Games People Play

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November 1988 †

Abstract

Büchi's approach to the determinacy of infinite games was original and not without a controversy. He resented determinacy proofs that did not live up to his standards of constructivity. We describe the infinite games in question, discuss the constructivity of determinacy proofs and comment on Büchi's contributions to determinacy.

1 Games and Strategies

The infinite games of interest to us here were introduced by Gale and Stewart [8]. In this section we recall the definition of the games. To simplify the exposition, we restrict attention to a special case when the set of possible moves in any position is a fixed finite set A.

We view A as an alphabet. As usual, A^* is the set of strings over A. Borrowing the terminology of Muchnik [12] we call functions from the set ω of natural numbers to A *superstrings*. Call a set U of superstrings *open* if every superstring $X \in U$ has a prefix $x \in A^*$ such that the *cone*

$$[x] = \{Y : \ Y \text{ is a superstring with a prefix } x \ \}$$

is included into U; this gives the well-known Cantor topology on the set of superstrings.

Each set W of superstrings gives rise to a game $G(W)$ between Player 0 and Player 1. In a course of the game, the two players build a superstring. Player 0 starts by choosing a letter a_0, then Player 1 chooses a letter a_1, then Player 0 chooses a letters a_2, and so on *ad infinitum*. If the resulting superstring (the *play*) $a_0 a_1 a_2 \ldots$ belongs to W then Player 0 wins; otherwise Player 1 wins. The *winning set* W_ε of a Player ε is the collection of plays where he is victorious. If $\varepsilon = 0$ then $W_\varepsilon = W$, and if $\varepsilon = 1$ then W_ε is the complement of W.

*The work was partially supported by NSF grant DCR 85-03275. Also, during the final stage of the work the author was with Stanford University and IBM Almaden Research Center (on a sabbatical leave from the University of Michigan).

†Meantime published in "Collected Works of J. Richard Büchi", ed. Saunders Mac Lane and Dirk Siefkes, Springer-Verlag, 517–524.

A game $G(W)$ is called *determined* if one of the players has a winning strategy. To formalize the notion of a strategy, let us notice that every string $x \in A^*$ can be viewed as a position in the game $G(W)$; if the length $|x|$ of x is even then Player 0 makes a move in position x, and if $|x|$ is odd then Player 1 does. For technical reasons, it is convenient to allow nondeterministic strategies. Formally, a *strategy* for a Player ε is any function F that assigns a nonempty subset $F(x)$ of A (the set of recommended moves) to each position x in the game where Player ε makes a move. A strategy F is *deterministic* if every $F(x)$ comprises a single letter (the recommended move). Consider the case when both players follow some strategies; if both strategies are deterministic then the play is uniquely determined, otherwise the number of plays consistent with the two strategies is in the range from one to the power of continuum. It should be clear what a *winning strategy* is; following a winning strategy, the player wins regardless of what the other player does. A winning strategy may be nondeterministic.

Each position x determines a remainder of a game $G(W)$ called the *x-remainder* and denoted $G_x(W)$. The remainder is a game in its own right; if $|x|$ is even then Player 0 starts $G_x(W)$, otherwise Player 1 does. However, the initial position of $G_x(W)$ is x rather than the empty string; the plays of $G_x(W)$ form the cone $[x]$. Player ε wins a play X of $G_x(W)$ if $X \in W_\varepsilon$. The definitions of strategies and winning strategies for Player ε in the remainder game $G_x(W)$ should be clear.

2 Open Games

A game $G(W)$ is called *open*, $\mathcal{F}_\sigma$, *Borel*, etc. if W is so (in the Cantor topology). If U is an open set of superstrings, then a *support* for U is any string set S such that $U = \bigcup_{x \in S}[x]$. The collection of all strings x with $[x] \subseteq U$ is the largest support for U.

Theorem 1 (Gale and Stewart) *Every open or closed game $G(W)$ is determined.*

We give two proofs of the theorem. One is a proof by contradiction (essentially, the original proof); the other proof — also well known — is more constructive.

