A New Generalized Cross Correlator

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Abstract—A new generalized cross correlator (GCC) for the passive time delay estimation problem is presented. The interpretation of this GCC is that of estimating the cross-correlation function by cross correlating the least mean-square estimates of the signal component in each of the observed waveforms. The implementation is simply a GCC with the weighting filter equal to the magnitude coherency squared. Numerical evaluation of the performance of this processor and a robust version indicate that they compare favorably to some of the well-known GCC procedures.

I. Introduction

THE estimation of propagation delay in a common signal arriving at two spatially separated sensors is a problem which has received much attention in the literature. Cross-correlation methods, among which the standard cross correlator (CC) is the most basic, are particularly popular because of the richness and variety of processors within this class and the general ease of implementation. Of these, the Hannan–Thomson (HT) [1], [5], the Eckart (EK) [1], and the Hassab–Boucher (HB) [2] are examples of "optimum" processors which maximize some performance criteria. On the other hand, the SCOT [5] is an example of an "ad hoc" or intuitive correlation-type processor. In general, the optimum processors are very sensitive to deviations from the assumed signal and noise characteristics. By way of contrast, the CC and SCOT appear to be more robust to these deviations from the nominal model. However, these latter processors can have very poor performance at the nominal point.

In the following development, a different approach to the problem yields another type of (generalized) cross correlator. This is the Wiener Processor (WP), which has a simple form. Yet, preliminary results indicate that it outperforms most of the other above-named processors when compared under various performance criteria for the few important cases considered in this paper. A "robust" version of the WP also indicates good performance relative to the others under spectral uncertainty.

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II. Problem Statement and Background

We first consider a system model generating the observations in Fig. 1. We observe Gaussian, ergodic, wide-sense stationary processes $x_1(t)$ and $x_2(t)$ over a time interval $[0, T]$ which contain uncorrelated broad-band noises $n_1(t)$ and $n_2(t)$ and signals $s(t)$ and $s_0(t)$, respectively. We assume that $c(t)$ is a linear time-invariant channel having a transfer function $C(\omega)$ with unknown linear phase so that $s_0(t)$ is a delayed but possibly distorted version of $s(t)$. Furthermore, we assume that the noises are uncorrelated with the signal and that T is much greater than the correlation time T_c of $x_1(t)$ and $x_2(t)$. The object is then to estimate the time delay D associated with the channel.

We define the sample cross correlation

$$\hat{R}_{12}(\tau) = \frac{1}{T} \int_0^T x_1(\sigma) x_2(\sigma + \tau)\, d\sigma \tag{1}$$

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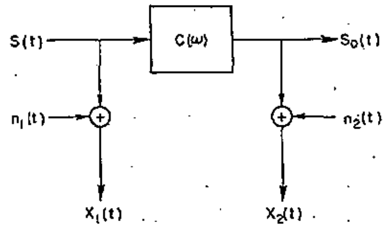

Fig. 1. Model governing the observations $x_1(t)$ and $x_2(t)$. $C(w)$ is the transfer function of a linear phase intersensor channel. The random signal $s(t)$ is independent of the noises $n_1(t)$ and $n_2(t)$.

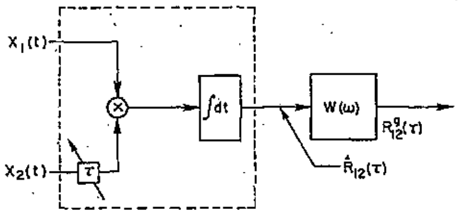

Fig. 2. Generalized cross correlator implemented as a filter on the sample cross-correlation function $\hat{R}_{12}$.

or for $T \gg T_c$ in the frequency domain,

$$\hat{R}_{12}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}_{12}(\omega)\, e^{j\omega\tau}\, d\omega. \tag{2}$$

Here $\hat{G}_{12} = (1/T)\, X_1^* X_2$ where X_1 and X_2 are the finite time Fourier transforms of x_1 and x_2. For large observation time, $\hat{R}_{12}(\tau)$ is a good approximation to the true cross-correlation function which has a global peak at D. In fact, if $c(t)$ is pure delay and $s(t)$ is white, the cross-correlation function is a delta function at the true delay. For finite observation time, we can decompose $\hat{R}_{12}(\tau)$ into the sum of four terms:

$$\hat{R}_{12}(\tau) = c(\tau) * \hat{R}_{ss}(\tau) + c(\tau) * \hat{R}_{n_1 s}(\tau) + \hat{R}_{sn_2}(\tau) + \hat{R}_{n_1 n_2}(\tau) \tag{3}$$

where "$*$" denotes convolution. Here $\hat{R}_{ss}(\tau)$ is an estimate of the signal autocorrelation function $R_{ss}(\tau)$ and $\hat{R}_{n_1 s}(\tau)$, $\hat{R}_{sn_2}(\tau)$, and $\hat{R}_{n_1 n_2}(\tau)$ are estimates of the cross correlation between the respective signal and noise terms in the observations. In the limit, the sample cross correlation converges to $c(\tau) * R_{ss}(\tau)$ which displays an absolute maximum at D. Thus, it is the last three terms in (3) which constitute zero-mean disturbances affecting peak resolution of the first term. This suggests prefiltering the sample cross correlation with a filter $W(\omega)$ to obtain better resolution of the peak at D, where $W(\omega)$ has zero phase. This scheme is referred to as the generalized cross-correlation method or the generalized cross correlator (GCC) and is illustrated in Fig. 2. We denote the GCC output waveform $R_{12}^g(\tau)$. Therefore, we have

$$R_{12}^g(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}_{12}(\omega)\, W(\omega)\, e^{j\omega\tau}\, d\omega. \tag{4}$$

When $W(\omega)$ is unity, the resulting GCC is called the simple cross correlator (CC). Considering the first term in (3) as a "signal" in additive noise, classical optimal filtering theory can be applied to derive filters $W(\omega)$ which maximize signal-to-noise ratio.

