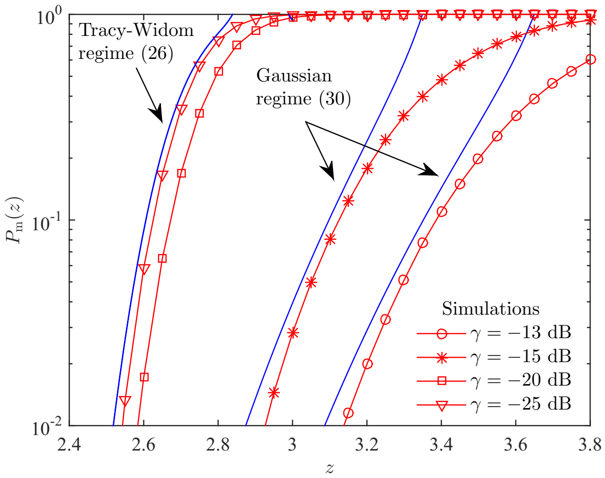

Fig. 2. Asymptotic expansions of misdetection probability (26) and (30).

Note that by incorporating higher order terms of the expansion (24) that are computed in a recursive manner, we could obtain refined estimates of $P_{\mathrm{m}}(z)$.

B. Gaussian Regime

The asymptotic expansion of the Gaussian tail is rather straightforward to derive. By the relation

$$F_{\mathrm{G}}(z) = \frac{1}{2}\left(1 - \mathrm{erf}\left(-\frac{z}{\sqrt{2}}\right)\right), \tag{27}$$

where $\mathrm{erf}(x)$ is Gauss error function and the fact that

$$\mathrm{erf}(x) = 1 - \frac{1}{\sqrt{\pi}x}\mathrm{e}^{-x^2}\left(1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right), \tag{28}$$

we have as $z \to -\infty$,

$$F_{\mathrm{G}}(z) = -\frac{1}{\sqrt{2\pi}z}\mathrm{e}^{-\frac{z^2}{2}}\left(1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right). \tag{29}$$

Consequently, for the case $\sigma_1 > \sigma_{\mathrm{crit}}$ the outage probability at the vanishing regime is approximated by

$$P_{\mathrm{m}}(z) \approx -\frac{b_{\mathrm{G}}}{\sqrt{2\pi}\,(z - a_{\mathrm{G}})}\mathrm{e}^{-\frac{(z-a_{\mathrm{G}})^2}{2b_{\mathrm{G}}^2}}. \tag{30}$$

C. Numerical Results

In Fig 2, we plot the asymptotic expansions of misdetection probability (26), (30) versus simulations. A single target MIMO radar system is considered, where we assume $m = 50$ sensors and each sensor collects $n = 100$ samples. In this case, the critical value equals $\sigma_{\mathrm{crit}} = 1.707$. The deterministic channel $\mathbf{h}$ follows a vector-valued complex Gaussian distribution with zero mean and covariance matrix $\mathbf{I}_m$. We consider the set of the signal powers -13 dB, -15 dB, -20 dB, and -25 dB, where the largest spikes equal 3.505, 2.581, 1.5, and 1.158, respectively. As a result, the signal powers -13 dB and -15 dB correspond to the Gaussian case (30) and the signal powers -20 dB and -25 dB correspond to the Tracy-Widom case (26). As can be observed, in the Tracy-Widom regime the misdetection probability can not be effectively reduced by increasing the transmit power. This observation is in line with the discussion in Section III. It is also seen that the derived asymptotic expansions capture the behavior of the misdetection probability in the regime of practical interest.

V. Conclusion and Future Work

In this paper, we studied the phase transition (detectability) phenomenon on the performance of GLRT for detecting a single target in a MIMO radar, equivalent to the SNR maximizing detector and also to the largest eigenvalue detector. It was shown that the detection performance is described by two distinct distributional laws: the Tracy-Widom distribution and the Gaussian distribution. In both regimes, we derived the scaling limits and asymptotic expansions of the misdetection probability. This analysis was made possible by recent progress in random matrix theory.

Future work include applying the proposed framework and analytical results to other related problems. In particular, we would like to generalize the results in [7] from a single to multiple observations, where the spiked model becomes relevant. We also plan to examine the phase transition of the false alarm probability in inverse covariance estimation [8].

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References

[1] J. Li and P. Stoica (Eds.), *MIMO radar signal processing.* Wiley-IEEE Press, 2008.

[2] E. Fishler, A. Haimovich, R. S. Blum, L. J. Cimini, D. Chizhik, and R. A. Valenzuela, "Spatial diversity in radars – models and detection performance," *IEEE Trans. Signal Process.,* vol. 54, no. 3, pp. 823-838, Mar. 2006.

[3] O. Besson, S. Kraut, and L. L. Scharf, "Detection of an unknown rank-one component in white noise," *IEEE Trans. Signal Process.,* vol. 54, no. 7, pp. 2835-2839, July 2006.

[4] A. Taherpour, M. N. Kenari, and S. Gazor, "Multiple antenna spectrum sensing in cognitive radios," *IEEE Trans. Wireless Commun.,* vol. 9, no. 2, pp. 814-823, Feb. 2010.

[5] L. Wei and O. Tirkkonen, "Cooperative spectrum sensing of OFDM signals using largest eigenvalue distributions," *in Proc. IEEE Int. Symp. Pers., Indoor Mobile Radio Commun.,* 2009

[6] J. Baik, G. Ben Arous, and S. Péché, "Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices," *Ann. Probab.,* vol. 33, no. 5, 2005.

[7] P.-Y. Chen and A. O. Hero, "Phase transitions in spectral community detection," *IEEE Trans. Signal Process.,* vol. 63, no. 16, pp. 4339-4347, Aug. 2015.

[8] A. O. Hero and B. Rajaratnam, "Hub discovery in partial correlation graphs," *IEEE Trans. Inf. Theory,* vol. 58, no. 9, pp. 6064-6078, Sept. 2012.

[9] Z. Bai and J.-F. Yao, "Central limit theorems for eigenvalues in a spiked population model," *Ann. Inst. Henri Poincaré Probab. Stat.,* vol. 44, no. 3, pp. 447-474, 2008.

[10] C. Tracy and H. Widom, "Level-spacing distributions and the Airy kernel," *Commun. Math. Phys.,* vol. 159, 1994.

[11] K. Johansson, "Discrete orthogonal polynomial ensembles and the Plancherel measure," *Ann. Math.,* vol. 153, no. 2, pp. 259-296, 2001.

[12] P. Deift, A. Its, and I. Krasovsky, "Asymptotics of the Airy-kernel determinant," *Commun. Math. Phys.,* vol. 278, Jan. 2008.

[13] J. Baik, R. Buckingham, and J. DiFranco, "Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II Function," *Commun. Math. Phys.,* vol. 280, June 2008.