Proof 1. Suppose that a Player ε, whose winning set W_ε is open, does not have a winning strategy (in the initial position). A winning strategy F of his opponent Player δ is designed to ensure that Player ε has no winning strategy in any position in the course of the game. If Player ε does not have a winning strategy in $G_x(W)$ (and Player δ makes a move in x) then $F(x)$ comprises letters a such that Player ε has no winning strategy in $G_{xa}(W)$; otherwise we don't care what $F(x)$ is.

Any play X consistent with F belongs to W_δ. For, if it belongs to W_ε then it has a prefix x such that W_ε contains the whole cone $[x]$. But then Player ε has a winning strategy in $G_x(W)$ which is impossible. **End of Proof 1.**

To give a more constructive proof of Theorem 1, we define the *rank function* of a Player ε corresponding to a given string set S. It is the unique partial function ρ from A^* to ω such that for every string x:

- $\rho(x) = 0$ if and only if x has a prefix in S.

- $\rho(x) \leq n+1$ if and only if either Player ε makes a move in position x and $\rho(xa) < n+1$ for some letter a or else his opponent makes a move in position x and $\rho(xa) < n+1$ for every letter a.

Proof 2. Again, we suppose that W_ε is open and $\delta = 1 - \varepsilon$. Let S be any support for W_ε. Player ε wins a play X if and only if X meets S, i.e., X has a prefix that belongs to S. Let ρ be the rank function of Player ε corresponding to S.

If the rank of the initial position is defined then Player ε has a winning strategy F that can be called "Decrease the rank": If $\rho(x)$ is defined and nonzero (and Player ε makes a move in x) then $F(x)$ comprises all letters a such that $\rho(xa) < \rho(x)$; otherwise we don't care what $F(x)$ is.

If the rank of the initial position is undefined then Player δ has a winning strategy F' that can be called "Keep off the ranked positions": If $\rho(x)$ is undefined (and Player δ makes a move in x) then $F'(x)$ comprises all letter a such that $\rho(xa)$ is undefined; otherwise we don't care what $F'(x)$ is. **End of Proof 2**

The second proof actually presents winning strategies. (These winning strategies will be used in the next section.) If the rank function ρ is computable then the second proof provides algorithms to decide who of the two players has a winning strategy and to execute the winning strategy.

3 $\mathcal{F}_\sigma$ games

Recall that an $\mathcal{F}_\sigma$ set is the intersection of a countable collection of open sets, and a $\mathcal{G}_\delta$ set is the complement of an $\mathcal{F}_\sigma$ set.

Theorem 2 (Wolfe) *Every $\mathcal{F}_\sigma$ or $\mathcal{G}_\delta$ game $G(W)$ is determined.*

Again we give two proofs. The first one is a proof by contradiction along the lines of the original proof. The second one, close to the proof of Wolfe's theorem in Moschovakis' book [11], is more constructive.

Proof 1. Fix open sets U_n such that the winning set of a Player ε is the intersection of the sets U_n; without loss of generality, each U_n includes U_{n+1}. Choose supports S_n for U_n in such a way that:

- Each S_n is an antichain. In other words, if x and y belong to the same S_n then the cones $[x]$ and $[y]$ are disjoint.

- Each S_n is below S_{n+1} (if A^* is viewed as a tree that grows upward). More exactly, if $x \in S_n$, $y \in S_{n+1}$ and the cones $[x]$ and $[y]$ are not disjoint then x is a proper prefix of y.

Let T be the union of the antichains S_n. An arbitrary superstring X belongs to W_ε if and only if it meets T infinitely many times, i.e., it has infinite many prefixes in T. For every string x, let T_x be set of nonempty strings xy such that $xy \in T$. Let D be the set of positions x such that Player δ, the opponent of Player ε, has a winning strategy in the x-remainder of the game.