Letting the last three terms of (3) be characterized as "noise," we can define a signal-to-noise ratio at the output of the GCC as the magnitude squared of the global peak of the "signal" term divided by the power of the "noise" which generates false peaks in $R_{12}^g(\tau)$. We will denote this SNR_1. For a sufficiently broad-band signal $s(t)$, the variance of the cross-correlation estimate (1) outside of the immediate vicinity of the true delay is given by

$$\text{var}\,(\hat{R}_{12}(\tau)) = \frac{1}{T2\pi} \int_{-\infty}^{\infty} G_{11}(\omega)\, G_{22}(\omega)\, d\omega \tag{5}$$

and the variance of $\hat{R}_{ss}$ is given by

$$\text{var}\,(\hat{R}_{ss}(\tau)) = \frac{1}{T2\pi} \int_{-\infty}^{\infty} |G_{12}(\omega)|^2\, d\omega \tag{6}$$

for $\tau \neq D$ [12]. $G_{11}(\omega)$ and $G_{22}(\omega)$ are the power spectral densities of the observations $x_1(t)$ and $x_2(t)$, respectively, and $G_{12}(\omega)$ is the cross spectrum. Using the above results, it is straightforward to derive the cross-correlation noise power which is given by

$$\sigma_n^2(\tau) = \frac{1}{2\pi T} \int_{-\infty}^{\infty} G_{11}(\omega)\, G_{22}(\omega)\,(1 - |\gamma_{12}(\omega)|^2) \cdot |W(\omega)|^2\, d\omega, \qquad \tau \neq D. \tag{7}$$

$|\gamma_{12}(\omega)|^2$ is the magnitude coherency squared

$$|\gamma_{12}(\omega)|^2 = \frac{|G_{12}(\omega)|^2}{G_{11}(\omega)\, G_{22}(\omega)}. \tag{8}$$

Then from the defining relation

$$\text{SNR}_1 = \frac{[E\{R_{12}^g(\tau)|_{\tau=D}\}]^2}{\sigma_n^2}, \tag{9}$$

we obtain

$$\text{SNR}_1 = \frac{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |G_{12}(\omega)|\, W(\omega)\, d\omega\right]^2}{\frac{1}{2\pi T} \int_{-\infty}^{\infty} G_{11}(\omega)\, G_{22}(\omega)(1 - |\gamma_{12}(\omega)|^2)\, |W(\omega)|^2\, d\omega}. \tag{10}$$

The maximum is obtained through the Schwarz inequality and yields the HT processor for the pure delay channel. The same result is derived in [4] as the result of minimizing the local variance of the delay estimate over the entire GCC class, and in [1] as the result of maximum likelihood estimation. The filter is

$$W_{\text{HT}}(\omega) = \frac{1}{|G_{12}(\omega)|} \frac{|\gamma_{12}(\omega)|^2}{1 - |\gamma_{12}(\omega)|^2}. \tag{11}$$

Neglecting the effect of the signal and noise cross terms, $c(\tau) * \hat{R}_{sn_1}(\tau)$ and $\hat{R}_{sn_2}(\tau)$ in (3) gives another characterization of the noise in the cross-correlation domain. With this definition of noise, another signal-to-noise ratio is defined in

[6], SNR_2, which is shown for pure delay to be maximized by the Eckart Processor $W_{EK}(\omega)$

$$SNR_2 = \frac{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{12}(\omega)|\, W(\omega)\, d\omega\right]^2}{\frac{1}{2\pi T}\int_{-\infty}^{\infty} G_{n_1}(\omega)\, G_{n_2}(\omega)\, |W(\omega)|^2\, d\omega} \tag{12}$$

$$W_{EK}(\omega) = \frac{|G_{12}(\omega)|}{G_{n_1}(\omega)\, G_{n_2}(\omega)} \tag{13}$$

where $G_{n_1}(\omega)$ and $G_{n_2}(\omega)$ are the autospectra of the noises $n_1(t)$ and $n_2(t)$, respectively. Note that in terms of the spectra of the observables $x_1(t)$ and $x_2(t)$, the filter takes the form

$$W_{EK}(\omega) = \frac{|G_{12}(\omega)|}{(G_{11}(\omega) - |G_{12}(\omega)|)\,(G_{22}(\omega) - |G_{12}(\omega)|)}.$$

Hassab and Boucher [2] take the approach of maximizing a singal-to-noise ratio SNR_3 defined as the ratio of the expected peak energy at the true delay to the total statistical variation of the output of the GCC. This, in a sense, lumps the "signal" $c(\tau) * \hat{R}_{ss}(\tau)$ variation into the noise terms and yields the HB filter $W_{HB}(\omega)$

$$SNR_3 = \frac{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_{12}(\omega)|\, W(\omega)\, d\omega\right]^2}{\frac{1}{2\pi T}\int_{-\infty}^{\infty} G_{11}(\omega)\, G_{22}(\omega)\, |W(\omega)|^2\, d\omega} \tag{14}$$

$$W_{HB}(\omega) = \frac{|G_{12}(\omega)|}{G_{11}(\omega) G_{22}(\omega)}. \tag{15}$$

The HB is similar to the SCOT introduced by Carter *et al.* [5] in that, for highly dynamic spectra, in addition to suppressing the cross-spectral estimate in ω-regions of low signal-to-noise ratio, high signal-to-noise ratio regions are also suppressed in an attempt to reject strong tonals in the observations.

