What does O(n) mean?

Yuri Gurevich[1]
Electrical Engineering and Computer Science
University of Michigan, Ann Arbor, HI 48109-1109

In "Crusade for Better Notation" [1], Gilles Brassard calls one-way equations involving big omicron, big omega, etc. "bad, irrational and confusing notation". He promotes the following alternative that seems to him better and more natural: define O(t(n)), Ω (t(n)), etc. as sets and "use them as such, in accordance with set theory". However, the new notation calls for extending arithmetic operations (and, in due course, some other operations) from functions to sets of functions, and gives rise, as Gilles Brassard honestly and immediately acknowledges, to a new type of ambiguous expressions like $[O(n)]^2$ or $2^{O(n)}$. This does not seem very convenient to me, and I will try to defend the traditional notation here.

I agree with Donald E. Knuth [4] that "we understand the meaning of our existing notation quite well". In particular, Knuth's explanation of one-way equations with O-expressions in the first edition of "The Art of Programming", Volume 1 [2] seems clear to me. However Knuth never told us there what O(t(n)) *is*, and in the second edition [3] he defines O(t(n)) as a set (but continues to use one-way equations). So the problem seems to be to provide the meaning for O(t(n)), as well as Ω(t(n)), o(t(n)), etc., which is consistent with the traditional notation.

By the way, indefinite integrals pose a similar problem. Textbooks do not explain what ∫f(x)dx is, and it was suggested to define indefinite integrals as sets. I propose to view O(t(n)), Ω(t(n)), etc., as well as ∫f(x)dx, as *common names.*

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There are proper names, like 'Donald E. Knuth' or 'Gilles Brassard', and there are common names, like 'apple' or 'triangle'. A set of possible values is associated with a common name *v*; let us call it the *range* of *v*. The meaning of a common name is different from its range. An apple does not mean the set of all apples, and a triangle does not mean the set of all triangles. The meaning of a common name *v* is an indefinite element of the range of *v*. A function, defined on the range of a common name, can be applied to an indefinite element; one speaks about the height of a triangle and the peel of an apple. If α is a proper name and β is a common name then 'α is β' means that α belongs to the range of β; consider for example the sentence 'Donald E. Knuth is an author'. If α is a common name with range A and β is a common name with range B then 'α is β' means $A \subseteq B$; consider for example the sentence 'A triangle is a polygon'.

The range of O(t(n)) is a set of functions. Respectively, 'n+2 is O(n)' means that the function n+2 belongs to the range of O(n), 'O(n) is $O(n^2)$' means that the range of O(n) is included in the range of $O(n^2)$, and '$O(n)+O(n^2)$' means an indefinite function in the range of O(n) plus an indefinite function in the range of $O(n^2)$. A question may arise whether O(n)•O(n) is necessarily a function with only positive values. (Suppose that we allow negatively valued functions in the range of O(n)). The answer is NO. Different occurrences of the same common name in the same expression do not necessarily mean the same object. Consider for example the expression 'An integer plus an integer'; it does not necessarily mean an even integer. Consider the statement 'An integer plus an integer is an integer'; the third of the mentioned integers is not necessarily equal to the first one.

First-order set theory does not capture the notion of common names. (The notion of variables is somewhat similar but definitely different.) In the course of formalization, common names disappear. This should not discredit common names. First-order set theory was set up to provide consistent formalization of mathematics, that's all. Set theory can be combined with a theory of names (but not in the frame of first-order logic). Why not? Also, there are alternative (though by far less popular) formalizations of mathematics. According to Andreas Blass, who kindly read and commented on this letter, Lesniewski's ontology is a formal theory of names whose only primitive predicate is 'is'.

I admit some peculiarity of O(n) as a common name. We say '2n is O(n)' rather then '2n is an O(n)'. One can argue that O(n) is an attribute rather than a common name; an expression 'O(n)+O(n)', for example, can be seen as an abbreviation for 'An O(n)-function plus an O(n)-function'. Well, this is possible. People that use O-expressions very modestly may indeed comprehend O(n) as an attribute. The common name semantics suits better the extensive use of O-expressions.

Finally, '$\alpha(n) = \beta(n)$' simply means that $\alpha(n)$ is $\beta(n)$. I am not going to praise one-way equations, but they do shorten phrases. For example, '2n is O(n) which is $O(n^2)$' becomes '$2n = O(n) = O(n^2)$'.

References

1. Gilles Brassard, Crusade for a Better Notation. SIGACT News 17:1, 1985, 60-64.

2. Donald E. Knuth, The Art of Computer Programming, Volume P Fundamental Algorithms. First edition. Addison-Wesley, 1968.

3. Donald E. Knuth, The Art of Computer Programming, Volume 1-Fundamental Algorithms. Second edition. Addison-Wesley, 1973.

4. Donald E. Knuth, Big Omicron and Big Omega and Big Theta. SIGACT News 18'2, 1976, 18-24.