For every position $x \notin D$, Player ε has a strategy F_x in the x-remainder $G_x(W)$ of the game $G(W)$ that allows him to reach a position in $T_x - D$. For, suppose that such F_x does not exist. By Theorem 1, Player δ has a strategy to avoid $T_x - D$ in $G_x(W)$. Following such a strategy, Player δ either avoids T_x and this way wins $G_x(W)$ or hits D at some position y where he can start playing a strategy winning $G_y(W)$. Thus, he has a winning strategy in $G_x(W)$ which is impossible.

We describe a winning strategy for Player ε in any $G_{x_0}(W)$ such that $x_0 \notin D$: Use F_{x_0} to reach a position x_1 in $T_{x_0} - D$, then use F_{x_1} to reach a position x_2 in $T_{x_1} - D$, then use F_{x_2} to reach a position x_3 in $T_{x_2} - D$, and so on. **End of Proof 1.**

Proof 2. Suppose again that W_ε is the intersection of open sets U_n and $\delta = 1 - \varepsilon$. Let S_n be the largest support for U_n. The goal of Player ε is to hit every S_n. All rank functions in this proof are of Player ε.

If a position x is unranked with respect to some S_n then Player δ has an obvious winning strategy of keeping off S_n-ranked positions. Let P_1 be the collection of positions x ranked with respect to all S_n. From a position in P_1, Player ε can reach some position in any S_n, but that position may be unranked with respect to some other S_m. Let P_2 be the set of positions x that are ranked with respect to all $P_1 \cap S_n$. From a position in P_2, Player ε can hit any $P_1 \cap S_n$ from where he can hit any S_m; this is again insufficient for a winning strategy.

By induction on k, define P_{k+1} to be the set of positions ranked with respect to all $P_k \cap S_n$. The index k may be seen as a measure of "potential" to hit sets S_n. Starting in a position of potential k and given any sequence $(n_1, \ldots, n_k)$ of natural numbers, Player ε can hit S_{n_1} in a position of potential $k-1$, from where he can hit S_{n_2} in a position of potential $k-2$, and so on. A winning strategy seems to require an infinite potential.

Define $P_\omega = \bigcap_k R_k$. The potential of ω gurantees that Player ε can hit any finite sequence of sets S_n, but it does not guarantee hitting an infinite sequence of sets S_n. This motivates the following definition by induction on a (countable) ordinal α.

- $P_0 = A^*$.
- $P_{\alpha+1}$ is the set of positions ranked with respect to $P_\alpha \cap S_n$ for all n.
- If α is a limit ordinal then $P_\alpha = \bigcap_{\beta<\alpha} P_\beta$.
- P is any P_α equal to $P_{\alpha+1}$.

It is easy to see that the sequence of P_α decreases; hence indeed some $P_\alpha = P_{\alpha+1}$. Notice that if $x \in P$ then it is ranked with respect to every $P \cap S_n$. This allows us to construct a strategy F for Player ε winning every x-remainder game such that $x \in P$. If $m = \min\{n : n \notin S_n\}$ (we don't care what $F(x)$ is in the case that x belongs to all S_n), ρ is the rank function corresponding to $P \cap S_m$ and Player ε makes a move in x then

$$F(x) = \{a : \rho(xa) < \rho(x)\}.$$

Next we construct a strategy F' for Player δ winning every x-remainder game such that $x \notin P$. We don't care what $F'(x)$ is when $x \in P$. If $x \notin P$, let $\alpha(x)$ be the least ordinal α such that $x \in P_\alpha - P_{\alpha+1}$ and $n(x)$ be the least number n such that x is not ranked with respect to $P_\alpha \cap S_n$. If Player δ makes a move in a position $x \notin P$, then

$$F'(x) = \{a : xa \text{ is not ranked with respect to } P_{\alpha(x)} \cap S_{n(x)}\}.$$