Note that the above performance criteria impose equal penalty on small and large errors. That is, the location of the false peak in the GCC output exerts no influence on the signal-to-noise ratios defined in (9), (12), and (14). Therefore, one can only rely on these criteria if the signal-to-noise ratio is sufficiently high to guarantee a low probability of large error. The behavior of this probability as a function of signal-to-noise ratio, observation time, and signal bandwidth is investigated elsewhere [7], [13].

III. The Wiener Processor

Here a different approach is taken to derive an optimal filter. We deal directly with the quantities in the observation time domain (i.e., Fig. 1). The procedure is motivated by the following argument. If we knew the signal $s(t)$ and the filtered version $s_0(t)$ exactly, then from the linearity of the phase of the channel, the time delay could be estimated exactly by detecting the peak of the sample cross correlation of $s(t)$ and $s_0(t)$. Therefore, we simply try to estimate the signal $s(t)$ as best we can from the observations $x_1(t)$ and the channel out-

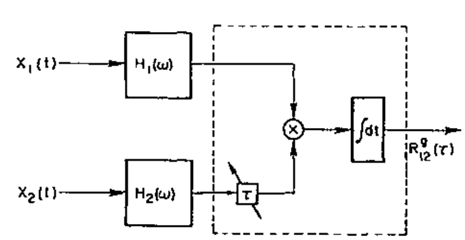

Fig. 3. Implementation of the generalized cross correlator as a preprocessor on the sensor waveforms. H_1 is the complex conjugate of H_2.

put signal $s_0(t)$ from $x_2(t)$ by minimizing the mean-square errors:

$$E\{(s(t) - \hat{s}(t))^2\} = \min \tag{16}$$

$$E\{(s_0(t) - \hat{s}_0(t))^2\} = \min \tag{17}$$

where

$$\hat{s}(t) = \int_{-T}^{T} x_1(\sigma)\, h_1(t - \sigma)\, d\sigma \tag{18}$$

$$\hat{s}_0(t) = \int_{-T}^{T} x_2(\sigma)\, h_2(t - \sigma)\, d\sigma. \tag{19}$$

The above procedure is illustrated in Fig. 3. Given the channel characteristic $C(\omega)$, the solutions to (16) and (17) are the Wiener filters $H_1(\omega)$ and $H_2(\omega)$:

$$H_1(\omega) = \frac{G_{ss}(\omega)}{G_{ss}(\omega) + G_{n_1}(\omega)} \tag{20}$$

$$H_2(\omega) = \frac{G_{ss}(\omega)\, |C(\omega)|^2}{G_{ss}(\omega)\, |C(\omega)|^2 + G_{n_2}(\omega)} \tag{21}$$

Noting that $G_{12}(\omega) = C(\omega)\, G_{ss}(\omega)$, we can express the above filters in terms of the quantities derived from the observables

$$H_1(\omega) = \frac{1}{C(\omega)}\, \frac{G_{12}(\omega)}{G_{11}(\omega)} \tag{22}$$

$$H_2(\omega) = C^*(\omega)\, \frac{G_{12}(\omega)}{G_{22}(\omega)} \tag{23}$$

where $C^*(\omega)$ is the complex conjugate of $C(\omega)$.

With these filters, the sample cross correlation of the least mean-square error estimates of $s(t)$ and $s_0(t)$ yields the estimate of the cross-correlation function:

$$R_{12}^{WP}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{T}\, \hat{S}^*(\omega)\, \hat{S}_0(\omega)\, e^{j\omega\tau}\, d\omega \tag{24}$$

where

$$\hat{S}(\omega) = H_1(\omega)\, X_1(\omega) \tag{25}$$

$$\hat{S}_0(\omega) = H_2(\omega)\, X_2(\omega). \tag{26}$$

Regrouping terms in (24), we obtain

$$R_{12}^{WP}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}_{12}(\omega)\, \frac{|G_{12}(\omega)|^2}{G_{11}(\omega)\, G_{22}(\omega)}\, e^{j\omega\tau}\, d\omega. \tag{27}$$

Comparing (27) to (4), we have the result that the WP is equivalent to using a generalized cross correlator with the filter $W(\omega)$ equal to the magnitude coherency squared.

It should be emphasized that even though the Wiener filters H_1 and H_2 involve the knowledge of the channel $C(\omega)$ itself, the GCC equivalent processor does not impose this requirement. In fact, as far as the cross-correlation estimate of time delay is concerned, the actual channel is immaterial to the peak detection procedure in the cross-correlation domain. Hence, the Wiener filter implementation (Fig. 3) with $C(\omega)$ arbitrarily set to unity in (22) and (23) is equivalent to any other choice of $C(\omega)$ for the time delay estimation problem.

From (5), the variance of the cross-spectrum estimate $\hat{G}_{12}(\omega)$ is proportional to the product of the autospectra of the observations $G_{11}(\omega)\ G_{22}(\omega)$. Fix the sample autocorrelation $\hat{R}_{ss}(\tau)$ in (3). Then the definition of "additive noise" leading to the signal-to-noise ratio SNR_1, (7), yields the interpretation of $1 - |\gamma_{12}(\omega)|^2$ as a measure of the cross-spectral estimator variance about the "desired signal" $c(\tau) * \hat{R}_{ss}(\tau)$. Thus, the WP deemphasizes those ω regions where the sample cross spectrum is likely to be a highly inaccurate estimate of the true cross spectrum. This is not surprising given the raison d'être of the WP which is to accurately estimate the smoothed sample autocorrelation $c(\tau) * \hat{R}_{ss}(\tau)$.

The WP does not, of course, maximize the signal-to-noise ratio in general. If we examine the optimal processor for SNR_1, the HT (11), we see that it has the additional ability to overemphasize as well as to deemphasize the cross-spectral estimate according to the function $|\gamma_{12}(\omega)|^2/(1 - |\gamma_{12}(\omega)|^2)$. (Actually, in [4], the above function is shown to be inversely proportional to the variance of the phase estimate $\hat{G}_{12}(\omega)/|\hat{G}_{12}(\omega)|$ with respect to the true phase of the cross spectrum.) However, in situations where the coherence is low and the signal spectrum is nearly flat, the HT and the WP are virtually identical and exhibit identical performance [(11) becomes proportional to $|\gamma_{12}|^2$].

It is also observed that the WP is equivalent to the HB for nearly flat signal spectra, and also to the Eckart if we add a low signal-to-noise ratio condition

$$W_{\mathrm{HB}}(\omega) = \frac{1}{|G_{12}(\omega)|}\,|\gamma_{12}(\omega)|^2 = \frac{1}{|G_{12}(\omega)|}\,W_{\mathrm{WP}}(\omega) \tag{28}$$

$$W_{\mathrm{EK}}(\omega) = \frac{G_{ss}(\omega)}{G_{n_1}(\omega)\,G_{n_2}(\omega)} \approx \frac{1}{|G_{12}(\omega)|}\,|\gamma_{12}(\omega)|^2. \tag{29}$$

The above signal-to-noise ratio condition is that $G_{ss}(\omega)$ be uniformly small as compared to $G_{n_1}(\omega)$ and $G_{n_2}(\omega)$.

IV. The Robust Wiener Processor for Unknown Spectra

The optimal GCC's all require knowledge of the signal and noise spectra underlying the observations. When the spectral quantities used in the filter function for the GCC do not match the true spectra, there is a consequent deterioration in performance. Two approaches to the problem of unknown spectra are of interest. We either estimate the spectra and substitute the estimates into the aforementioned filters (totally unknown spectra) or we search for a robust solution over a range of spectra perturbed from some nominal point (partially unknown spectra).

For the estimation approach, the sensitivity of the GCC filter to small deviations in the estimated spectra may be an important consideration. As applied to the HT, the substitution method yields a procedure which weights the phase of the sample cross spectrum $\hat{G}_{12}(\omega)$ with the function $\hat{W}_{\mathrm{HT}}(\omega) = |\hat{\gamma}_{12}(\omega)|^2/(1 - |\hat{\gamma}_{12}(\omega)|^2)$, $|\hat{\gamma}_{12}(\omega)|^2$, a magnitude coherency squared estimate. A simple local analysis of the estimation error associated with $\hat{W}_{\mathrm{HT}}$ yields the variance

$$\mathrm{var}\,(\hat{W}_{\mathrm{HT}}(\omega)) \approx \frac{1}{(1 - |\gamma_{12}(\omega)|^2)^4}\,\mathrm{var}\,(|\hat{\gamma}_{12}(\omega)|^2). \tag{30}$$

$\hat{W}_{\mathrm{HT}}$ may critically underweight the phase estimate over frequencies where $|\gamma_{12}|^2$ is high, that is, where the phase estimate is apt to be the most accurate. On the other hand, substitution of $|\hat{\gamma}_{12}|^2$ for the WP gives only as much error as the estimation error of $|\hat{\gamma}_{12}|^2$ itself. An improved filter estimate could translate into improved performance of the time delay estimate. Naturally, these comments must be verified through a future simulation study.

In practice, the spectra may be only partially unknown, and a different strategy can be used to design the GCC. This is the "robust" approach which has been applied to classical matched and Wiener filtering with some success [8]-[10]. The resultant filters are robust in the maximim sense, e.g., the filter maximizes the minimum output signal-to-noise ratio as the spectra are allowed to vary over their regions of uncertainty. A more precise formulation is as follows.

It is assumed that the uncertainty corresponding to our partial knowledge of signal and noise spectra G_{ss}, G_{n_1}, and G_{n_2} is such that these quantities can be described as belonging to the uncertainty classes of spectra σ, η_1, and η_2, respectively. Frequently, the above classes represent neighborhoods centered around some nominal spectra G_{ss}^0, $G_{n_1}^0$, and $G_{n_2}^0$ where the size of the classes is chosen to reflect our overall confidence in the nominals as underlying the observations. Two common classes for the signal spectrum are the $\epsilon d - d$ mixture for $\epsilon > 0$ [9],

$$\sigma = \left\{G_{ss}\colon G_{ss} = (1 - \epsilon)\,G_{ss}^0 + \epsilon G_{ss}' \quad \text{and} \quad \int_{-\infty}^{\infty} G_{ss}(\omega)\,d\omega = 2\pi v_s^2\right\} \tag{31}$$

and the total-variation class for $\delta > 0$ [9],

$$\sigma = \left\{G_{ss}\colon \int_{-\infty}^{\infty} |G_{ss}(\omega) - G_{ss}^0(\omega)|\,d\omega \leq \delta \quad \text{and} \quad \int_{-\infty}^{\infty} G_{ss}(\omega)\,d\omega = 2\pi v_s^2\right\} \tag{32}$$

where, in (31), G_{ss}' is an arbitrary spectrum and v_s^2 is a known signal power. Analogous expressions can hold for the noise spectra.