Say that a position $x \notin P$ is δ-preferable to a position $y \notin P$ if either $\alpha(x) < \alpha(y)$ or else $\alpha(x) = \alpha(y)$ and $n(x) < n(y)$. Now suppose that $x \notin P$ and consider a play of $G_x(W)$ consistent with F'. Player δ steers clear of $P_{\alpha(x)} \cap S_{n(x)}$-ranked positions until a δ-preferable position y is reached. Then Player δ steers clear of $P_{\alpha(y)} \cap S_{n(y)}$-ranked positions until a δ-preferable position z is reached. And so on. From some moment on, Player δ sticks to steering clear of positions ranked with respect to a fixed $P_\alpha \cap S_n$. From that moment, all positions in the play belong to P_α and are not $P_\alpha \cap S_n$-ranked. This means that the play never hits S_n. **End of Proof 2.**

4 Constructive Determinacy

In a pioneering paper [2], Büchi and Landweber proved a constructive determinacy theorem for games $G(W)$, where the winning set W is a set of superstrings accepted by some finite automaton A. (The phenomenon of finite automata on superstrings is explained in McNaughton's article [10] in this volume.) Their proof yields an algorithm that, given an automaton A, decides which player has a winning strategy and constructs a finite automaton that executes a winning strategy. Later, Büchi tried to extend this result. In the case of more complicated winning sets, it was impossible to obtain strategies executable by finite automata, and Büchi sought strategies that can be played by finite automata which have the game conditions given by an oracle. This seems to be the essence of Büchi's constructivism in the field of game determinace.

By the way, the term "constructive" is used very informally here. Büchi was not an adherent of the school of constructive mathematics as far as I know. Problems of decidability of monadic second-order theories and some related issues led him to game determinacy. His intention was to give sharpened determinacy results. He was interested in winning strategies that can be defined or executed by certain restricted means; he was especially interested in winning strategies executable by finite automata. The constructivity we are talking about is more related to definability and complexity than to the philosophy of constructive mathematics. (Today, computational complexity folks improve Büchi's algorithms and complain that his papers are difficult to read).

In [4], Büchi and Klein proved a constructive determinacy theorem for $\mathcal{F}_\sigma$ games. The paper was never published. "The referee said it was not worth the trouble", complained Büchi in [7]. It was an exciting time for the "determinacy community" [11]. People tried to extend determinacy to larger and larger collections of sets and studied set-theoretic worlds where every Gale-Stewart game is determined. The complexity of winning strategies was very much in the center of attention. People have computed exactly the complexity (in the sense of descriptive set theory) of the simplest winning strategies for various classes of games, and constructive, substantially shorter proofs of the $\mathcal{F}_\sigma$ determinacy were well known to the community; one such proof is reproduced in the previous section. So, the referee's decision can be justified. Still, the special kind of constructivity of Büchi-Klein's proof is not found in other known proofs as far as I know.

An important application of determinacy is discussed in [6]. Let $\mathcal{B}$ be the collection of boolean combinations of $\mathcal{F}_\sigma$ sets. Büchi stated in [6] (for the proof the reader is referred to [5]) his result on constructive $\mathcal{B}$ determinacy and then sketched how this result implies Rabin's complementation lemma which is by far the hardest part in the famous proof by Rabin of the decidability of the monadic second-order theory of standard binary tree. This application is further discussed in paper [7] whose declared goal is to prove a constructive $\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}$ determinacy. Unfortunately, the papers [6] and [7] are indeed very hard to understand. ($\mathcal{B}$ games are analyzed in [9]; simpler proofs of Rabin's complementation lemma can be found in [9] and [12].)

The writing of Büchi cannot be called boring. He held strong views and was not shy to express them. In particular, Büchi spoke sharply against "fancy model-theoretical proofs". In this connection, let me notice that a "fancy" and nonconstructive proof may be a shortcut to a constructive result. Examples of such phenomenon can be found in papers like [13] or [9] that seemed to irritate Büchi. It is most important however that — even though Büchi's papers are difficult to reconstruct or verify — their ideas are worth exploring.

Acknowledgements

The title of this article is stolen from [1]. I am thankful to Martín Abadi, Alekos Kechris, Saunders MacLane, Dirk Siefkes, Wolfgang Thomas and Moshe Vardi for their reactions on a draft of this article.

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