Let W be a function in the class of filter functions $\boldsymbol{W}$. Now define ρ, a GCC performance criterion, e.g., signal-to-noise ratio, as a function of the three spectral quantities and the filter

$$\rho \equiv \rho(G_{ss}, G_{n_1}, G_{n_2}, W). \tag{33}$$

It is desired to find a $W(\omega)$ that maintains good performance as the spectra vary over the uncertainty classes. The idea is to design an optimum filter under adversity, i.e., one that assures us of a minimum level of "good" performance regardless of the underlying spectra. The *robust processor* for ρ is defined as a W, say W^R, that arises from the solution of

$$\max_{W} \min_{\sigma,\eta_1,\eta_2} \rho(G_{ss}, G_{n_1}, G_{n_2}, W). \tag{34}$$

Related to robust processing is the concept of least favorable spectra discussed by Lehmann [14]. These are points in the uncertainty classes where the worst possible performance occurs subject to optimal filtering strategy. Specifically, let W^l be a filter which maximizes ρ for the set of spectra $(G_{ss}^l, G_{n_1}^l, G_{n_2}^l) \in \sigma \times \eta_1 \times \eta_2$. Then $(G_{ss}^l, G_{n_1}^l, G_{n_2}^l)$ is a *least favorable set of spectra* for ρ if, for any other set $(G_{ss}, G_{n_1}, G_{n_2}) \in \sigma \times \eta_1 \times \eta_2$,

$$\rho(G_{ss}, G_{n_1}, G_{n_2}, W^l) \geq \rho(G_{ss}^l, G_{n_1}^l, G_{n_2}^l, W^l). \tag{35}$$

Note that if (35) holds, we have a saddlepoint solution of (34) and we can take $W^R = W^l$.

Unfortunately, no results are known to us concerning the solution of the above robust time delay estimation problem, i.e., (34). Short of this, the only known published result in maximin filters for time delay is that of Kassam and Hussaini [11] for the pure delay case. In [11], they used the fact that the Eckart processor maximizes a classically defined signal-to-noise ratio [see (12)] to relate the filtering problem to robust hypothesis testing. This is achievable only by associating uncertainty classes with the spectral product $G_{n_1}(\omega)\, G_{n_2}(\omega)$ rather than with the individual noise spectra themselves.

An alternate approach to performance degradation under spectral uncertainty is suggested by recent work in robust Wiener filtering [9], [10] when applied to the WP. Assume that the uncertainty classes σ, η_1, and η_2 govern the individual spectra as above, and that $|C(w)| = 1$. Here, one simple constructs two robust Wiener filters for the signal s, each acting on x_1 and x_2, respectively. This procedure can then be related to a GCC in the same manner as the Wiener filters were related to the WP in Section III.

Consider first the waveform $x_1(t)$ in Fig. 1. Define the mean-square error of an estimate of $s(t)$ obtained by linearly filtering $x_1(t)$ with $H_1 \in \boldsymbol{H}$, a class of filter functions, by

$$e \equiv e(G_{ss}, G_{n_1}, H_1). \tag{36}$$

Following Poor [9], $H_1 = H_1^R$ is called a *robust Wiener filter* if it is the solution of

$$\min_{H} \max_{\sigma,\eta_1} e(G_{ss}, G_{n_1}, H_1). \tag{37}$$

Let H_1^l be the Wiener filter for $(G_{ss}^l, G_{n_1}^l)$, i.e., H_1^l minimizes the mean-square error for these spectra. The pair $(G_{ss}^l, G_{n_1}^l) \in \sigma \times \eta_1$ is called *least favorable for Wiener filtering* if, for any other spectra $(G_{ss}, G_{n_1}) \in \sigma \times \eta_1$,

$$e(G_{ss}, G_{n_1}, H_1^l) \leq e(G_{ss}^l, G_{n_1}^l, H_1^l). \tag{38}$$

If the uncertainty classes given by (31) and (32) govern the signal and noise spectra, it is shown in [9] that least favorable spectra exist and that H_1^l is the robust Wiener filter

$$H_1^R = \frac{G_{ss}^l}{G_{ss}^l + G_{n_1}^l}. \tag{39}$$

The same discussion carries over to the waveform $x_2(t)$ with the analogous robust Wiener filter

$$H_2^R = \frac{G_{ss}^l}{G_{ss}^l + G_{n_2}^l}. \tag{40}$$

With regard to the original formulation of the WP, we can use the robust filters H_1^R and H_2^R in place of H_1 and H_2 [see (22) and (23)]. Finally, we implement these filters in the cross-correlation domain as a GCC, a scheme which we will call the *Robust Wiener Processor*, or the RWP. In the next section, we give a detailed example of least favorable spectra and the resulting robust filters. For the example considered, there is a 3 dB processing gain at low signal-to-noise ratio using the RWP.

V. Numerical Comparisons

At the present time, no simulation results concerning the experimental performance of the WP and RWP as opposed to the other GCC's are available. In their absence, a preliminary investigation of the relative merits of the above processors was performed based on the various signal-to-noise ratio criteria defined in Section II for some specific observation spectra and for the pure delay channel.

Example 1

Figs. 6-10 show the relative performance of the HT, HB, Eckart, SCOT, and CC under the criteria SNR_1, SNR_2, SNR_3, and local variance var_L of the time delay estimate [3] for a third-order Markov signal in first-order Markov noises with the noise 3 dB bandwidth a factor of ten greater than that of the signal (see Figs. 4 and 5). Specifically, the signal spectrum has the form

$$G_{ss}(\omega) = S\,\frac{3}{2\pi B_s}\,\frac{1}{1 + \left(\dfrac{\omega}{2\pi B_s}\right)^6} \tag{41}$$

with $B_s = 10$ Hz, and the noise spectra $G_{n_1}(\omega)$ and $G_{n_2}(\omega)$ are of identical form:

$$G_n(\omega) = N\,\frac{1}{2\pi B_n}\,\frac{1}{1 + \left(\dfrac{\omega}{2\pi B_n}\right)^2} \tag{42}$$

with $B_n = 100$ Hz. Here S and N are the input signal and noise powers, respectively. These spectra were chosen for their tail behavior to avoid certain degeneracies in the local variance criterion. The interesting thing to note is that under SNR_1 and SNR_2, the WP exhibits better performance than all of the other suboptimum GCC's for that particular definition of SNR. Under SNR_3, it is a close second next to the MLE. In fact, under the criterion SNR_1, performance of the WP is virtually

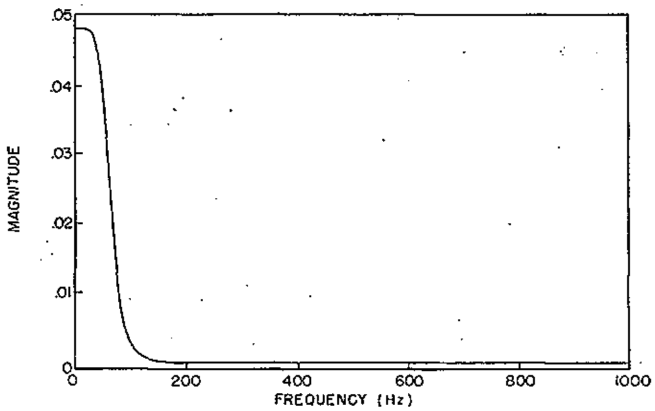

Fig. 4. Third-order Markov signal spectrum for Example 1.

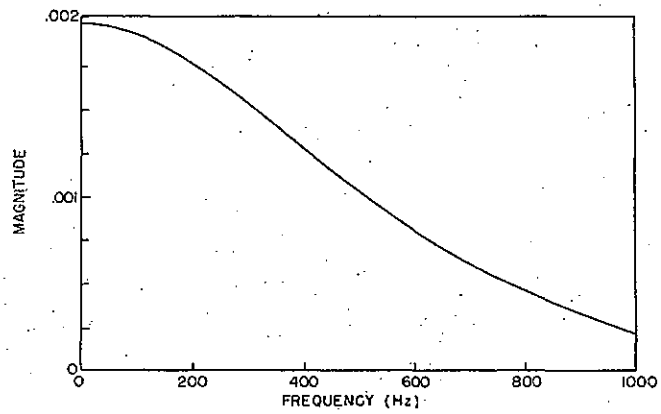

Fig. 5. First-order Markov noise spectra for Example 1.

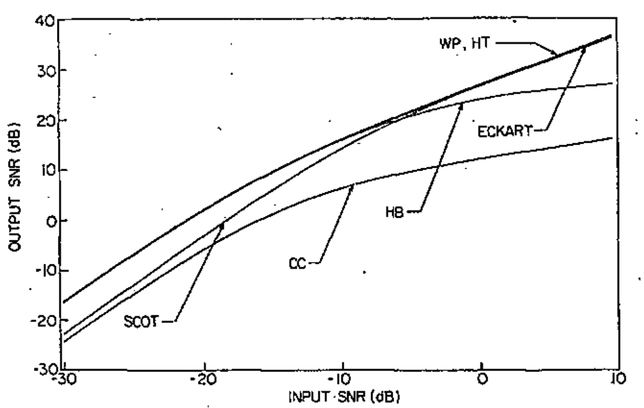

Fig. 6. Relative performance of various processors under criterion SNR_1.

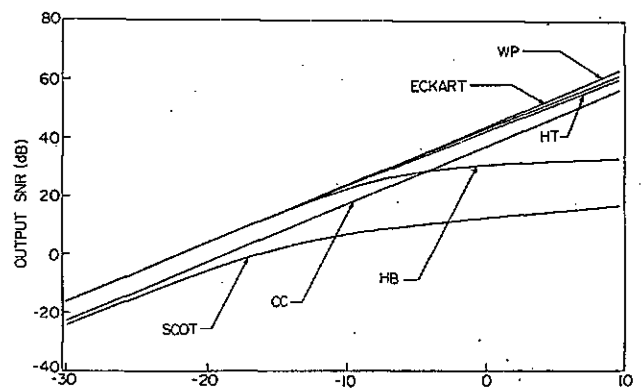

Fig. 7. Relative performance of various processors under criterion SNR_2.

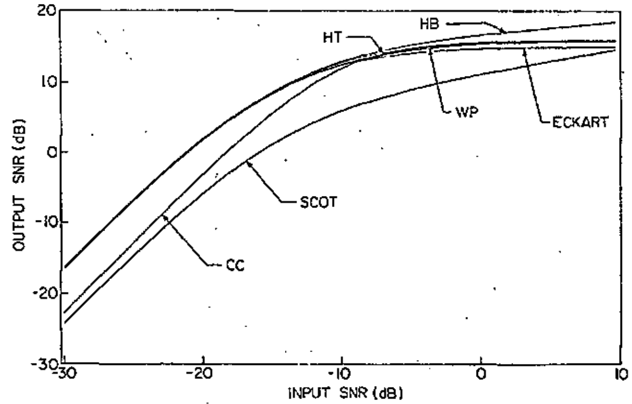

Fig. 8. Relative performance of various processors under criterion SNR_3.

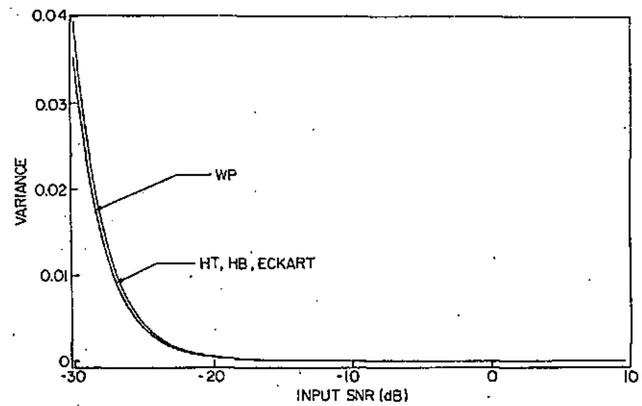

Fig. 9. Local variance for various processors.

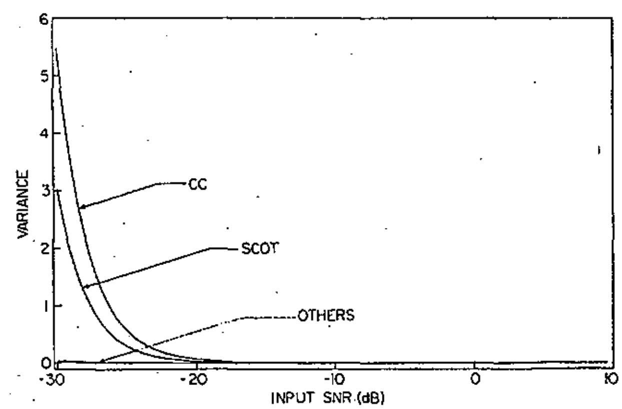

Fig. 10. Local variance for various processors (magnification of Fig. 9).

identical to the optimal HT processor. Although the local variance ranks the WP behind the HT, HB, and Eckart (see Figs. 9 and 10), it only marginally disfavors the WP at low signal-to-noise ratios. (It is to be noted from Fig. 9 that the SCOT and the CC have local variance orders of magnitude worse than the WP and are off scale. They are shown in Fig. 10.)

Example 2

Here the performance of the RWP, WP, and other GCC's are compared using SNR_1 for the ϵ-contaminated uncertainty class on the specific spectra in the example outlined in Kassam and Lim's paper on robust Wiener filtering [9]. Specifically, under the nominal assumption, at each sensor we have a signal with

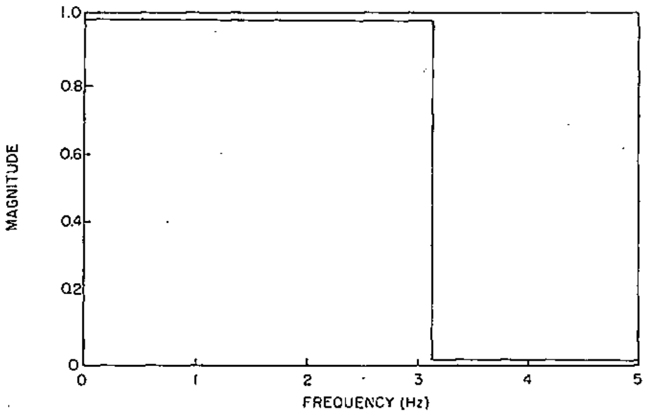

Fig. 11. Signal spectrum for Example 2.

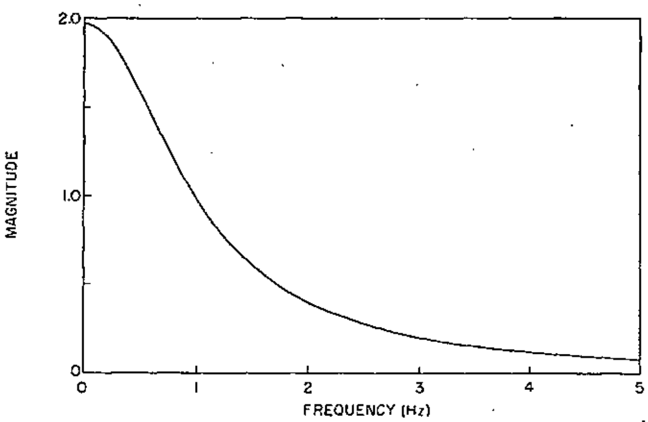

Fig. 12. Noise spectrum for Example 2.

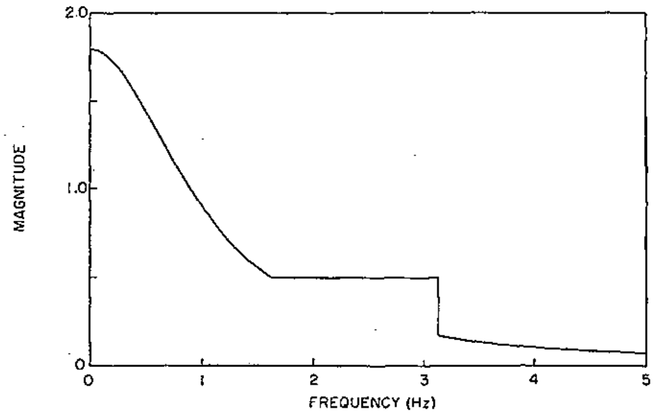

Fig. 13. Least favorable signal spectrum for Example 2.

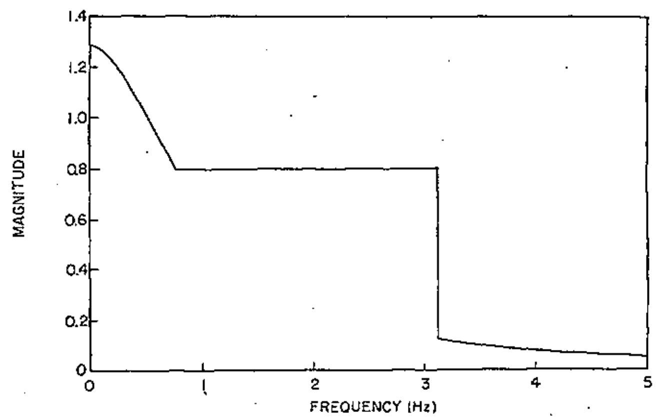

Fig. 14. Least favorable noise spectrum for Example 2.

the flat band-limited spectrum $G_{ss}^0(\omega)$ in first-order Markov noise with the spectrum $G_n^0(\omega)$ where the signal and noises are of comparable bandwidths (see Figs. 11 and 12). The uncertainty on the signal and noise spectra are modeled as the ϵ-mixtures

$$(1 - \epsilon_1)\, G_{ss}^0(\omega) + \epsilon_1 G_{ss}'(\omega) \tag{43}$$

and

$$(1 - \epsilon_2)\, G_n^0(\omega) + \epsilon_2 G_n'(\omega), \tag{44}$$

respectively, with $G_{ss}'(\omega)$ and $G_n'(\omega)$ arbitrary spectra having the same mass as the nominal and ϵ_1 and ϵ_2 lying in the interval $[0, 1]$. When $\epsilon_1 = 0.2$ and $\epsilon_2 = 0.1$, the least favorable signal and noise spectra are plotted in Figs. 13 and 14, respectively. This corresponds to the case where one may have more confidence in the nominal noise than in the nominal signal. The least favorable spectra for this example illustrate a typical attribute of least favorables in that the worst performance of a Wiener filter occurs when the signal masquerades as the noise and vice versa, i.e., when we get a minimum separation of hypotheses concerning the presence or absence of the signal within the uncertainty classes (43) and (44). Fig. 15 shows the relative performance for the nominal spectra and Fig. 16 the performance for the least favorable signal and noise spectra for Wiener filtering. Looking at the nominal case, we note that the use of the RWP entails a loss of about 3 dB at low SNR (below about 0 dB) over the optimal for the least favorable pair. However, when the true signal and noise spectra are

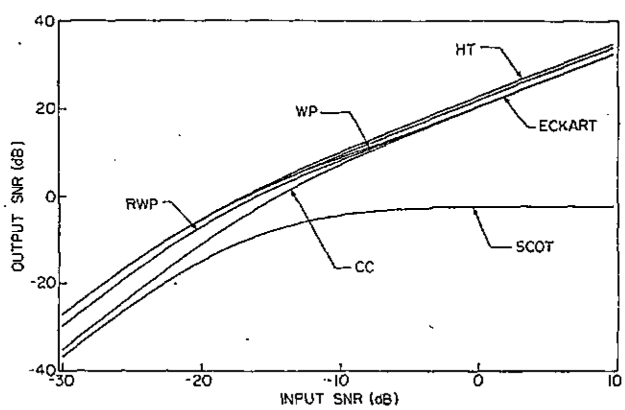

Fig. 15. Output SNR_1 for various processors at the nominal spectrum.

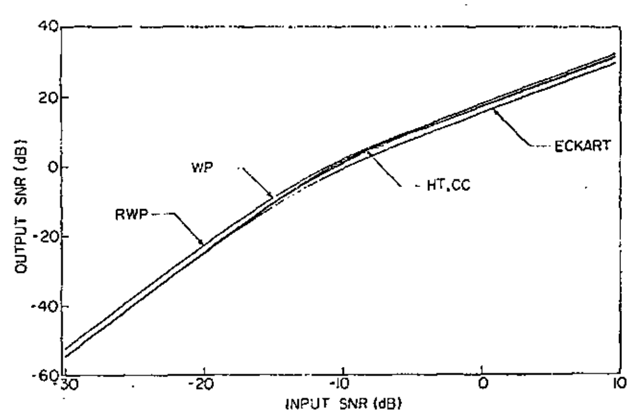

Fig. 16. Output SNR_1 for various processors at the least favorable spectrum.

least favorable for Wiener filtering, the RWP displays uniformly better relative performance, gaining about 3 dB over the other processors at low signal-to-noise ratios. Note that the pair in Figs. 13 and 14 is not necessarily the least favorable pair for HT filtering, so that no conclusive result is indicated here. However, Fig. 16 does suggest that, at least for some spectra in the above uncertainty class, we can expect better performance with the RWP than with the optimal scheme for the nominal spectra.

VI. Conclusion

We have outlined the development of several of the most popular GCC implementations and have introduced a simple GCC, the WP, which is believed to be new. Although the new GCC was motivated by intuition rather than any overall optimality considerations, it has been shown that the WP can perform well with respect to the other optimal schemes, as measured by their respective optimality criteria. For cases where the spectral uncertainty can be modeled as restricted to a specific class, classical robustness theory leads to a simple modification of the WP, which we called the RWP. The evaluation of its theoretical performance for a typical uncertainty class of signal and noise spectra was performed, which indicated a potential gain in output signal-to-noise ratio relative to the other GCC implementations. The above results suggest that the WP and the RWP may be viable alternatives to existing time delay estimation schemes. Further experimental- and simulation-based performance evaluation is required before any general conclusions can be drawn.

Acknowledgment

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